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Non-parametric identification of the extent of downward wage rigidities

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Abstract

We study the problem of identification of measures of the extent of individual types of downward wage rigidity from micro-level data on nominal wage growth rates, in the context of a wage adjustment process that may feature any number such rigidity types. For that purpose we develop a comprehensive framework for the modelling and measurement of wage rigidities. We show that the presence of measurement error does not alter fundamentally the nature of this identification problem, and develop an identification strategy that is applicable with measurement-error-free and measurement-error-contaminated data. This relies on weaker restrictions than those usually employed in the literature, including being non-parametric.

JEL Classification: J31, C24 **Keywords:** downward wage rigidity, identification, censoring, measurement error.

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1 Introduction

Real wage inflexibility has important theoretical implications for the functioning of the labour market and for macroeconomic outcomes. For example, firms that find it difficult to cut real wages in the face of an adverse demand shock may end up downsizing their workforce in order to reduce labour costs; accordingly, downward real wage inflexibility is invoked by the (neo-)Keynesian macroeconomic theory to explain the failure of labour markets to clear and the existence of (involuntary) unemployment. Also, as postulated by the neoclassical general equilibrium theory, wage and price flexibility are necessary for optimal resource allocation, including that for labour.

Given the theoretical importance of real wage flexibility, a corresponding empirical literature has developed over time that aims at verifying the existence and measuring the extent of wage (in-)flexibility. The oldest strand, which goes back to Dunlop (1938) and Tarshis (1939), is concerned with the measurement of the degree of responsiveness of the level of real wages to the cyclical variation in economic conditions. The methods typically employed by those studies seek to estimate the elasticity of the (mean) real wage with respect to some cyclical indicator, such as the unemployment rate and aggregate measures of output, using macro- or micro-level data on wage rates (see, for example, Abraham and Haltiwanger (1995) and Brandolini (1995) for an extensive survey of these methods). Following the seminal paper by McLaughlin (1994), a second line of empirical research uses micro-level data on nominal wage growth rates to investigate the nature of the (micro-level) mechanisms that could induce real wage inflexibility. Knowledge about such mechanisms can be particularly useful for the conduct of monetary policy, as a higher inflation level would "grease the wheels of the labour market" by allowing firms to implement larger real wage cuts when faced with resistance from workers to implement nominal wage cuts, i.e., due to Downward Nominal Wage Rigidity (DNWR)[1] – see Tobin (1972) for a discussion of this idea, and Akerlof et al. (1996) for formalisation and extensions to it. McLaughlin's approach hinges on that the existence of wage outcomes

[1]Note that pursuing such policy would, on the other hand, be ineffective if downward real wage inflexibility were instead induced *directly*, as a result of the workers' resistance to accept nominal wage increases below the prevailing anticipated inflation rate, i.e., due to Downward Real Wage Rigidity.

(of individual wage adjustments) that are restricted by downward wage rigidities would induce distortions to the shape of the distribution of the realised nominal wage growth rates associated with those wage adjustments, relative to the shape which would prevail in the absence of such rigidities. Accordingly, to the extent that different combinations of rigidity types distort differently the shape of that distribution, the existence and features of such distortions could potentially provide information about the types of rigidity affecting the wage outcomes; also, their size, about the extent of each individual rigidity type.

The identification of such distortions given knowledge of the shape of the distribution of the *observed* nominal wage growth rates, i.e., the factual distribution, can be particularly challenging, as is also the identification of the rigidity measures from the size of those distortions. Moreover, both tasks become harder when the data on the nominal wage growth rates are contaminated with measurement error, as that would induce additional distortions to the shape of the factual distribution that alter the distortions generated by wage rigidities (which, by definition, contain identifying information about the nature of those rigidities). The current literature views measurement error as noise, whose usual treatment involves imposing stronger identifying restrictions than those typically employed to obtain identification when there is no measurement error. Furthermore, identification results based on this approach are currently available only for measures of the extent of two particular types of downward wage rigidity, namely DNWR and Downward Real Wage Rigidity (DRWR), within two particular modelling contexts: one where all wage adjustments are assumed to be negotiated under DNWR, and another where DNWR is allowed to co-exist with DRWR.

In this paper we seek to generalise those results by considering the problem of identification of rigidity measures in the context of a wage adjustment process that can feature *any number* of downward rigidity types. Our key result is that, given data on nominal wage growth rates that are contaminated with classical measurement error, the nature of this problem is essentially the same as that given measurement-error-free data. Based on this we are then able to develop a new identification strategy that is applicable to both types of data, i.e., there is no special treatment of the measurement-error-contaminated data, involving additional restrictions, as in the existing literature. Specifically, this uses information about the size of the distortions in the shape of the corresponding (measurement-error-free or -contaminated) factual distribution relative to the shape of the relevant counterfactual distribution, the latter defined as the factual distribution that would prevail in the absence of rigidities. Using this strategy we are able to show that it is possible to obtain the same identification results from either type of data under the same set of restrictions, for any number of admissible downward wage rigidity regimes; furthermore these restrictions, which are non-parametric, are weaker than those employed by the current literature, especially for the case of measurement-error-contaminated data.

In addition to establishing new identification results, the work presented here contributes to this literature by developing a generic framework for the modelling and measurement of wage rigidities. This framework enables us to present a rigorous identification analysis, which in turn helps to clarify the contribution of individual restrictions to the process of obtaining identification. It further helps to unify models and results in the literature; for example to reconcile the alternative model-dependent rigidity measures that have been considered so far by establishing their relation to the generic (model-independent) measure proposed here.

The remainder of the paper is organised as follows: In Section 2 we provide a short overview of the related literature. In Section 3 we specify our model for the process that generates the observed nominal wage growth rates, and define the measure of rigidity that we seek to identify. Section 4 examines the effect of measurement error on the nature of the identification problem, and Section 5 presents our identification results. Section 6 summarises and concludes. In Appendix A we demonstrate how our identification results specialise to the standard case considered by the literature, where any given wage adjustment may be negotiated under DNWR, or DRWR, or the Flexible regime; we also compare our results to those in the literature. Proofs for all reported results are provided in Appendix B.

2 Overview of the literature

McLaughlin (1994) was the first study to consider extracting information relating to the existence of downward wage rigidities from features of the shape of the distribution of the observed (annual) nominal wage growth rates of individual workers. Subsequent studies that follow this approach[2] may be assigned to one of two groups. The first includes those concerned with establishing the *existence* of individual types of downward wage rigidity; Christofides and Stengos (2002) and Elsby (2009) are among those that focus on DNWR, whereas Christofides and Nearchou (2007, 2010) on DNWR and DRWR (NB: Some of these studies have also examined the existence of rigidity generated by menu costs). The second group includes studies that seek to estimate measures of the *extent* of the individual types of downward wage rigidity. A subset of those assumes that all wage adjustments are negotiated under DNWR, and in that context the only rigidity measure to be specified, identified, and estimated concerns the extent of DNWR; Card and Hyslop (1997), Kahn (1997), Altonji and Devereux (2000), Fehr and Goette (2005), Knoppik (2006), and Holden and Wulfsberg (2008) are among the studies that develop alternative methods for that purpose. Another subset assumes the co-existence of DNWR with DRWR, and in that context the tasks of specification, identification, and estimation concern measures for each of the two rigidity types; Dickens and Goette (2006), Dickens et al. (2007), and Goette et al. (2007) describe the three methods currently available that produce estimates of those measures.

Focusing next on the second group of studies, a notable feature of these is that different rigidity measures have been typically adopted in the two modelling contexts considered, even though the way nominal wage growth rates are assumed to be restricted by a given rigidity type, i.e., the "rigidity mechanisms", are the same. With regard to the latter, this mechanism is typically assumed to be a left-, partial-censoring mechanism, hereafter referred to as the "Standard DWR Mechanism".[3] This specifies that the realised nominal wage growth rate associated with a given wage adjustment is constrained by the relevant

[2] See Kramarz (2001), Stiglbauer (2002) and Palenzuela et al. (2003) for a survey of its earlier part.

[3] NB: Notable exceptions are the mechanisms considered by Altonji and Devereux (2000), Holden (2004), Dickens and Goette (2006) and Elsby (2009).

rigidity type – with a certain (censoring) probability – only if its unconstrained value lies below a threshold; that threshold, known as the "rigidity bound", is also assumed to be the constrained value of the realised nominal wage growth rate in the event that rigidity is binding. In the case of DNWR the relevant rigidity bound is the value of zero, whereas in the case of DRWR this is the inflation rate that is anticipated – at the time of wage bargaining – to prevail during the effective period of the associated wage agreement. It therefore follows that, in both the cases of the "Standard DNWR Process" and the "Standard D-NR-WR Process",[4] DNWR may prevent some of the nominal wage cuts that would take place in its absence, inducing instead nominal wage freezes. Similarly, and only in the context of the "Standard D-NR-WR Process", DRWR could prevent some of the anticipated real wage cuts that would take place in its absence, inducing instead anticipated real wage freezes.

With regard to the rigidity measures typically adopted in the two modelling contexts, in that of the Standard DNWR Process the measure for the extent of DNWR, given observable characteristics, is the proportion of unfulfilled nominal wage cuts; accordingly, this measure relates to the incidence of constrained wage adjustments (by DNWR). On the other hand, in the context of the Standard D-NR-WR Process the standard measure of the extent of a given type of rigidity (DNWR or DRWR), given observable characteristics, is the proportion of wage adjustments negotiated under that type, which is usually referred to as its "coverage". Accordingly in the latter context the measure for the extent of DNWR is the proportion of wage adjustments negotiated under DNWR, which is conceptually different from the incidence-type measure specified in the former modelling context.

The identification of either type of rigidity measure is inherently challenging, as neither the event of a wage adjustment being constrained nor the rigidity type under which a wage adjustment is negotiated are observable – meaning that neither type is directly identifiable from the available data. An indirect identification approach is followed instead, which relies on information derived from the discrepancies in the shape of the distribution of

[4] NB: The "Standard DNWR Process" refers to the wage adjustment process that features only DNWR where this is also described by the Standard DWR Mechanism; the "Standard D-NR-WR Process" refers to the wage adjustment process that features DNWR and DRWR where both are described by the Standard DWR Mechanism (and also, possibly, the Flexible regime that precludes any form of rigidity).

the realised nominal wage growth rates (the "actual distribution") relative to the shape of the notional distribution – of such wage growth rates – that would prevail in the absence of rigidities (the "flexible" distribution). This approach is based on that the existence of wage adjustments that are constrained by any one of the rigidity types admitted by the postulated model would result to the relocation of some of the probability mass of the "actual distribution", inducing distortions to its shape whose size reflect the *incidence* of wage adjustments that are constrained. Accordingly that size could convey information about incidence-type rigidity measures, such as the proportion of unrealised nominal wage cuts adopted in the context of the Standard DNWR Process. At the same time, when multiple rigidity types are admitted by the postulated model then that size would *also* reflect the prevalence of the wage adjustments negotiated under each rigidity type, and therefore could *also* convey information about "coverage"-type measures, such as the one considered in the context of the Standard D-NR-WR Process.

The implementation of this indirect identification approach entails, on the one hand, linking the unknown values of the rigidity measures to the distortions in the shape of the "actual distribution" and, on the other, identifying those distortions from the information provided by the data, i.e., from the knowledge of the distribution of the *observed* nominal wage growth rates – the factual distribution. With regard to the first task one may have to deal with that a particular distortion in the shape of the "actual distribution" is induced by the existence of wage adjustments constrained by *several* co-existing types of rigidity, and therefore not possible to associate its size to any particular one. Even if this is not the case, there is also the possibility that a given distortion is the result of probability mass being shifted *to* and *from* that part of the "actual distribution" by a given type of rigidity, resulting to its size understating the incidence of wage adjustments constrained by that.

The second task, i.e., of identifying the distortions in the shape of the "actual distribution", also poses challenges. In the case of *measurement-error-free* data there is the issue of identifying the shape of the – unobservable – "flexible distribution" from the factual distribution, where the latter coincides with the "actual distribution". Note that in that context the "flexible distribution" plays the role of the counterfactual. In the case of *measurement-error-contaminated* data the factual distribution does not coincide with the "actual distribution", and as a result the identification of the relevant distortions (in the shape of the "actual" distribution) becomes much harder; specifically, it would require to identify – parts of the – shapes of *both* the "actual" and "flexible" distributions from the (measurement-error-contaminated) factual distribution.

There are currently two approaches in the literature that seek to address the additional complications induced by the presence of measurement error (NB: Both share the same assumption of a classical measurement error, i.e., one that is additive to the actual value of the nominal wage growth rate variable, and stochastically independent of all variables in the model). The first seeks to purge measurement error from the data in a preliminary stage, followed by a second stage where the analysis of the "cleaned" data takes place as if they were in fact error-free (even though these are estimates of the corresponding error-free data that are produced under specific assumptions). There are two variants of this approach: a non-parametric one, developed by Gottschalk (2005), and another which is parametric, developed by Dickens et al. (2007). The second approach, on the other hand, proceeds with the specification of a fully parametric model of the wage adjustment process that allows explicitly for the presence of measurement error (NB: Normality is the typical assumption for all continuously distributed unobservable variables in the models specified). This works by imposing restrictions on the wage adjustment process that are sufficiently strong to identify the measurement-error distribution along with the "actual" and "flexible" distributions, the latter two jointly identifying the rigidity measures. This method was originally developed by Altonji and Devereux (2000) for the case where only DNWR is assumed to exist, and was later extended to the case where DNWR is allowed to co-exist with DRWR – see Goette et al. (2007) and Bauer et al. (2007a).

3 The model and rigidity measures

Next we proceed with the description of our model for the process that generates the observed nominal wage growth rates, and define the measure of rigidity that we seek to identify.

3.1 The observed wage adjustment process

We consider wage rigidities to be a feature of the wage adjustments of "job stayers".[5] Such (nominal) wage adjustments are typically observed to occur periodically over the course of the occupation of the job position; accordingly, in this context, a given wage adjustment is defined with respect to the relevant employer-employee-job combination and effective period of the associated wage agreement. Here we assume that any such wage adjustment may be negotiated under one of several wage adjustment regimes, which may feature some form of rigidity that could lead to an incomplete adjustment of the nominal wage *level* in response to changes in the economic conditions that are relevant for such employment relationships. The "Flexible" regime, which allows for complete adjustment, could also be among those regimes.

For our purposes, the outcome of interest from a given wage adjustment is the realised nominal wage growth rate, here denoted by $\dot{w}^* \in \mathbb{R}$ and referred to as the "actual rate".[6] This may be measured with error and therefore may be different from its observed value, here denoted by $\dot{w} \in \mathbb{R}$ and referred to as the "observed rate".[7] Assumptions 1-5 describe the process that generates $\dot{w}$, i.e., the "observed (wage adjustment) process". In particular, Assumptions 1-4 describe the process that generates $\dot{w}^*$, i.e., the "actual (wage adjustment) process", and Assumption 5 the measurement error model that links $\dot{w}$ to $\dot{w}^*$.

[5] For our purposes, these are workers in "stable" employment relationships who occupy the same job position over a substantial period of time during which they experience wage adjustments (NB: Here these are equivalent to *wage reviews*, as the outcome could be to leave the (nominal) wage unchanged). Accordingly our notion of rigidity is not associated with wage adjustments that might be incurred due to job changes.

[6] NB: Our notation omits the observation unit identifier since our analysis is restricted to the study of the identification problem.

[7] In practice the observed rate is constructed as the log-difference between the nominal wage rate reported to apply during the effective period of the given wage agreement and the rate reported to apply in the immediately preceding period. In practice the effective period typically lasts for a year, and therefore the calculated nominal wage growth rates are annual.

3.1.1 The modelling framework

Assumption 1. *(a) The actual rate for a given wage adjustment is determined according to the following mechanism:*

$$\dot{w}^* \;=\; \delta \cdot \dot{w}^{c*} + (1-\delta)\, \dot{w}^{N*} \;, \quad \dot{w}^{c*} \neq \dot{w}^{N*} \tag{1}$$

where δ is a binary variable that indicates whether the wage adjustment is constrained by rigidity or not, with $\delta = 1$ corresponding to the event that the wage adjustment is constrained. Accordingly, $\dot{w}^{N*} \in \mathbb{R}$ records the unconstrained value of $\dot{w}^*$ (the "(true) flexible rate"), and $\dot{w}^{c*} \in \mathbb{R}$ its constrained value (the "(true) constrained rate").

(b) $\dot{w}^{N*}$, $\dot{w}^{c*}$ and δ are unobservable; their joint probability distribution is described by the probability density function (PDF) $f_{\dot{w}^{N*},\dot{w}^{c*},\delta|R,x}$, where $x \in \mathcal{X}$ is a vector of observed heterogeneity characteristics and $R \in \mathcal{R}$ a categorical variable that records the unobservable wage adjustment regime that applies (given x).

Assumption 1 provides a *generic* description of the actual wage adjustment process that may feature rigidities. Assumption 1(a) simply allows for the possibility that the wage adjustment is constrained; Assumption 1(b) specifies that all right-hand-side (RHS) variables in equation (1) are unobservable, with their joint distribution exhibiting heterogeneity with respect to observed characteristics, as well as *additional* unobservable heterogeneity which we associate with the notion of the wage adjustment regime.

Given this setup, we can specify the features of a given actual wage adjustment process starting from the specification of the set of admissible wage adjustment regimes $\mathcal{R}$ and the set of observable characteristics $\mathcal{X}$. This defines a partition of the population into heterogeneity groups, each one characterised by the values of R and x. We can then specify the features of the wage adjustment mechanism associated with each such group by imposing restrictions on the associated distribution of unobservables $f_{\dot{w}^{N*},\dot{w}^{c*},\delta|R,x}$, which may be decomposed as follows:

$$\begin{aligned} f_{\dot{w}^{N*},\dot{w}^{c*},\delta|R,x}\left(\dot{w}^{N*},\dot{w}^{c*},\delta|R,x\right) &= f_{\dot{w}^{N*}|R,x}\left(\dot{w}^{N*}|R,x\right)\times \\ &\quad \times f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}\left(\dot{w}^{c*}|\dot{w}^{N*},R,x\right) \\ &\quad \times \Pr\left(\delta|\dot{w}^{N*},\dot{w}^{c*},R,x\right) \end{aligned} \tag{2}$$

The role of each of the RHS components in (2) becomes clear if we think of $\dot{w}^*$ (associated with a given wage adjustment with characteristics x, that is negotiated under R) as being determined in the following sequence of steps, given R and x:

Step 1: the value of $\dot{w}^{N*}$ is drawn from $f_{\dot{w}^{N*}|R,x}$,

Step 2: the value of $\dot{w}^{c*}$ is drawn from $f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}$, which depends on the value of $\dot{w}^{N*}$ drawn in Step 1,[8]

Step 3: the value of δ is drawn from a Bernoulli distribution with parameter:

$$\rho_x^{R*}\left(\dot{w}^{N*},\dot{w}^{c*}\right) \equiv \Pr\left(\delta=1|\dot{w}^{N*},\dot{w}^{c*},R,x\right) \tag{3}$$

which depends on the values of $\left(\dot{w}^{N*},\dot{w}^{c*}\right)$ drawn in Steps 1 and 2,[9] here referred to as the "intensity of rigidity" of type R at point $\left(\dot{w}^{N*},\dot{w}^{c*}\right)$,

Step 4: $\dot{w}^*$ is determined according to (1) given the drawn values of $\left(\dot{w}^{N*},\dot{w}^{c*},\delta\right)$.

3.1.2 The Standard DWR Process

Next using the above framework, we specify the features of the "Standard DWR Process", which encompasses the downward wage rigidity processes that feature the Standard DWR Mechanism, considered in this literature. We leave $\mathcal{R}$ and $\mathcal{X}$ unspecified, therefore allow for any number of downward rigidity regimes. Assumptions 2-4 impose restrictions on the distributions involved in Steps 1-3 of the above mechanism:

[8]NB: Taken together, Steps 1 and 2 are equivalent to drawing a pair of values $\left(\dot{w}^{N*},\dot{w}^{c*}\right)$ from $f_{\dot{w}^{N*},\dot{w}^{c*}|R,x}$.

[9]We further note that this is treated as a function of $\dot{w}^{N*}$ and $\dot{w}^{c*}$, whose domain is the support of $f_{\dot{w}^{N*},\dot{w}^{c*}|R,x}$, and also that it may be heterogeneous across R and x.

Assumption 2. *$f_{\dot{w}^{N*}|R,x}$ satisfies the following set of restrictions:*

(a) $\dot{w}^{N*}$ and R are conditionally independent given x; therefore $f_{\dot{w}^{N*}|R,x}$ is identical to $f_{\dot{w}^{N*}|x}$ and has support $\mathcal{W}_{Rx}^{N*} = \mathcal{W}_{x}^{N*}$, where $\mathcal{W}_{x}^{N*} \equiv \left[\dot{w}_{x0}^{N*}, \dot{w}_{x1}^{N*}\right]$ is the support of $f_{\dot{w}^{N*}|x}$.

(b) $f_{\dot{w}^{N*}|R,x}$ (therefore, also, $f_{\dot{w}^{N*}|x}$) is continuous.

(c) $f_{\dot{w}^{N*}|R,x}$ (therefore, also, $f_{\dot{w}^{N*}|x}$) is symmetric.

Assumption 3. *(a) Given R and x, the constrained rate $\dot{w}^{c*}$ may be either known or unknown:*

If unknown then this is treated as a random variable; its conditional distribution $f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}$ is continuous and has the same support as $f_{\dot{w}^{c*}|R,x}$ for all values of $\dot{w}^{N*}$, denoted by $\mathcal{W}_{Rx}^{c*} \equiv [\dot{w}_{Rx0}^{c*}, \dot{w}_{Rx1}^{c*}]$.

If known then this is treated as fixed with value denoted by $\dot{w}_{Rx1}^{c*}$, i.e., $\dot{w}^{c*} = \dot{w}_{Rx1}^{c*}$ for all values of $\dot{w}^{N*}$, and therefore $\mathcal{W}_{Rx}^{c*} \equiv \{\dot{w}_{Rx1}^{c*}\}$; for practical purposes we also define $\dot{w}_{Rx0}^{c*} = \dot{w}_{Rx1}^{c*}$.

(b) There may be only two configurations with regard to the relative position of $\mathcal{W}_{Rx}^{N*}$ and $\mathcal{W}_{Rx}^{c*}$: in the first $\mathcal{W}_{Rx}^{c*}$ lies to the left of $\mathcal{W}_{Rx}^{N*}$, i.e., $\dot{w}_{Rx1}^{c*} < \dot{w}_{x0}^{N*}$, and in the second $\mathcal{W}_{Rx}^{c*}$ is a proper subset of $\mathcal{W}_{Rx}^{N*}$ (i.e., $\mathcal{W}_{Rx}^{c*} \subset \mathcal{W}_{Rx}^{N*}$).

Assumption 4. *Let $\mathcal{W}_{Rx}^{Nc*} = \left[\dot{w}_{x0}^{N*}, \dot{w}_{x1}^{N*}\right] \times [\dot{w}_{Rx0}^{c*}, \dot{w}_{Rx1}^{c*}]$ denote the support of $f_{\dot{w}^{N*},\dot{w}^{c*}|R,x}$. Given R and x, the intensity of rigidity at point $\left(\dot{w}^{N*}, \dot{w}^{c*}\right) \in \mathcal{W}_{Rx}^{Nc*}$ satisfies the following restrictions:*

(a) $\rho_x^{R*}\left(\dot{w}^{N*}, \dot{w}^{c*}\right)$ is positive only if $\dot{w}^{N*} < \dot{w}^{c*}$, and

(b) if positive then $\rho_x^{R*}\left(\dot{w}^{N*}, \dot{w}^{c*}\right)$ is fixed and equal to $\overline{\varrho}_x^R$.

Accordingly:

$$\rho_x^{R*}\left(\dot{w}^{N*}, \dot{w}^{c*}\right) = \begin{cases} \overline{\varrho}_x^R \geq 0 & , \dot{w}^{N*} < \dot{w}^{c*} \\ 0 & , o/w \end{cases} \tag{4}$$

Under this specification the admissible wage adjustment regimes may only differ with respect to the distribution of constrained rates $f_{\dot{w}^{c*}|R,x}$ and the value of $\overline{\varrho}_x^R$. At the same time all are described by the same structural equation:[10]

$$\dot{w}^{*} = \begin{cases} \dot{w}^{N*} & , \dot{w}^{N*} \geq \dot{w}^{c*} \\ \delta \cdot \dot{w}^{c*} + (1-\delta)\,\dot{w}^{N*} & , \dot{w}^{N*} < \dot{w}^{c*} \end{cases} \tag{5}$$

which we refer to as the "Standard DWR Mechanism" due to its prevalence in the literature. Also, they share the same distribution of flexible (unconstrained) rates.

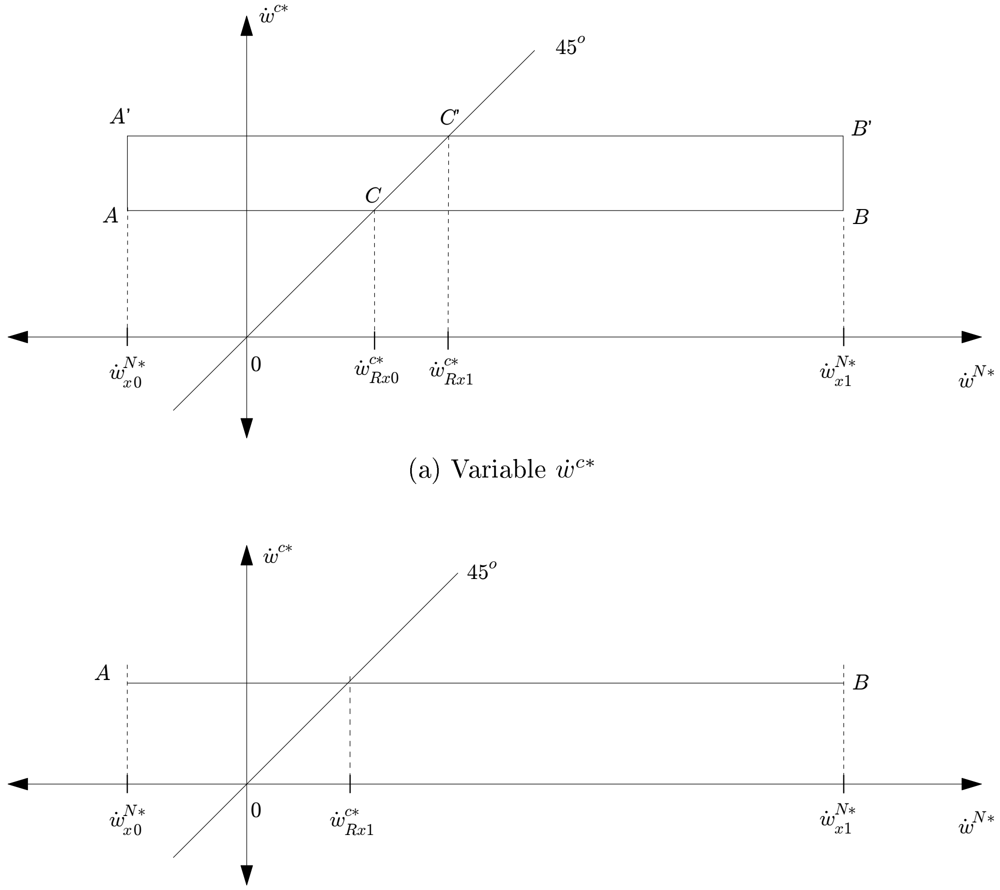
We further note that Assumptions 2-3 impose relatively mild restrictions on the joint distribution of the flexible and constrained rates. In particular, these are nonparametric and permit any form of dependence between the flexible and constrained rates. The flexible distribution $f_{\dot{w}^{N*}|R,x}$ is continuous and symmetric, while if non-degenerate (i.e., the case of variable constrained rate) $f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}$ is only restricted to be continuous.[11] As we discuss further below, Assumption 2(a) is essential for obtaining our identification results when $\mathcal{R}$ includes more than one regimes, Assumption 2(b) simplifies the notation and also provides additional identifying relationships when we work with error-free and there are case of fixed $\dot{w}^{c*}$, while Assumption 2(c) contributes to the identification the features of the counterfactual distribution although other types of shape restrictions on $f_{\dot{w}^{N*}|R,x}$ could also be used. Assumption 4(b) can be viewed as a generalisation of the "Proportionality" assumption used in the context of the Standard DNWR Process.

In Figures 1a and 1b we depict examples of the support of $f_{\dot{w}^{N*},\dot{w}^{c*}|R,x}$, i.e., $\mathcal{W}^{Nc*}_{Rx}$, for the cases of $\dot{w}^{c*}$ variable and fixed, respectively. In the former case this specialises to $\mathcal{W}^{Nc*}_{Rx} = \left[\dot{w}^{N*}_{x0}, \dot{w}^{N*}_{x1}\right] \times \left[\dot{w}^{c*}_{Rx0}, \dot{w}^{c*}_{Rx1}\right]$ (from Assumptions 2 and 3), which corresponds to the rectangle $ABB'A'$; in the latter case to $\mathcal{W}^{Nc*}_{Rx} = \left[\dot{w}^{N*}_{x0}, \dot{w}^{N*}_{x1}\right] \times \{\dot{w}^{c*}_{Rx1}\}$, which corresponds to the line segment AB.

Based on the definition of $\rho^{R}_{x}(\cdot,\cdot)$ we consider a wage adjustment with characteristics $\left(\dot{w}^{N*}, \dot{w}^{c*}, R, x\right)$ to be a *candidate to be constrained (by R)* if $\rho^{R}_{x}\left(\dot{w}^{N*}, \dot{w}^{c*}\right) > 0$. Accord-

[10]This follows from incorporating the restrictions of Assumption 4(a) onto equation (1).

[11]NB: In Assumption 3(a), and without loss of generality, we restrict the support of $f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}$ to be the same as that of $f_{\dot{w}^{c*}|R,x}$ in order to simplify notation. In general, the support of $f_{\dot{w}^{c*}|R,x}$ is given by the union of the supports of $f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}$ across all values of $\dot{w}^{N*}$.



(a) Variable $\dot{w}^{c*}$

(b) Fixed $\dot{w}^{c*}$

Figure 1: Support of $f_{\dot{w}^{N*},\dot{w}^{c*}|R,x}$ (for the case where $\mathcal{W}^{c*}_{Rx} \subset \mathcal{W}^{N*}_{Rx}$)

ingly, the set:

$$\mathcal{C}_{Rx}^{*} \equiv \left\{ \left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{W}_{Rx}^{Nc*} : \rho_x^R \left(\dot{w}^{N*}, \dot{w}^{c*} \right) > 0 \right\} \tag{6}$$

is the subset of the support of $f_{\dot{w}^{N*},\dot{w}^{c*}|R,x}$ that includes the pairs of values of flexible and constrained rates associated with the wage adjustments (with characteristics x) that are candidates to be constrained (by R). Under Assumption 4 this specialises to:

$$\mathcal{C}_{Rx}^{*} \equiv \left\{ \left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{W}_{Rx}^{Nc*} : \dot{w}^{N*} < \dot{w}^{c*}, \overline{\varrho}_x^R > 0 \right\} \tag{7}$$

Geometrically, and given $\overline{\varrho}_x^R > 0$, this set includes the subset of values of $\mathcal{W}_{Rx}^{Nc*}$ that lie to the left of the 45° line. For the case depicted in Figure 1a (variable $\dot{w}^{c*}$) this corresponds to the trapezoid $ACC'A'$ excluding the line segment CC', and for that depicted in Figure 1b (fixed $\dot{w}^{c*}$) to the line segment AC excluding point C; for *all* those values $\rho_x^R \left(\dot{w}^{N*}, \dot{w}^{c*} \right)$ is equal to $\overline{\varrho}_x^R$, and equal zero everywhere else.

3.1.3 The measurement error model

Assumption 5 completes our specification of the observed process by allowing for the possibility that the observed rate is contaminated with classical measurement error:

Assumption 5. *(Measurement error model)*

The observed rate relates to the actual rate as follows:

$$\dot{w} = \dot{w}^{*} + \dot{\varepsilon} \tag{8}$$

where $\dot{\varepsilon}$ denotes measurement error. This is distributed independently of all the unobservables in the model and the vector of observable characteristics; its PDF, denoted by $f_{\dot{\varepsilon}}$, is symmetric around point zero and has support $[-\dot{\varepsilon}_1, \dot{\varepsilon}_1]$.

It follows that for $f_{\dot{\varepsilon}}$ non-degenerate, i.e., the case of error-contaminated observed rates, $\dot{\varepsilon}$ has zero mean and can take equally likely positive and negative values of the same absolute size. In the case of error-free data ($\dot{\varepsilon} = 0$) then $f_{\dot{\varepsilon}}$ is degenerate, with all its probability mass concentrated at point zero.

3.2 Rigidity measure

We adopt as measure of (the extent of) the rigidity associated with the wage adjustment regime R, within the sub-population of wage adjustments with observable characteristics x, the proportion of wage adjustments constrained among those that are candidates to be constrained (by R). We denote this measure by ϱ_x^R.

Formally, let $\mathcal{R}_x$ be the subset of the admissible wage adjustment regimes under which there exist – with non-zero probability – negotiated wage adjustments, with characteristics x, that are candidates to be constrained:

$$\mathcal{R}_x \equiv \left\{ R \in \mathcal{R} : \Pr\left(\left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{C}^*_{Rx} | R, x \right) > 0 \right\} \tag{9}$$

We note that from the earlier discussion follows that if $\mathcal{W}^{c*}_{Rx} \subset \mathcal{W}^{N*}_{Rx}$ then $R \in \mathcal{R}_x$, while if $\mathcal{W}^{c*}_{Rx}$ lies to the left of $\mathcal{W}^{N*}_{Rx}$ (i.e., $\dot{w}^{c*}_{Rx1} < \dot{w}^{N*}_{x0}$) then $R \notin \mathcal{R}_x$. Then, for each $R \in \mathcal{R}_x$:[12]

$$\begin{aligned} \varrho_x^R &\equiv \Pr\left(\delta = 1 | \left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{C}^*_{Rx}, R, x \right) & (10) \\ &= \frac{\Pr\left(\delta = 1 | R, x \right)}{\Pr\left(\left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{C}^*_{Rx} | R, x \right)} & (11) \end{aligned}$$

We observe that this definition is *generic* in the sense that it does not depend on the postulated features of the underlying wage adjustment process, such as the number of admissible wage adjustment regimes and the particular features of the corresponding distributions of the unobservables. Furthermore that ϱ_x^R is a *conditional incidence* measure as it records the incidence of the wage adjustments constrained *among* those that are candidates to be constrained (conditional on R and x); accordingly it has the advantage of being independent of the "size" of the set of candidates associated with R and x, measured by $\Pr\left(\left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{C}^*_{Rx} | R, x \right)$, and therefore its values are comparable across R and x even when this "size" varies.

[12] Note that (11) follows from writing $\Pr\left(\delta = 1 | \left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{C}^*_{Rx}, R, x \right) = \frac{\Pr\left(\delta=1, \left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{C}^*_{Rx} | R, x \right)}{\Pr\left(\left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{C}^*_{Rx} | R, x \right)}$, and using that $\Pr\left(\delta = 1, \left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{C}^*_{Rx} | R, x \right) = \Pr\left(\delta = 1 | R, x \right)$ since by definition, and given R and x, the set of constrained wage adjustments ($\delta = 1$) is a subset of those that are candidates to be constrained ($\left(\dot{w}^{N*}, \dot{w}^{c*} \right) \in \mathcal{C}^*_{Rx}$). We further note that, for $R \in \mathcal{R} \setminus \mathcal{R}_x$ the denominator in (11) is equal to zero.

The following results hold under our assumptions:

Lemma 1. *(a) Under Assumption 4(a)* $\mathcal{R}_x = \left\{R \in \mathcal{R} : \Pr\left(\dot{w}^{N*} < \dot{w}^{c*}|R, x\right) > 0\right\}$. *Furthermore,* ϱ_x^R *specialises as follows for each* $R \in \mathcal{R}_x$ *:*

$$\varrho_x^R = \Pr\left(\delta = 1|\dot{w}^{N*} < \dot{w}^{c*}, R, x\right) \tag{12}$$

$$= \frac{\Pr\left(\delta = 1|R, x\right)}{\Pr\left(\dot{w}^{N*} < \dot{w}^{c*}|R, x\right)} \tag{13}$$

which is the proportion of unrealised wage adjustments with actual rate below $\dot{w}^{c*}$*, given* R *and* x*.*

(b) Under Assumptions 1 and 4, ϱ_x^R *is equal to* $\overline{\varrho}_x^R$*. Accordingly, for the process considered here, the rigidity measure coincides with a key model parameter.*

4 Nature of the identification problem

We assume that we have data on $(\dot{w}, x)$, where $x \in \mathcal{X}$, and $\dot{w}$ generated by the observed wage adjustment process described by Assumptions 1-5, thus possibly error-contaminated. Given such data the identification problem associated with the heterogeneity group with characteristics x is that of learning about the value of ϱ_x^R for each $R \in \mathcal{R}_x$ based on the knowledge of $f_{\dot{w}|x}$ for all $x \in \mathcal{X}$, where $f_{\dot{w}|x}$ is the factual distribution associated with heterogeneity group x.

In this section we explore the impact of measurement error on the nature of this problem: first we examine its impact on the properties of the observed process, and then on the interpretation of the rigidity measure.

4.1 Properties of the observed wage adjustment processes

Let $\dot{w}^N \equiv \dot{w}^{N*} + \dot{\varepsilon}$ and $\dot{w}^c \equiv \dot{w}^{c*} + \dot{\varepsilon}$ be the (possibly) error-contaminated unconstrained and constrained values of the observed rate, respectively.

Lemma 2. *Under Assumptions 1 and 5 the observed rate is given by the following ex-*

pression:

$$\dot{w} \;=\; \delta \cdot \dot{w}^c + (1-\delta)\,\dot{w}^N \;, \quad \dot{w}^c \neq \dot{w}^N \tag{14}$$

This result follows from the additivity of the rigidity mechanism equation (1) and of the measurement error model in (8).[13] By construction, $\dot{w}^N$, $\dot{w}^c$ and δ are unobservable; like in the error-free case the PDF of their joint distribution, conditional on R and x, may be decomposed as follows:

$$\begin{aligned} f_{\dot{w}^N,\dot{w}^c,\delta|R,x}\left(\dot{w}^N,\dot{w}^c,\delta|R,x\right) \;&=\; f_{\dot{w}^N|R,x}\left(\dot{w}^N|R,x\right) \times \\ &\quad \times f_{\dot{w}^c|\dot{w}^N,R,x}\left(\dot{w}^c|\dot{w}^N,R,x\right) \times \\ &\quad \times \Pr\left(\delta|\dot{w}^N,\dot{w}^c,R,x\right) \end{aligned} \tag{15}$$

where:

$$\rho_x^R\left(\dot{w}^N,\dot{w}^c\right) \;\equiv\; \Pr\left(\delta=1|\dot{w}^N,\dot{w}^c,R,x\right) \tag{16}$$

is the intensity of rigidity of type R at point $\left(\dot{w}^N,\dot{w}^c\right)$ in the support of $f_{\dot{w}^N,\dot{w}^c|R,x}$, denoted by $\mathcal{W}_{Rx}^{Nc}$ and given by:

$$\mathcal{W}_{Rx}^{Nc} = \left\{\left(\dot{w}^{N*}+\dot{\varepsilon},\dot{w}^{c*}+\dot{\varepsilon}\right) : \left(\dot{w}^{N*},\dot{w}^{c*}\right) \in \mathcal{W}_{Rx}^{Nc*}, \dot{\varepsilon} \in \left[-\dot{\varepsilon}_1,\dot{\varepsilon}_1\right]\right\}$$

Lemma 3 summarises the properties of the RHS terms in (15):

Lemma 3. *Under Assumptions 1-5 the following results hold:*
(a) $\dot{w}^N$ *is conditionally independent of* R *given* x*, therefore* $f_{\dot{w}^N|R,x}$ *coincides with* $f_{\dot{w}^N|x}$*; the latter is the error-contaminated* $f_{\dot{w}^{N*}|x}$*, and its value is given by:*

$$f_{\dot{w}^N|x}\left(\dot{w}|x\right) = \int_{-\dot{\varepsilon}_1}^{\dot{\varepsilon}_1} f_{\dot{w}^{N*}|x}\left(\dot{w}-\epsilon|x\right) f_{\dot{\varepsilon}}\left(\epsilon\right) d\epsilon \tag{17}$$

Accordingly, $f_{\dot{w}^N|x}$ *is continuous, symmetric, and with support* $\mathcal{W}_x^N \equiv \left[\dot{w}_{x0}^N,\dot{w}_{x1}^N\right] =$

[13] Clearly, in the absence of measurement error (14) reduces to (1).

$\left[\dot{w}_{x0}^{N*} - \dot{\varepsilon}_1, \dot{w}_{x1}^{N*} + \dot{\varepsilon}_1\right]$. *Also its mean and variance are given by:*

$$E\left(\dot{w}^N|x\right) = E\left(\dot{w}^{N*}|x\right) \tag{18}$$

$$Var\left(\dot{w}^N|x\right) = Var\left(\dot{w}^{N*}|x\right) + Var\left(\dot{\varepsilon}\right) \geq Var\left(\dot{w}^{N*}|x\right) \tag{19}$$

(b-1) Given R and x, let $\dot{w}^{c*} = \dot{w}_{Rx1}^{c*}$ for all values of $\dot{w}^{N*}$, i.e., fixed. Then $\dot{w}^c = \dot{w}_{Rx1}^{c*} + \dot{\varepsilon}$, therefore $f_{\dot{w}^c|\dot{w}^N,R,x}$ coincides with $f_{\dot{w}^c|R,x}$ which is the location-shifted measurement-error distribution. Its value is given by:

$$f_{\dot{w}^c|R,x}\left(\dot{w}|R,x\right) = f_{\dot{\varepsilon}}\left(\dot{w} - \dot{w}_{Rx1}^{c*}\right) \tag{20}$$

Accordingly $f_{\dot{w}^c|R,x}$ is continuous, with support $\mathcal{W}_{Rx}^c \equiv [\dot{w}_{Rx0}^c, \dot{w}_{Rx1}^c] = [\dot{w}_{Rx1}^{c*} - \dot{\varepsilon}_1, \dot{w}_{Rx1}^{c*} + \dot{\varepsilon}]$, and mean and variance given by:

$$E\left(\dot{w}^c|R,x\right) = \dot{w}_{Rx1}^{c*} \tag{21}$$

$$Var\left(\dot{w}^c|R,x\right) = Var\left(\dot{\varepsilon}\right) \geq 0 \tag{22}$$

(b-2) Given R, x, and $\dot{w}^N$, let $\dot{w}^{c*}$ be variable. Then $\dot{w}^c = \dot{w}^{c*} + \dot{\varepsilon}$ and distributed according to $f_{\dot{w}^c|\dot{w}^N,R,x}$, which is the error-contaminated $f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}$. Its value is given by:

$$f_{\dot{w}^c|\dot{w}^N,R,x}\left(\dot{w}|\dot{w}^N,R,x\right) = \int_{-\dot{\varepsilon}_1}^{\dot{\varepsilon}_1} f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}\left(\dot{w} - \epsilon|\dot{w}^{N*} - \epsilon, R, x\right) f_{\dot{\varepsilon}}\left(\epsilon\right) d\epsilon \tag{23}$$

Accordingly $f_{\dot{w}^c|\dot{w}^N,R,x}$ is continuous, with support $\mathcal{W}_{Rx}^c \equiv [\dot{w}_{Rx0}^c, \dot{w}_{Rx1}^c] =$ $[\dot{w}_{Rx0}^{c*} - \dot{\varepsilon}_1, \dot{w}_{Rx1}^{c*} + \dot{\varepsilon}_1]$, and mean and variance given by:

$$E\left(\dot{w}^c|\dot{w}^N,R,x\right) = E\left(\dot{w}^{c*}|\dot{w}^{N*},R,x\right) \tag{24}$$

$$Var\left(\dot{w}^{c}|\dot{w}^{N},R,x\right) = Var\left(\dot{w}^{c*}|\dot{w}^{N*},R,x\right)+Var\left(\dot{\varepsilon}\right) \geq Var\left(\dot{w}^{c*}|\dot{w}^{N*},R,x\right) \quad (25)$$

(c) $\rho_x^R\left(\dot{w}^N,\dot{w}^c\right)$ *is positive only if* $\dot{w}^N<\dot{w}^c$ *and, if so, equal to* $\overline{\varrho}_x^R$*, i.e.:*

$$\rho_x^R\left(\dot{w}^N,\dot{w}^c\right) = \begin{cases}\overline{\varrho}_x^R\geq 0 & ,\dot{w}^N<\dot{w}^c\\ 0 & ,o/w\end{cases} \quad (26)$$

Accordingly, (14) *is restricted as follows:*

$$\dot{w} = \begin{cases}\dot{w}^N & ,\dot{w}^N\geq\dot{w}^c\\ \delta\cdot\dot{w}^c+(1-\delta)\,\dot{w}^N & ,\dot{w}^N<\dot{w}^c\end{cases} \quad (27)$$

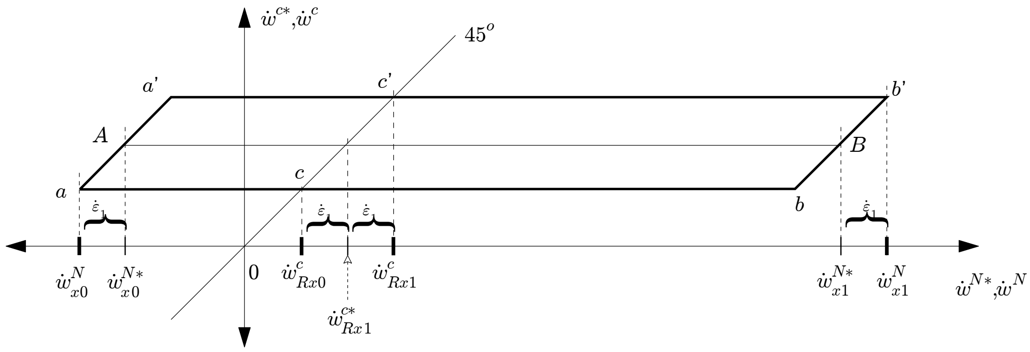
Taken together Lemmas 2 and 3 describe the properties of the observed wage adjustment process that may be error-contaminated.[14] From these follows that the only effect that measurement error has on the features of the observed wage adjustment process (for given x) is the increase in the *spreads* of $f_{\dot{w}^N|x}$ and $f_{\dot{w}^c|\dot{w}^N,R,x}$ relative to those of $f_{\dot{w}^{N*}|x}$ and $f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}$, respectively. By construction this, also, implies an increase in the spread of the support of $f_{\dot{w}^N,\dot{w}^c|R,x}$ (i.e., $\mathcal{W}_{Rx}^{Nc}$) relative to that of $f_{\dot{w}^{N*},\dot{w}^{c*}|R,x}$ (i.e., $\mathcal{W}_{Rx}^{Nc*}$) in both directions of $\dot{w}^N$ and $\dot{w}^c$.[15]

We depict these effects in Figures 2 and 3 for the cases of fixed and variable constrained rates, respectively. In Figure 2a, for $\mathcal{W}_{Rx}^{Nc}$ in the error-free case (i.e., $\mathcal{W}_{Rx}^{Nc*}$) corresponding to the line segment AB, then with the introduction of measurement error this expands into to the area of the parallelogram $abb'a'$.[16] In Figure 3a), for $\mathcal{W}_{Rx}^{Nc*}$ corresponding to

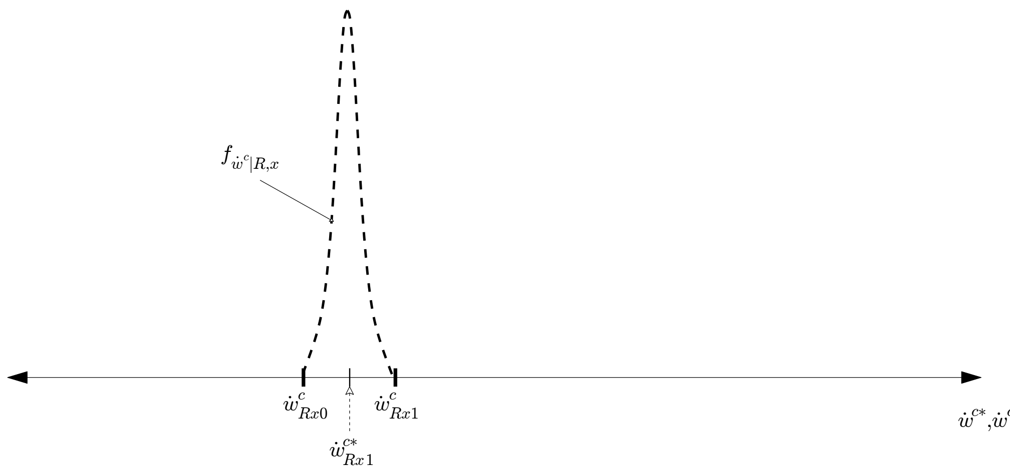
[14]It is easy to verify that in the error-free case (i.e., $f_{\dot{\varepsilon}}$ degenerate) these properties coincide with the restrictions on the actual wage adjustment process that follow from Assumptions 1-4. Specifically, setting $\dot{\varepsilon}=0$ then (14) in Lemma 2 reduces to equation (1) in Assumption 1. Also the results given in Lemma 3, parts (a), (b) and (c), give the restrictions on $f_{\dot{w}^{N*},\dot{w}^{c*}|R,x}$ imposed by Assumptions 2, 3 and 4, respectively.

[15] Accordingly it has no effect on the respective *location* of these distributions, nor on $\rho_x^R\left(\dot{w}^N,\dot{w}^c\right)$ relative to $\rho_x^{R*}\left(\dot{w}^{N*},\dot{w}^{c*}\right)$.

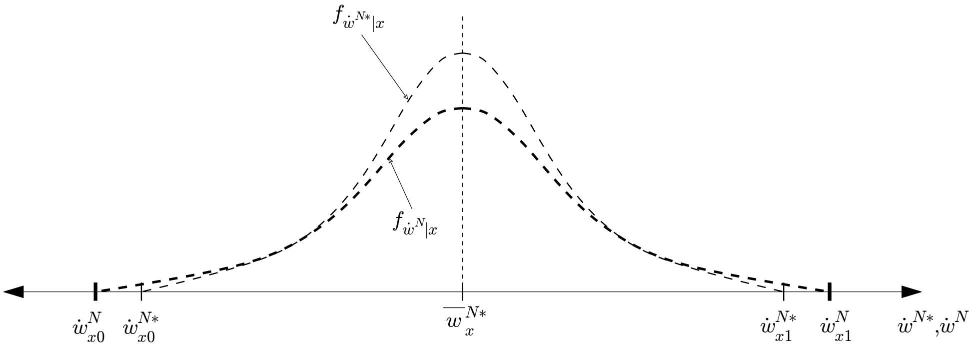
[16]This result follows from that the error-contaminated values of $\left(\dot{w}^N,\dot{w}^c\right)$ that correspond to a particular pair of error-free values of $\left(\dot{w}^{N*},\dot{w}^{c*}\right)$ belong to the set $\left\{\left(\dot{w}^{N*}+\dot{\varepsilon},\dot{w}^{c*}+\dot{\varepsilon}\right):\dot{\varepsilon}\in[-\dot{\varepsilon}_1,\dot{\varepsilon}_1]\right\}$. Geo-



(a) Support of $f_{\dot{w}^N, \dot{w}^c | R, x}$

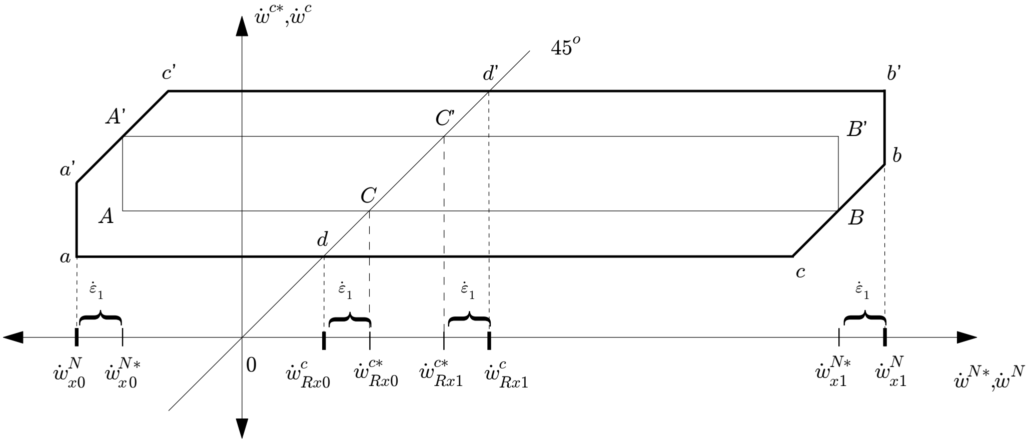


(b) Error-contaminated distribution of the constrained rate

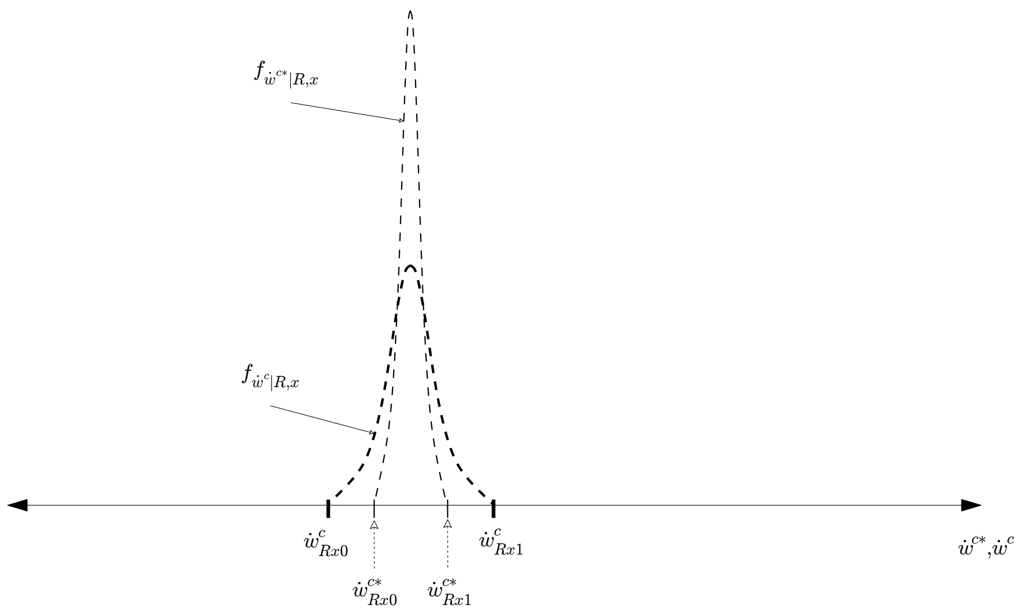


(c) Error-free and error-contaminated distributions of the unconstrained rate

Figure 2: Effect of measurement error, fixed constrained rate



(a) Support of $f_{\dot{w}^N, \dot{w}^c|R,x}$



(b) Error-free and error-contaminated distributions of the constrained rate

Figure 3: Effect of measurement error, variable constrained rate

the area of the rectangle $ABB'A'$, then with measurement error $\mathcal{W}_{Rx}^{Nc}$ expands to the area of $acbb'c'a'$. Also in Figures 2b (fixed constrained rate) and 3b (variable constrained rate) we can see the change (increase) in the spread of the corresponding graphs of $f_{\dot{w}^c|R,x}$ for the cases of error-free and error-contaminated data, and similarly in Figure 2c for $f_{\dot{w}^N|x}$. In all these cases the location of the distributions remains unchanged.

From the definition of $\rho_x^R(\cdot,\cdot)$ given in equation (16) follows that we can characterise wage adjustments as candidates to be constrained according to their associated values of $(\dot{w}^N, \dot{w}^c)$ in the same spirit as wage adjustments were be characterised (in Section 3) as candidates to be constrained according to the sign of $\rho_x^{R*}(\cdot,\cdot)$ given their associated values of $(\dot{w}^{N*}, \dot{w}^{c*})$. Specifically, a wage adjustment – with characteristics x that is negotiated under R – is a candidate to be constrained (by R) if, and only if the associated values $(\dot{w}^N, \dot{w}^c)$ are such that $\rho_x^R(\dot{w}^N, \dot{w}^c) > 0$. From (26) follows that under our assumptions this is equivalent to the pair $(\dot{w}^N, \dot{w}^c)$ satisfying the condition $\dot{w}^N < \dot{w}^c$. Accordingly:

$$\mathcal{C}_{Rx} \equiv \left\{(\dot{w}^N, \dot{w}^c) \in \mathcal{W}_{Rx}^{Nc} : \dot{w}^N < \dot{w}^c\right\} \tag{28}$$

is the subset of the support of $f_{\dot{w}^N,\dot{w}^c|R,x}$ that includes the pairs of unconstrained and constrained values of the *observed* rate associated with the wage adjustments with characteristics x that are candidates to be constrained by R.[17] Geometrically, this is the subset of the support $f_{\dot{w}^N,\dot{w}^c|R,x}$ that lies to the left of the 45° line. In the example depicted in Figure 2a (fixed constrained rate) this corresponds to the area of the trapezoid $acc'a'$, excluding the points along cc'. Also, in the example depicted in Figure 3a (variable constrained rate) this corresponds to the area of $add'c'a'$, excluding the points along dd'. By construction, $\rho_x^R(\dot{w}^N, \dot{w}^c)$ is equal to $\overline{\varrho}_x^R$ at each point in $\mathcal{C}_{Rx}$, and zero everywhere else.

metrically this is the *segment* of the 45° line that goes through point $(\dot{w}^{N*}, \dot{w}^{c*})$ and has end-points $(\dot{w}^{N*} - \dot{\varepsilon}_1, \dot{w}^{c*} - \dot{\varepsilon}_1)$ and $(\dot{w}^{N*} + \dot{\varepsilon}_1, \dot{w}^{c*} + \dot{\varepsilon}_1)$. For example, in the case of point $A = (\dot{w}_{x0}^{N*}, \dot{w}_{Rx1}^{c*})$, the corresponding pairs of error-contaminated values $(\dot{w}^N, \dot{w}^c)$ will lie on the line segment aa'. Taking the union of the sets of points that lie along such line segments, corresponding to each point on AB, gives the area of $abb'a'$. Using similar arguments we can derive the shape of $\mathcal{W}_{Rx}^{Nc}$ in the case of a variable constrained rate, as in Figure 3a.

[17]NB: This specialises to $\mathcal{C}_{Rx}^*$ in the error-free case.

4.2 Generalisation of the definition of the rigidity measure

Since $\dot{w}^{N*} < \dot{w}^{c*}$ implies, and is implied by, $\dot{w}^{N*} + \dot{\varepsilon} < \dot{w}^{c*} + \dot{\varepsilon}$ for any value of $\dot{\varepsilon}$, i.e., $\dot{w}^{N} < \dot{w}^{c}$, it follows that $\Pr\left(\dot{w}^{N} < \dot{w}^{c}|R,x\right) = \Pr\left(\dot{w}^{N*} < \dot{w}^{c*}|R,x\right)$. Substituting this into the denominator of (13) leads to the generalisation of the definition of ϱ_x^R, now stated in terms of the constrained and unconstrained values of the *observed* rate that might be error-contaminated:[18]

Lemma 4. *Under Assumptions 1-5 the rigidity measure associated with R and x satisfies the following:*

$$\varrho_x^R = \Pr\left(\delta = 1|\dot{w}^{N} < \dot{w}^{c}, R, x\right) \quad (29)$$

$$= \frac{\Pr\left(\delta = 1|R,x\right)}{\Pr\left(\dot{w}^{N} < \dot{w}^{c}|R,x\right)} \quad (30)$$

Given all the above, we can recast the identification problem as follows:

Proposition 1. *Given data on $(\dot{w}, x)$, where $x \in \mathcal{X}$, and $\dot{w}$ generated by the observed wage adjustment process described by Assumptions 1-5, the identification problem associated with the heterogeneity group with characteristics x is that of learning about the value of ϱ_x^R described in (29), for each $R \in \mathcal{R}_x$, based on the knowledge of $f_{\dot{w}|x}$ whose properties follow from the results reported in Lemmas 2 and 3.*

We observe that for this identification problem the presence of measurement error has the same relevance as would have an increase in the spread of $f_{\dot{w}^{N*},\dot{w}^{c*}|R,x}$ if working with error-free data.

5 Identification results

From Proposition 1 follows that the same identification strategy could be effective in both cases of error-free and error-contaminated data. Next we consider such strategy. This exploits the information that can be derived from the distortions in the shape of the

[18]NB: This simplifies to the original definition of ϱ_x^R if no measurement error, i.e., $\dot{w}^{N} = \dot{w}^{N*}$ and $\dot{w}^{c} = \dot{w}^{c*}$.

factual distribution $f_{\dot{w}|x}$ relative to that of $f_{\dot{w}^N|x}$, which is the factual distribution that would prevail in the absence of rigidities and here plays the role of the *counterfactual distribution*. In the case of error-free data this strategy specialises to that also used in the existing literature, i.e., it exploits the information from the distortions in the shape of the *actual* distribution ($f_{\dot{w}^*|x}$) relative to that of the *flexible* distribution ($f_{\dot{w}^{N*}|x}$). In the case of error-contaminated data this specialises to that which uses the information from the distortions in the shape of the *error-contaminated* actual distribution relative to that of the *error-contaminated* flexible distribution.

5.1 Factual Vs counterfactual distributions

5.1.1 Decomposition of the factual distribution

From the Law of Total Probability follows that we can write:[19]

$$f_{\dot{w}|x}\left(\dot{w}|x\right) = \sum_{\vartheta\in\mathcal{R}} \Pr\left(R=\vartheta|x\right) f_{\dot{w}|R,x}\left(\dot{w}|\vartheta,x\right) \tag{31}$$

Lemma 5 presents the decomposition of $f_{\dot{w}|R,x}$:

Lemma 5. *Under Assumptions 1 and 5, and for any $R\in\mathcal{R}$, $f_{\dot{w}|R,x}$ may be decomposed as follows:*

$$f_{\dot{w}|R,x}\left(\dot{w}|R,x\right) = f_{\dot{w}^N|R,x}\left(\dot{w}|R,x\right) + L_x^R\left(\dot{w}\right) + G_x^R\left(\dot{w}\right) \tag{32}$$

where:

$$L_x^R\left(\dot{w}\right) \equiv -f_{\dot{w}^N,\delta|R,x}\left(\dot{w},1|R,x\right) \leq 0 \tag{33}$$

$$G_x^R\left(\dot{w}\right) \equiv f_{\dot{w}^c,\delta|R,x}\left(\dot{w},1|R,x\right) \geq 0 \tag{34}$$

From the above follows that $L_x^R\left(\dot{w}\right)$ is the decrease ("loss") and $G_x^R\left(\dot{w}\right)$ the increase ("gain") in the value of $f_{\dot{w}|R,x}$ relative to that of $f_{\dot{w}^N|R,x}$, at point $\dot{w}$, caused by the existence of wage adjustments (with characteristics x) that are constrained (by R).

[19]NB: The analysis in this section applies to both cases of error-free and error-contaminated data.

Lemma 6 links $L_x^R(\cdot)$ and $G_x^R(\cdot)$ to $\mathcal{C}_{Rx}$ and $f_{\dot{w}^N,\dot{w}^c,\delta|R,x}$, i.e., the distribution of unobservables:[20]

Lemma 6. *Let $\mathcal{L}_{Rx}$ and $\mathcal{G}_{Rx}$ be subsets of $\mathcal{W}_{Rx}^N$ and $\mathcal{W}_{Rx}^c$, respectively, defined as follows:*

$$\mathcal{L}_{Rx} \equiv \left\{\dot{w}^N \in \mathcal{W}_{Rx}^N : \left(\dot{w}^N, \dot{w}^c\right) \in \mathcal{C}_{Rx}\right\} \tag{35}$$

$$\mathcal{G}_{Rx} \equiv \left\{\dot{w}^c \in \mathcal{W}_{Rx}^c : \left(\dot{w}^N, \dot{w}^c\right) \in \mathcal{C}_{Rx}\right\} \tag{36}$$

i.e., $\mathcal{L}_{Rx}$ is the projection of $\mathcal{C}_{Rx}$ on $\mathcal{W}_{Rx}^N$, and $\mathcal{G}_{Rx}$ the projection of $\mathcal{C}_{Rx}$ on $\mathcal{W}_{Rx}^c$. Also let $\mathcal{L}_{Rx}(\dot{w})$ and $\mathcal{G}_{Rx}(\dot{w})$ be the subsets of $\mathcal{L}_{Rx}$ and $\mathcal{G}_{Rx}$, respectively, defined as follows:

$$\mathcal{L}_{Rx}(\dot{w}) \equiv \left\{\dot{w}^N \in \mathcal{L}_{Rx} : \left(\dot{w}^N, \dot{w}\right) \in \mathcal{C}_{Rx}\right\} \tag{37}$$

$$\mathcal{G}_{Rx}(\dot{w}) \equiv \left\{\dot{w}^c \in \mathcal{G}_{Rx} : \left(\dot{w}, \dot{w}^c\right) \in \mathcal{C}_{Rx}\right\} \tag{38}$$

Then, under Assumptions 1 and 5, $L_x^R(\cdot)$ and $G_x^R(\cdot)$ are given by:

$$L_x^R(\dot{w}) = \begin{cases} -\int_{v\in\mathcal{G}_{Rx}(\dot{w})} \rho_x^R(\dot{w}, v)\, f_{\dot{w}^N,\dot{w}^c|R,x}(\dot{w}, v|R,x)\, dv \le 0 & , \dot{w} \in \mathcal{L}_{Rx} \\ 0 & , o/w \end{cases} \tag{39}$$

$$G_x^R(\dot{w}) = \begin{cases} \int_{\varpi\in\mathcal{L}_{Rx}(\dot{w})} \rho_x^R(\varpi, \dot{w})\, f_{\dot{w}^N,\dot{w}^c|R,x}(\varpi, \dot{w}|R,x)\, d\varpi \le 0 & , \dot{w} \in \mathcal{G}_{Rx} \\ 0 & , o/w \end{cases} \tag{40}$$

Corollary 1. *(a) $L_x^R(\dot{w}) \neq 0$ only if $\dot{w} \in \mathcal{L}_{Rx}$, and $G_x^R(\dot{w}) \neq 0$ only if $\dot{w} \in \mathcal{G}_{Rx}$.*
(b) If $\mathcal{C}_{Rx} = \emptyset$ then $\mathcal{L}_{Rx} = \mathcal{G}_{Rx} = \emptyset$, and $L_x^R(\dot{w}) = G_x^R(\dot{w}) = 0$ everywhere.
(c) If $\mathcal{W}_{Rx}^c$ is located to the left of $\mathcal{W}_{Rx}^N$, therefore $R \notin \mathcal{R}_x$, then $L_x^R(\dot{w}) = G_x^R(\dot{w}) = 0$ everywhere. If $\mathcal{W}_{Rx}^c \subset \mathcal{W}_{Rx}^N$, therefore $R \in \mathcal{R}_x$, then $L_x^R(\dot{w}) \ge 0$ and $G_x^R(\dot{w}) \ge 0$.

From (39) and (40) further follows that $\mathcal{L}_{Rx}$ is the subset of $\mathcal{W}_{Rx}^N$ from which there is relocation of probability mass of $f_{\dot{w}|R,x}$ (as loss) to $\mathcal{G}_{Rx}$ (as gain). We observe that the

[20]NB: The results reported in Lemma 5, as well as those in Lemmas 6 and 7, and in Corollary 1, all rely on Assumptions 1 and 5 only, and therefore apply for any specification of the distribution of the unobservables $f_{\dot{w}^{N*},\dot{w}^{c*},\delta|R,x}$.

direction of this relocation is always to the right since the constrained rate $\dot{w}^c$ is always larger than the corresponding unconstrained rate $\dot{w}^N$.

Combining all the above provides the following link between the factual and counterfactual distributions:

Lemma 7. *Under Assumptions 1 and 5 the factual distribution may be decomposed as follows:*

$$f_{\dot{w}|x}(\dot{w}|x) = f_{\dot{w}^N|x}(\dot{w}|x) + \sum_{\vartheta \in \mathcal{R}_x} \Pr(R = \vartheta|x) \left[L_x^{\vartheta}(\dot{w}) + G_x^{\vartheta}(\dot{w}) \right] \tag{41}$$

5.1.2 Distortions in the shape of the factual distribution

Lemma 8. *Let $f_{\dot{w}|x}(\dot{w}|x) - f_{\dot{w}^N|x}(\dot{w}|x)$ be the "distortion" in the shape of the factual distribution (relative to that of the counterfactual) at point $\dot{w}$. Under Assumptions 1 and 5 this satisfies:*

$$\begin{aligned} & f_{\dot{w}|x}(\dot{w}|x) - f_{\dot{w}^N|x}(\dot{w}|x) = \\ & = \sum_{\vartheta \in \mathcal{R}_x} \Pr(R = \vartheta|x) \left[f_{\dot{w}|R,x}(\dot{w}|R,x) - f_{\dot{w}^N|R,x}(\dot{w}|R,x) \right] && (42) \\ & = \sum_{\vartheta \in \mathcal{R}_x} \Pr(R = \vartheta|x) \left[L_x^{\vartheta}(\dot{w}) + G_x^{\vartheta}(\dot{w}) \right] && (43) \end{aligned}$$

Let $f_{\dot{w}|R,x}(\dot{w}|R,x) - f_{\dot{w}^N|R,x}(\dot{w}|R,x)$ be the distortion in the shape of $f_{\dot{w}|R,x}$ at point $\dot{w}$. From (42) follows that the distortion in the shape of $f_{\dot{w}|x}$ at point $\dot{w}$ is the weighted average, across $R \in \mathcal{R}$, of the distortions in $f_{\dot{w}|R,x}$ at the same point.[21] Furthermore from (43) follows that the size of this distortion will reflect the "loss" ($= L_x^R(\dot{w})$) and "gain" ($= G_x^R(\dot{w})$) in the value of $f_{\dot{w}|R,x}$ at that point for each $R \in \mathcal{R}_x$, as well as the *prevalence* of wage adjustments negotiated under each $R \in \mathcal{R}$ given x, which is measured by $\Pr(R|x)$.

Next we examine how $L_x^R(\cdot)$ and $G_x^R(\cdot)$ specialise under our modelling assumptions:

Lemma 9. *$L_x^R(\cdot)$ and $G_x^R(\cdot)$ specialise as follows under Assumptions 1-5:*

[21] Note that, for each $R \in \mathcal{R} \setminus \mathcal{R}_x$, the distortion in the shape of $f_{\dot{w}|R,x}$ at each point in its support is equal to zero.

(a) Let $\mathcal{W}^c_{Rx}$ be located to the left of $\mathcal{W}^N_{Rx}$. Then $\mathcal{L}_{Rx} = \mathcal{G}_{Rx} = \emptyset$, and $L^R_x(\dot{w}) = G^R_x(\dot{w}) = 0$ everywhere.

(b-1) Let $\mathcal{W}^c_{Rx} \subset \mathcal{W}^N_{Rx}$ and $\dot{w}^c$ variable, i.e., $\dot{w}^c \in \mathcal{W}^c_{Rx} = [\dot{w}^c_{Rx0}, \dot{w}^c_{Rx1}]$. Then:

$$L^R_x(\dot{w}) = \begin{cases} -\overline{\varrho}^R_x \int_{v \in \mathcal{G}_{Rx}(\dot{w})} f_{\dot{w}^N, \dot{w}^c|R,x}(\dot{w}, v|R, x)\, dv & , \dot{w} \in \mathcal{L}_{Rx} \\ 0 & , o/w \end{cases} \tag{44}$$

$$G^R_x(\dot{w}) = \begin{cases} \overline{\varrho}^R_x \int_{\varpi \in \mathcal{L}_{Rx}(\dot{w})} f_{\dot{w}^N, \dot{w}^c|R,x}(\varpi, \dot{w}|R, x)\, d\varpi & , \dot{w} \in \mathcal{G}_{Rx} \\ 0 & , o/w \end{cases} \tag{45}$$

where $\mathcal{L}_{Rx} = \left[\dot{w}^N_{x0}, \dot{w}^c_{Rx1}\right)$ and $\mathcal{G}_{Rx} = [\dot{w}^c_{Rx0}, \dot{w}^c_{Rx1}]$, and therefore $\mathcal{L}_{Rx} \cap \mathcal{G}_{Rx} = [\dot{w}^c_{Rx0}, \dot{w}^c_{Rx1})$.

(b-2) Let $\mathcal{W}^c_{Rx} \subset \mathcal{W}^N_{Rx}$ and $\dot{w}^c$ fixed, i.e., $\dot{w}^c \in \mathcal{W}^c_{Rx} = \{\dot{w}^c_{Rx1}\}$. Then:

$$L^R_x(\dot{w}) = \begin{cases} -\overline{\varrho}^R_x f_{\dot{w}^N|x}(\dot{w}|x) & , \dot{w} \in \mathcal{L}_{Rx} \\ 0 & , o/w \end{cases} \tag{46}$$

$$G^R_x(\dot{w}) = \begin{cases} \overline{\varrho}^R_x F_{\dot{w}^N|x}(\dot{w}_{Rx1}|x) & , \dot{w} \in \mathcal{G}_{Rx} \\ 0 & , o/w \end{cases} \tag{47}$$

where $\mathcal{L}_{Rx} = \left[\dot{w}^N_{x0}, \dot{w}^c_{Rx1}\right)$ and $\mathcal{G}_{Rx} = \{\dot{w}^c_{Rx1}\}$, and therefore $\mathcal{L}_{Rx} \cap \mathcal{G}_{Rx} = \emptyset$.

Accordingly, if $\mathcal{W}^c_{Rx} \subset \mathcal{W}^N_{Rx}$ (and $\overline{\varrho}^R_x > 0$) then $L^R_x(\dot{w})$ is non-zero at all values of $\mathcal{W}^N_{Rx}$ smaller than $\dot{w}^c_{Rx1}$. Furthermore it is proportional to the probability mass allocated to the wage adjustments that that satisfy $\dot{w}^N = \dot{w}$ *and* are candidates to be constrained. Also $G^R_x(\dot{w})$ is non-zero at all values of $\mathcal{W}^c_{Rx}$, and proportional to the probability mass allocated to the wage adjustments that satisfy the condition $\dot{w}^c = \dot{w}$ *and* are candidates to be constrained.

In Figures 4 and 5 we provide examples of $L^R_x(\dot{w})$ and $G^R_x(\dot{w})$ based on the expressions given in Lemma 9(b-1, b-2), for $\dot{w}^c$ variable and fixed, respectively. In Figure 4 we

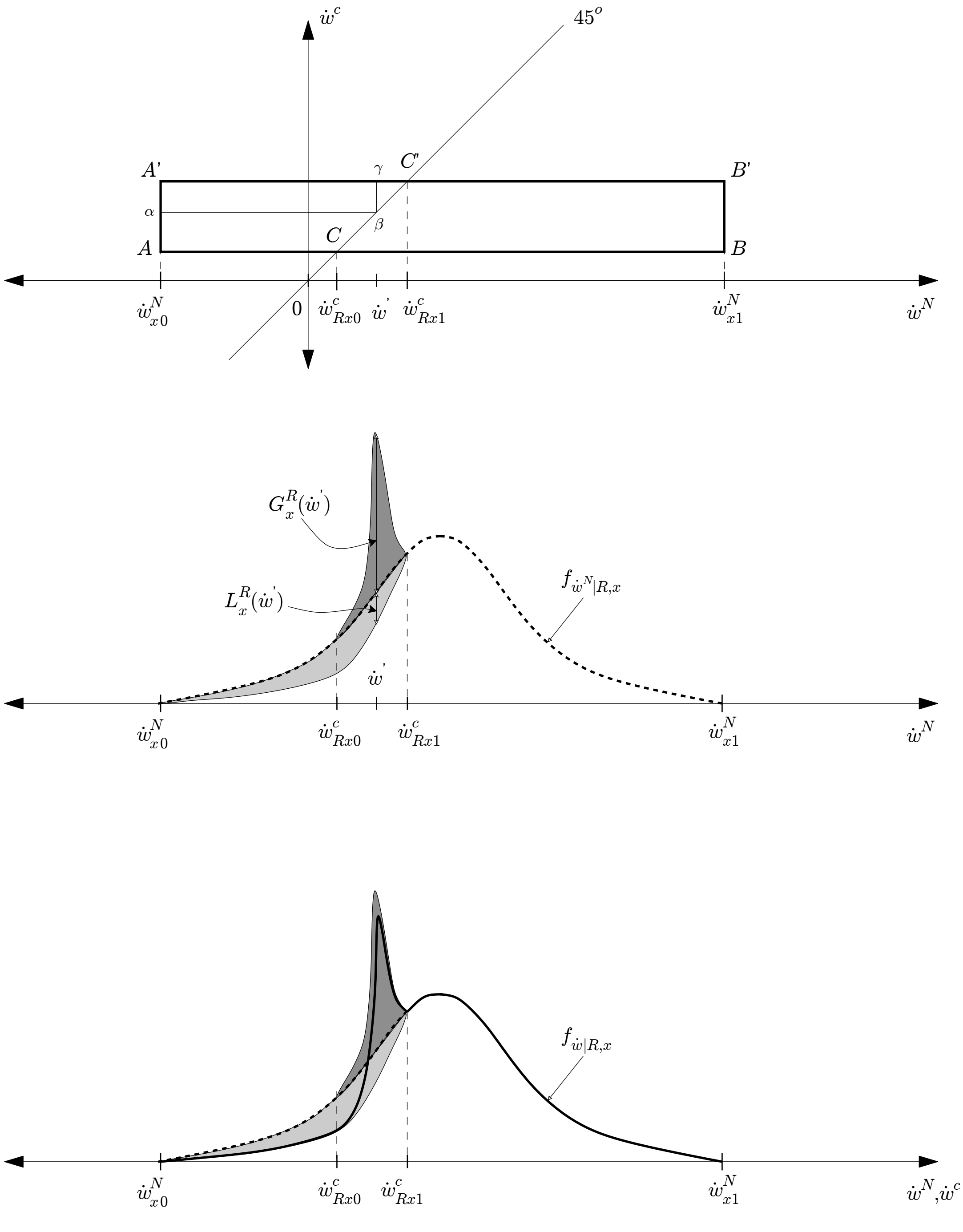


Figure 4: $L^R_x(\cdot)$, $G^R_x(\cdot)$, and $f_{\dot{w}|R,x}$ ($\dot{w}^c$ variable)

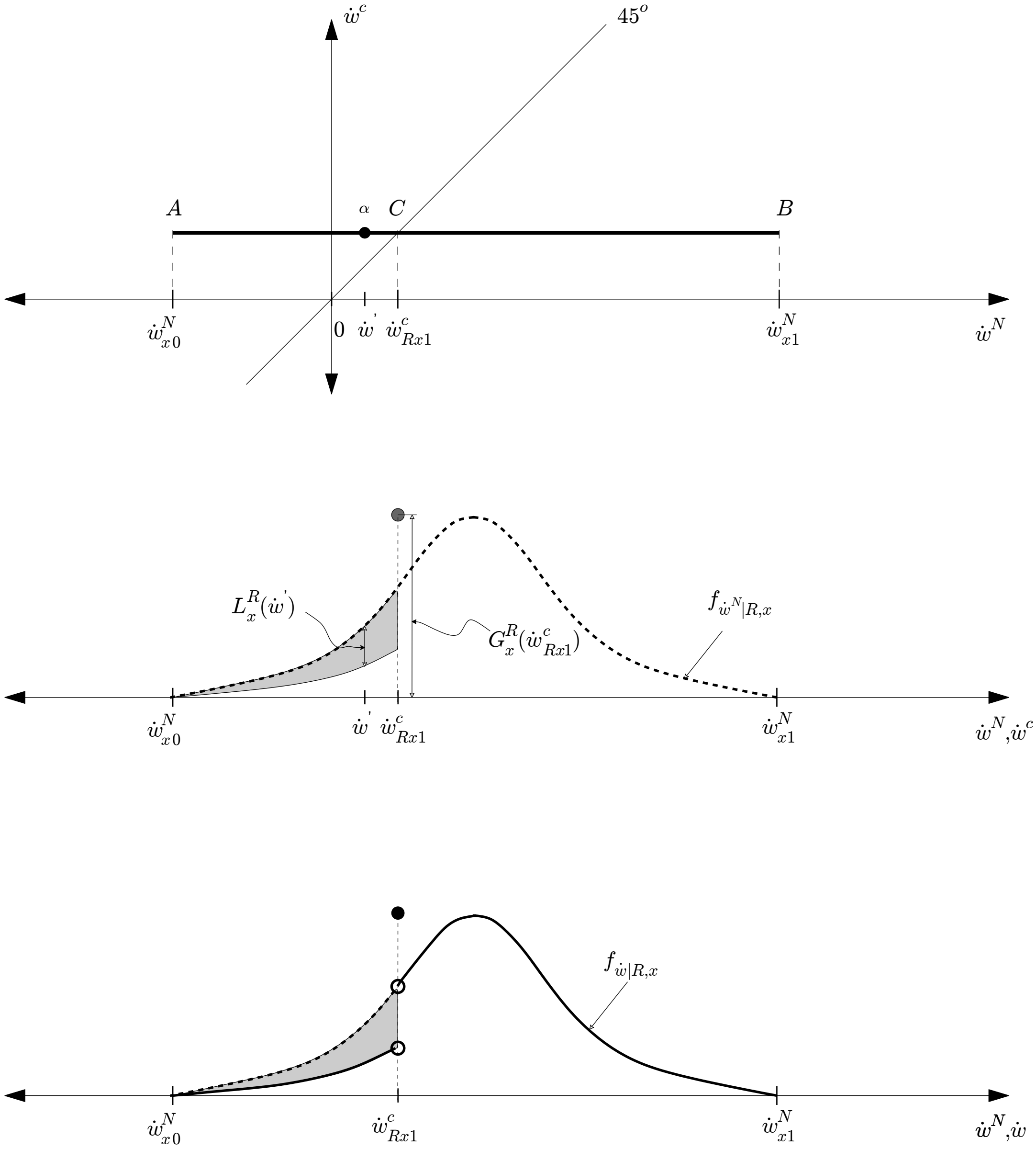


Figure 5: $L^R_x(\cdot)$, $G^R_x(\cdot)$, and $f_{\dot{w}|R,x}$ ($\dot{w}^c$ fixed)

have $\mathcal{W}^N_{Rx} = \left[\dot{w}^N_{x0}, \dot{w}^N_{x1}\right]$ and $\mathcal{W}^c_{Rx} = [\dot{w}^c_{Rx0}, \dot{w}^c_{Rx1}]$ such that $\dot{w}^N_{x0} < \dot{w}^c_{Rx0} < \dot{w}^c_{Rx1} < \dot{w}^N_{x1}$, and $\mathcal{W}^{Nc}_{Rx} = \mathcal{W}^N_{Rx} \times \mathcal{W}^c_{Rx}$ that corresponds to the area of the rectangle $AA'B'B$ (top diagram).[22] The value of $L^R_x(\cdot)$ at a given point $\dot{w} \in \mathcal{L}_{Rx} = \left[\dot{w}^N_{x0}, \dot{w}^c_{Rx1}\right)$ corresponds to the thickness of the light-grey-shaded area (middle diagram) located beneath the graph of $f_{\dot{w}^N|R,x}$ (depicted by the dotted line).[23] Furthermore, the value of $G^R_x(\cdot)$ at a given point $\dot{w} \in \mathcal{G}_{Rx} = [\dot{w}^c_{Rx0}, \dot{w}^c_{Rx1}]$ corresponds to the thickness of the dark-grey-shaded area located above the graph of $f_{\dot{w}^N|R,x}$.[24] It can be easily shown that the (total) light-grey- and dark-grey-shaded areas have the same size that is equal to $\Pr(\delta = 1|R, x)$, i.e., to the total probability mass of $f_{\dot{w}|R,x}$ that is relocated due to the existence of constrained wage adjustments.

In Figure 5 we have $\mathcal{W}^N_{Rx} = \left[\dot{w}^N_{x0}, \dot{w}^N_{x1}\right]$ and $\mathcal{W}^c_{Rx} = \{\dot{w}^c_{Rx1}\}$ such that $\dot{w}^N_{x0} < \dot{w}^c_{Rx1} < \dot{w}^N_{x1}$, and $\mathcal{W}^{Nc}_{Rx} = \mathcal{W}^N_{Rx} \times \mathcal{W}^c_{Rx}$ that corresponds to the line segment AB (top diagram).[25] Here too the value of $L^R_x(\cdot)$ at a given point $\dot{w} \in \mathcal{L}_{Rx} = \left[\dot{w}^N_{x0}, \dot{w}^c_{Rx1}\right)$ corresponds to the thickness of the light-grey-shaded area (middle diagram) located beneath the graph of $f_{\dot{w}^N|R,x}$ (depicted by the dotted line).[26] Furthermore $G^R_x(\dot{w}^c_{Rx1})$ corresponds to the height of the bullet point located above the graph of $f_{\dot{w}^N|R,x}$ at $\dot{w}^c = \dot{w}^c_{Rx1}$.[27]

From (32) follows that we can derive the graph of $f_{\dot{w}|R,x}$ by adding the thickness of the

[22]Given our earlier discussion, this is clearly the case where $\dot{w}^{c*}$ is variable such that $\mathcal{W}^{c*}_{Rx} \subset \mathcal{W}^{N*}_{Rx}$, and where also the data are free from measurement error. We note that the same qualitative results can be reached for the other two cases that can arise under our modelling assumptions, where $\dot{w}^c$ is variable and $\mathcal{W}^c_{Rx} \subset \mathcal{W}^N_{Rx}$: the first, when $\dot{w}^{c*}$ is variable such that $\mathcal{W}^{c*}_{Rx} \subset \mathcal{W}^{N*}_{Rx}$ and working with error-contaminated data, and the second when $\dot{w}^{c*}$ is fixed such that $\mathcal{W}^{c*}_{Rx} \subset \mathcal{W}^{N*}_{Rx}$ and working with error-contaminated data.

[23]For example in the case of point $\dot{w}'$ that lies in $\mathcal{G}_{Rx}$, therefore also in the overlap of $\mathcal{L}_{Rx}$ and $\mathcal{G}_{Rx}$, then $\mathcal{G}_{Rx}\left(\dot{w}'\right) = \left(\dot{w}', \dot{w}^c_{Rx1}\right]$, corresponding to the line segment $\beta\gamma$ excluding point β (see top diagram). From (44) follows that $L^R_x\left(\dot{w}'\right)$ is the value of the definite integral of $f_{\dot{w}^N,\dot{w}^c|R,x}\left(\dot{w}', \cdot|R, x\right)$ evaluated over those points, scaled by $-\overline{\varrho}^R_x$.

[24]For example, in the case of point $\dot{w}'$, then $\mathcal{L}_{Rx}\left(\dot{w}'\right) = \left[\dot{w}^N_{x0}, \dot{w}'\right)$, corresponding to the line segment $\alpha\beta$ excluding point β (see top diagram). From (45) follows that $G^R_x\left(\dot{w}'\right)$ is the value of the definite integral of $f_{\dot{w}^N,\dot{w}^c|R,x}\left(\cdot, \dot{w}'|R, x\right)$ evaluated over those points, scaled by $\overline{\varrho}^R_x$.

[25]From our earlier discussion it follows that this is the case where $\dot{w}^{c*}$ is fixed, $\mathcal{W}^{c*}_{Rx} \subset \mathcal{W}^{N*}_{Rx}$, and working with error-free data.

[26]For example, in the case of point $\dot{w}'$ that lies in $\mathcal{L}_{Rx}$, then $\mathcal{G}_{Rx}\left(\dot{w}'\right) = \{\dot{w}^c_{Rx1}\}$, corresponding to point α (see top diagram). From (46) follows that $L^R_x\left(\dot{w}'\right)$ is equal to $f_{\dot{w}^N|R,x}\left(\dot{w}'|R, x\right)$, scaled by $-\overline{\varrho}^R_x$.

[27]Note that $\mathcal{L}_{Rx}(\dot{w}^c_{Rx1}) = [\dot{w}^N_{x0}, \dot{w}^c_{Rx1})$, corresponding to the line segment AC excluding point C (see top diagram). From (47) follows that $G^R_x(\dot{w}^c_{Rx1})$ is given by the value of the definite integral of $f_{\dot{w}^N|R,x}(\cdot|R, x)$ evaluated over those points, scaled by $\overline{\varrho}^R_x$.

dark-grey-shaded area at any given point to the corresponding height of $f_{\dot{w}^N|R,x}$, and then subtracting from it the corresponding thickness of the light-grey-shaded area. Doing so for the cases considered in Figures 4 and 5 gives the solid lines depicted in their respective bottom diagrams. In the case of $\dot{w}^c$ being variable (Figure 4) $f_{\dot{w}|R,x}$ features a "deficit" (i.e., negative distortion) at each point in the interval to the left of point $\dot{w}^c_{Rx0}$ (i.e., in $\mathcal{L}_{Rx}\backslash\mathcal{G}_{Rx}$), resulting to a "cumulative deficit" in that interval. This is coupled with a "cumulative surplus" (i.e., positive distortion) of equal size in $\mathcal{G}_{Rx} = [\dot{w}^c_{Rx0}, \dot{w}^c_{Rx1}]$,[28] which appears as a "hump" in the shape of $f_{\dot{w}|R,x}$ in that interval. In the case of $\dot{w}^c$ being fixed (Figure 5) $f_{\dot{w}|R,x}$ also features a deficit at each point in the interval to the left of point $\dot{w}^c_{Rx0} = \dot{w}^c_{Rx1}$ (i.e., in $\mathcal{L}_{Rx}$, in this case), which results to a cumulative deficit in that interval; this is coupled with a surplus of equal size at point $\dot{w}^c_{Rx1}$, which manifests itself as a spike.

Lemma 10 provides results about the support of $f_{\dot{w}|R,x}$:

Lemma 10. *Let $\mathcal{W}_{Rx}$ denote the support of $f_{\dot{w}|R,x}$. The following hold under Assumptions 1-5:*

(a) if $\mathcal{W}^c_{Rx}$ is located to the left of $\mathcal{W}^N_{Rx}$ then $\mathcal{W}_{Rx}$ is equal to $\mathcal{W}^N_x = \left[\dot{w}^N_{x0}, \dot{w}^N_{x1}\right]$.

(b) if $\mathcal{W}^c_{Rx} \subset \mathcal{W}^N_{Rx}$ then, for both cases of $\dot{w}^c$ variable and fixed $\mathcal{W}_{Rx}$ is equal to $\mathcal{W}^N_x = \left[\dot{w}^N_{x0}, \dot{w}^N_{x1}\right]$ if $\overline{\varrho}^R_x < 1$, and equal to $\mathcal{W}^N_x \backslash \left[\dot{w}^N_{x0}, \dot{w}^c_{Rx1}\right) = \left[\dot{w}^N_{Rx0}, \dot{w}^N_{x1}\right]$ if $\overline{\varrho}^R_x = 1$.

From (31) follows, further, that we can derive the graph of $f_{\dot{w}|x}$ by adding vertically the *scaled* graphs of $f_{\dot{w}|R,x}$, $R \in \mathcal{R}$, where the mixing probabilities $\Pr(R|x)$ are the corresponding scale factors. The graph of $f_{\dot{w}^N|x}$ can be derived in a similar way using the graphs of $f_{\dot{w}^N|R,x}$, $R \in \mathcal{R}$ (NB: See examples in Section 5 for a particular specification of $\mathcal{R}$).[29]

Lemma 11 provides results about the support of $f_{\dot{w}|x}$:

Lemma 11. *Let $\mathcal{W}_x$ denote the support of $f_{\dot{w}|x}$. The following hold under Assumptions 1-5:*

[28] This follows from probability accounting since there are no distortions elsewhere other than in $\mathcal{L}_{Rx} \cup \mathcal{G}_{Rx}$. From where exactly in $\mathcal{L}_{Rx}$ the probability mass is re-allocated to where exactly in $\mathcal{G}_{Rx}$ depends on $f_{\dot{w}^N,\dot{w}^c|R,x}$.

[29] NB: In our case this is simpler as $f_{\dot{w}^N|R,x}$ is assumed to be the same as $f_{\dot{w}^N|x}$ (Assumption 2(a)).

(a) if $\mathcal{R}_x \subset \mathcal{R}$ then $\mathcal{W}_x$ is equal to $\mathcal{W}_x^N$.

(b) if $\mathcal{R}_x = \mathcal{R}$ then $\mathcal{W}_x$ is equal to $\mathcal{W}_x^N$ iff $\overline{\varrho}_x^R < 1$ for some $R \in \mathcal{R}$.

5.2 Measures Vs distortions

Given that the proportionality parameter $\overline{\varrho}_x^R$ that appears in the expressions for $L_x^{\vartheta}(\cdot)$ and $G_x^{\vartheta}(\cdot)$ that are given in Lemma 9 is equal to the rigidity measure ϱ_x^R (from Lemma 1), it follows that (43) can be used to examine how the latter relates to the distortions in the shape of the factual distribution.

5.2.1 Measures Vs spikes

Proposition 2 focuses on the case of spikes:

Proposition 2. *Given x, let $\mathcal{R}_x$ be non-empty. Furthermore let $\overline{\mathcal{R}}_x$ be the non-empty subset of $\mathcal{R}_x$ such that for each $R \in \overline{\mathcal{R}}_x$ the associated constrained rate $\dot{w}^{c*}$ is fixed and equal to $\dot{w}_{Rx1}^{c*}$, where $\dot{w}_{Rx1}^{c*} \neq \dot{w}_{R'x1}^{c*}$ for all $R' \in \overline{\mathcal{R}}_x$, $R' \neq R$. If f_ε is degenerate (error-free data) then, under Assumptions 1-5, $f_{\dot{w}|x}$ features a mass point at each value $\dot{w}_{Rx1}^{c*}$, $R \in \overline{\mathcal{R}}_x$, which attracts probability mass equal to:*

$$\Pr\left(\dot{w} = \dot{w}_{Rx1}^{c*}|x\right) = \varrho_x^{RR} F_{\dot{w}^N|x}\left(\dot{w}_{Rx1}^{c*}|x\right) \tag{48}$$

where:

$$\varrho_x^{RR} \equiv \Pr\left(R|x\right) \varrho_x^R \tag{49}$$

is the scaled rigidity measure associated with wage adjustment regime R.

We observe that the height of the spike at a given point $\dot{w}_{Rx1}^{c*}$ is proportional to the probability mass of the counterfactual distribution (here this is $f_{\dot{w}^{N*}|x}$) that lies to the left of it: specifically, if $\mathcal{R}$ admits a single wage adjustment regime then the scale factor is ϱ_x^R, whereas if it a admits several regimes then this is ϱ_x^{RR}. This result reflects the fact demonstrated in equation (42) that, given x, the size of the distortion in the shape of factual distribution at a given point reflects the size of the distortion in the shape of $f_{\dot{w}|R,x}$

at the same point, which depends on ϱ_x^R,[30] as well as the prevalence of wage adjustments negotiated under R, measured by $\Pr(R|x)$.

Also from (48) follows that, given error-free data, knowledge of the height of the spike at point $\dot{w}_{Rx1}^{c*}$, $\Pr(\dot{w} = \dot{w}_{Rx1}^{c*}|x)$, and of the amount of probability mass of the counterfactual distribution that lies to the left of it, $F_{\dot{w}^N|x}(\dot{w}_{Rx1}^{c*}|x)$, would be sufficient to point identify ϱ_x^R in the case of a single wage adjustment regime, and ϱ_x^{RR} in the case of multiple wage adjustment regimes.

5.2.2 Measures Vs cumulative distortions

Lemma 12 gives the expression for the cumulative distortion in the shape of the factual distribution to the left of any given point in its support:

Lemma 12. *Given x, let $\mathcal{R}_x$ be non-empty. Under Assumptions 1-5 the cumulative distortion in the shape of the factual distribution to the left of, and including, any point $\dot{w} \in \mathcal{W}_x$ is given by:*

$$F_{\dot{w}|x}(\dot{w}|x) - F_{\dot{w}^N|x}(\dot{w}|x) \quad = \quad -\sum_{\vartheta \in \mathcal{R}_x} \varrho_x^{\vartheta\vartheta} c_x^{\vartheta}(\dot{w}) \tag{50}$$

where:

$$c_x^R(\dot{w}) \equiv \iint_{(\varpi,v)\in\mathcal{C}_{Rx}:\varpi<\dot{w},v>\dot{w}} f_{\dot{w}^N,\dot{w}^c|R,x}(\varpi, v|R, x)\, dv d\varpi \geq 0 \tag{51}$$

is the probability mass of $f_{\dot{w}^N,\dot{w}^c|R,x}$ allocated to the points in the locus of $\mathcal{C}_{Rx}$ that lie to the left and above of $\dot{w}$.

Corollary 2. *(a) If $\mathcal{W}_{Rx}^c \subset \mathcal{W}_{Rx}^N$ and $\dot{w}^c$ is variable then $c_x^R(\dot{w})$ is given by:*

$$c_x^R(\dot{w}) = \begin{cases} F_{\dot{w}^N|x}(\dot{w}|x) & , \dot{w}_{x0}^N \leq \dot{w} < \dot{w}_{Rx0}^c \\ \int_{\dot{w}_{x0}^N}^{\dot{w}} \int_{\dot{w}}^{\max \mathcal{G}_{Rx}(\varpi)} f_{\dot{w}^N,\dot{w}^c|R,x}(\varpi, v|R, x)\, dv d\varpi & , \dot{w}_{Rx0}^c \leq \dot{w} < \dot{w}_{Rx1}^c \\ 0 & , \dot{w}_{Rx1}^c \leq \dot{w} \leq \dot{w}_{x1}^N \end{cases} \tag{52}$$

[30] As shown in Lemma 9.

where $\min \mathcal{L}_{Rx}(\dot{w})$ *is the minimum value in* $\mathcal{L}_{Rx}(\dot{w})$ *and* $\max \mathcal{G}_{Rx}(\dot{w})$ *is the maximum value in* $\mathcal{G}_{Rx}(\dot{w})$*.*

(b) If $\mathcal{W}^c_{Rx} \subset \mathcal{W}^N_{Rx}$ *and* $\dot{w}^c$ *is fixed then* $c^R_x(\dot{w})$ *is given by:*

$$c^R_x(\dot{w}) = \begin{cases} F_{\dot{w}^N|x}(\dot{w}|x) & , \dot{w}^N_{x0} \leq \dot{w} < \dot{w}^c_{Rx1} \\ 0 & , \dot{w}^c_{Rx1} \leq \dot{w} \leq \dot{w}^N_{x1} \end{cases} \tag{53}$$

We observe that the cumulative distortion to the left of a given point is negative, indicating a deficit, if this is located to the left of the maximum of $\dot{w}^c_{Rx1}$ across all $R \in \mathcal{R}_x$, and equal to zero elsewhere. The fact that, if non-zero, this can only be negative reflects that the direction of relocation of probability mass is always to the right.

The RHS of (50) is a linear combination of ϱ^{RR}_x for those values of $R \in \mathcal{R}_x$ that the corresponding coefficients $c^R_x(\dot{w}')$ are non-zero; from (52) and (53) follows further that this is the case for R such that $\dot{w}' < \dot{w}^c_{Rx1}$. It follows that with the appropriate choice of points $\dot{w}'$ in $\mathcal{W}_x$ we can derive a set of relationships that link different linear combinations of ϱ^{RR}_x to the cumulative distortion to the left of those points. For this purpose we define $K_x \geq 0$ to be the number of elements of $\mathcal{R}_x$. Furthermore, if $K_x > 1$ we write $\mathcal{R}_x = \{R_k : k = 1, \ldots, K_x\}$ such that for $\{R_{k-1}, R_k\} \subseteq \mathcal{R}_x$, $k = 2, \ldots, K_x$, holds that $\dot{w}^c_{R_{k-1}x1} \leq \dot{w}^c_{R_kx1}$; if $K_x = 1$ then $\mathcal{R}_x = \{R_1\}$.[31]

Proposition 3. *Given* x *let* $K_x \geq 1$*, therefore* $\mathcal{R}_x = \{R_1, \ldots, R_{K_x}\}$ *non-empty, where* $R \in \mathcal{R}_x \Leftrightarrow \mathcal{W}^c_{Rx} \subset \mathcal{W}^N_{Rx}$*. Furthermore if* $K_x > 1$ *we assume that for* $k = 2, \ldots, K_x$ *and* $\{R_{k-1}, R_k\} \subseteq \mathcal{R}_x$ *that* $\dot{w}^c_{R_kx1} \neq \dot{w}^c_{R_{k-1}x1}$*. Then under Assumptions 1-5, and given that* $\mathcal{W}_x = \mathcal{W}^N_x$*, there exist points* $\dot{w}'_1, \ldots, \dot{w}'_{K_x}$ *such that* $\dot{w}_{x0} < \dot{w}'_1 < \dot{w}^c_{R_1x1}$*, and* $\dot{w}^c_{R_{k-1}x1} < \dot{w}'_k < \dot{w}^c_{R_kx1}$ *for* $2 \leq k \leq K_x$*, for which the cumulative distortion in the shape of* $f_{\dot{w}|x}$ *to*

[31] Accordingly, the values of k indicate the relative position of the right limits of $\mathcal{W}^c_{Rx}$, $R \in \mathcal{R}_x$.

the left of, and including, each one of them is given by the following expressions:

$$\begin{aligned} F_{\dot{w}|x}\left(\dot{w}_1'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) &= -\sum_{j=1}^{K_x} \varrho_x^{R_jR_j} c_x^{R_j}\left(\dot{w}_1'\right) \\ F_{\dot{w}|x}\left(\dot{w}_2'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_2'|x\right) &= -\sum_{j=2}^{K_x} \varrho_x^{R_jR_j} c_x^{R_j}\left(\dot{w}_2'\right) \\ &\cdots \\ F_{\dot{w}|x}\left(\dot{w}_{K_x}'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_{K_x}'|x\right) &= -\varrho_x^{R_{K_x}R_{K_x}} c_x^{R_{K_x}}\left(\dot{w}_{K_x}'\right) \end{aligned} \tag{54}$$

The above is a triangular system of equations with respect to ϱ_x^{RR}, $R \in \mathcal{R}_x$, therefore can be solved for these parameters in terms of the values of $c_x^R(\cdot)$, $F_{\dot{w}|x}(\cdot)$ and $F_{\dot{w}^N|x}(\cdot)$ at points $\dot{w}_1', \ldots, \dot{w}_K'$. Accordingly, under Assumptions 1-5, knowledge of those values would be sufficient to point identify ϱ_x^{RR} for all $R \in \mathcal{R}_x$.

For the special case where the supports of the distributions of constrained rates (i.e., $\mathcal{W}_{Rx}^c$) associated with $R \in \mathcal{R}_x$ do not overlap, choosing points $\dot{w}_1', \ldots, \dot{w}_{K_x}'$ to be located in the "gaps" between those distributions would lead to solutions for ϱ_x^{RR}, $R \in \mathcal{R}_x$, that only depend on the values of $F_{\dot{w}|x}(\cdot)$ and $F_{\dot{w}^N|x}(\cdot)$ at points $\dot{w}_1', \ldots, \dot{w}_K'$:

Proposition 4. *Given x let $K_x \geq 1$, therefore $\mathcal{R}_x = \{R_1, \ldots, R_{K_x}\}$ non-empty, where $R \in \mathcal{R}_x \Leftrightarrow \mathcal{W}_{Rx}^c \subset \mathcal{W}_{Rx}^N$. Furthermore if $K_x > 1$ then for $k = 2, \ldots, K_x$ and $\{R_{k-1}, R_k\} \subseteq \mathcal{R}_x$ holds that $\dot{w}_{R_{k-1}x1}^c < \dot{w}_{R_kx0}^c$, i.e., $\mathcal{W}_{R_{k-1}x}^c \cap \mathcal{W}_{R_kx}^c = \emptyset$. Then under Assumptions 1-5 there exist points $\dot{w}_1', \ldots, \dot{w}_{K_x}'$ such that $\dot{w}_{x0} < \dot{w}_1' < \dot{w}_{R_1x0}^c$, and $\dot{w}_{R_{k-1}x1}^c < \dot{w}_k' < \dot{w}_{R_kx0}^c$ for $2 \leq k \leq K_x$, for which the cumulative distortion in the shape of $f_{\dot{w}|x}$ to the left of, and including, each one of them is given by the following expressions:*

$$\begin{aligned} F_{\dot{w}|x}\left(\dot{w}_1'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) &= -\left(\sum_{j=1}^{K_x} \varrho_x^{R_jR_j}\right) F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) \\ F_{\dot{w}|x}\left(\dot{w}_2'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_2'|x\right) &= -\left(\sum_{j=2}^{K_x} \varrho_x^{R_jR_j}\right) F_{\dot{w}^N|x}\left(\dot{w}_2'|x\right) \\ &\cdots \\ F_{\dot{w}|x}\left(\dot{w}_{K_x}'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_{K_x}'|x\right) &= -\varrho_x^{R_{K_x}R_{K_x}} F_{\dot{w}^N|x}\left(\dot{w}_{K_x}'|x\right) \end{aligned} \tag{55}$$

Accordingly, under , Assumptions 1-5, knowledge of the values of $F_{\dot{w}|x}$ and $F_{\dot{w}^N|x}$ at points $\dot{w}_1^{'}, \ldots, \dot{w}_K^{'}$ as specified is sufficient to point identify ϱ_x^{RR} for all $R \in \mathcal{R}_x$.

The width of the support of the measurement error distribution – thus the value of $\dot{\varepsilon}_1$, – determines which gaps, among those that exist in the case of error-free data, are preserved in the presence of measurement error. This is because the limits of such gaps move towards each other as $\dot{\varepsilon}_1$ increases, allowing for the possibility that any given gap to be eliminated for sufficiently large values of $\dot{\varepsilon}_1$. In such case also the identifying relationship associated with the point located in that gap would seize to hold. This also means that for sufficiently small values of $\dot{\varepsilon}_1$ all identifying relationships that hold in the absence of measurement error are preserved in its presence under no additional assumptions, allowing the same identification results to be derived.

5.3 Implementation issues

5.3.1 Extraneous information

Given data on $(\dot{w}, x)$ additional knowledge of the limits of W_{Rx}^c would be required in order to select points $\dot{w}_1^{'}, \ldots, \dot{w}_{K_x}^{'}$, which are needed to specify the equations (i.e., identifying relationships) given in Propositions 3 and 4. As can be easily verified, knowledge of the bounds of W_{Rx}^c would also be sufficient for that purpose, given that the results stated in those Propositions would hold if $\dot{w}_{Rx0}^c$ were substituted with a corresponding lower and $\dot{w}_{Rx1}^c$ with a corresponding upper bound. Using bounds instead of the exact limits of W_{Rx}^c can be useful in practice as obtaining the former is likely to be much easier than obtaining the latter. In such case one should, however, use the *greatest* lower bound of $\dot{w}_{Rx0}^c$ and *least* upper bound of $\dot{w}_{Rx1}^c$ available in order to avoid eliminating any of the gaps that may exist between adjacent distributions of constrained rates – and in doing so, avoid eliminating any of the identifying relationships that follow from Proposition 4 given knowledge of the actual limits of W_{Rx}^c.

5.3.2 Identification of counterfactual CDF values

The identifying relationships stated in Propositions 2-4 can only be applied in practice if the values of the CDF of the counterfactual distribution that appear in those relationships can be identified from the available data. That might be possible by imposing restrictions on the shape of the counterfactual distribution $f_{\dot{w}^N|x}$ (through restrictions on the shape of the flexible distribution $f_{\dot{w}^{N*}|x}$) such that knowledge of the shape of the part of $f_{\dot{w}^N|x}$ that can be identified from the undistorted part of the factual distribution $f_{\dot{w}|x}$ is sufficient for that purpose. Here we assume that $f_{\dot{w}^{N*}|x}$ is symmetric, which implies that $f_{\dot{w}^N|x}$ is also symmetric irrespective of the presence of measurement error – see Lemma 3(a). As stated below, if the upper half of the factual distribution is not distorted by rigidities then this is sufficient to ensure identification of the relevant values of $F_{\dot{w}^N|x}$ from the data on $(\dot{w}, x)$:

Lemma 13. *Given x, let $\mathcal{R}_x$ be non-empty. Furthermore let $\dot{w}^c_{R_{K_x}x1} < m_x$ where m_x denotes the median of $f_{\dot{w}|x}$. Then, under Assumption 2(c) the following result holds for any point $\dot{w} < \dot{w}^c_{R_{K_x}x1}$:*

$$F_{\dot{w}^N|x}(\dot{w}|x) \quad = \quad 1 - F_{\dot{w}|x}(2m_x - \dot{w}|x) \tag{56}$$

Shape homogeneity of $f_{\dot{w}^{N*}|x}$ across heterogeneity groups (x) is another non-parametric restriction that has been used in the literature, e.g. by Kahn (1997) and Holden and Wulfsberg (2008, 2009) (in the latter case, only up to scale). The typical parametric restriction is Normality, used among others by Fehr and Goette (2005) and Bauer et al. (2007b). Another parametric restriction is the Generalised Hyperbolic distribution, considered by Behr and Pötter (2010).

5.4 Set identification results

From the previous discussion follows that obtaining point identification of ϱ^R_x would require first to point identify $\Pr(R|x)$. On the other hand ϱ^{RR}_x can be informative as a measure of the "significance" of regime R among wage adjustments with characteristics x, since it combines the incidence of constrained wage adjustments by R among those negotiated under R (i.e., ϱ^R_x) and the prevalence of wage adjustments negotiated under R among x (i.e., $\Pr(R|x)$).

Nevertheless, it is possible to derive set identification results for ϱ^R_x – as well as for

$\Pr(R|x)$ – based on the knowledge of the identified values of ϱ_x^{RR} by exploiting the restrictions that follow from the fact that these parameters are probabilities:

Proposition 5. *Given x, let $\mathcal{R}_x$ be non-empty and assume knowledge of ϱ_x^{RR} for all $R \in \mathcal{R}_x$. Then:*

$$\varrho_x^{R} \in \left[\frac{\varrho_x^{RR}}{1-\sum_{\vartheta\in\mathcal{R}_x\setminus\{R\}}\varrho_x^{\vartheta\vartheta}}, 1\right] \quad , \quad R \in \mathcal{R}_x \tag{57}$$

$$\Pr(R|x) \in \begin{cases} \left[\varrho_x^{RR}, 1-\sum_{\vartheta\in\mathcal{R}_x\setminus\{R\}}\varrho_x^{\vartheta\vartheta}\right] & , \quad R \in \mathcal{R}_x \\ \left[0, 1-\sum_{\vartheta\in\mathcal{R}_x\setminus\{R\}}\varrho_x^{\vartheta\vartheta}\right] & , \quad R \in \mathcal{R}\setminus\mathcal{R}_x \end{cases} \tag{58}$$

Accordingly from (57) follows that knowledge of ϱ_x^{RR}, for all $R \in \mathcal{R}_x$, reduces the uncertainty about the lower bound of ϱ_x^{R} for R in $\mathcal{R}_x$. Also, from (58), that this knowledge provides information about the lower and upper bounds of $\Pr(R|x)$ for R in $\mathcal{R}_x$, but only for its upper bound for R not in $\mathcal{R}_x$.

6 Summary and conclusion

In this paper we study the problem of identification of measures of the extent of individual types of downward wage rigidity from micro-level data on nominal wage growth rates. This is undertaken within the context of a wage adjustment process that can feature any number of downward wage rigidity types. It therefore encompasses the models considered by the literature, which only allow for Downward Nominal Wage Rigidity and/or Downward Real Wage Rigidity.

A key finding of our analysis is that the existence of classical measurement error in the observed nominal wage growth rates does not alter fundamentally the nature of the identification problem. In turn, this allows to develop a common identification strategy for both the cases of measurement-error-free and measurement-error-contaminated data, which seeks to identify the rigidity measures from the size of the distortions in the shape of the corresponding factual distribution relative to the shape of the (rigidity-free) counterfactual distribution. Using this strategy we are then able to show that identification of the rigidity measures can be achieved under weaker restrictions than those employed by the existing literature, especially in the case of measurement-error-contaminated data. These restrictions are non-parametric, and sufficient to produce the same identification results in both cases if the support of the measurement error distribution is sufficiently narrow.

The work presented here also makes broader methodological contributions by developing a generic framework for the modelling and measurement of wage rigidities. This allows us to undertake a rigorous identification analysis, which provides new insights on the nature of the identification problem and its solution; also, to unify seemingly different models, types of rigidity measures, and identification results in the existing literature.

As much as enhancing our understanding of how the identification of measures of rigidity can be achieved, the identification results reported here could also be of practical use by providing the basis for the development of new estimation methods for such measures. As these identification results rely on weaker restrictions than those in the existing literature, that could potentially lead to estimation methods that also rely on weaker restrictions than the existing ones.

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A Example: the D-NR-WR process

A.1 Set up

In this appendix we examine how the generic identification results discussed in the main text specialise to the case of the Standard D-NR-WR process. This process allows for any given wage adjustment to be negotiated under one of three wage adjustment regimes, namely DNWR, DRWR, and the Flexible regime, where the first two regimes are described by the Standard DWR Mechanism.

Assumptions 6 and 7 stated below complement Assumptions 1-5 to provide a full description of that process:[32]

Assumption 6. *(Admissible wage adjustment regimes)*

Given observed characteristics x, a wage adjustment may be negotiated under one of three wage adjustment regimes: DNWR ($R = n$), DRWR ($R = r$), and the Flexible regime ($R = f$); accordingly, $\mathcal{R} = \{n, r, f\}$.

Assumption 7. *(Constrained rates, by wage adjustment regime)*

(a) If $R = n$ (DNWR) then $\dot{w}^{c*}$ is equal to zero. Accordingly $f_{\dot{w}^{c*}|\dot{w}^{N*},R=n,x}$ is degenerate, with all its probability mass concentrated at point zero (see also Assumption 3b); furthermore, $\mathcal{W}^{c*}_{Rx} = \mathcal{W}^{c*}_{nx} = \{0\}$.

(b) If $R = r$ (DRWR) then $\dot{w}^{c*}$ is equal to the anticipated inflation rate $\dot{P}^{e*}$. Accordingly $f_{\dot{w}^{c*}|\dot{w}^{N*},R=r,x}$ specialises to $f_{\dot{P}^{e*}|\dot{w}^{N*},R=r,x}$, whose properties are described in Assumption 3a; furthermore, $\mathcal{W}^{c*}_{Rx} = \mathcal{W}^{c*}_{rx} = \left[\dot{P}^{e*}_{x0}, \dot{P}^{e*}_{x1}\right]$.

(c) If $R = f$ (Flexible) then $\dot{w}^{c*}$ is, conventionally, set equal to $-\infty$. Accordingly, $f_{\dot{w}^{c*}|\dot{w}^{N*},R=f,x}$ is undefined.

By definition, there are no wage adjustments that are candidates to be constrained under the Flexible regime, therefore, given x, only n (DNWR) and r (DRWR) can be the elements of $\mathcal{R}_x$. Here we consider three cases of anticipated inflation regimes that determine the composition of $\mathcal{R}_x$:[33]

[32] Note that $\mathcal{X}$ is left unspecified.

[33] These definitions are consistent with the relationship $\dot{w}^{N*} = \dot{P}^{e*} + \dot{\tau}$, where $\dot{\tau}$ is the cumulative

HIGH Anticipated Inflation Regime ($x \in \mathcal{X}^H$): This is the case where the mean anticipated inflation rate is high enough so that all flexible rates and all anticipate inflation rates are positive:

$$0 < \dot{w}_{x0}^{N*} < \dot{P}_{x0}^{e*} \leq \dot{P}_{x1}^{e*} < \dot{w}_{x1}^{N*} \tag{59}$$

It follows that $\mathcal{R}_x = \{r\}$.[34]

MODERATE Anticipated Inflation Regime ($x \in \mathcal{X}^M$): This is the case where the mean anticipated inflation rate is positive but at moderate levels so that there exist positive as well as negative flexible rates, although only positive anticipate inflation rates:

$$\dot{w}_{x0}^{N*} < 0 < \dot{P}_{x0}^{e*} \leq \dot{P}_{x1}^{e*} < \dot{w}_{x1}^{N*} \tag{60}$$

It follows that $\mathcal{R}_x = \{n, r\}$.[35]

LOW Anticipated Inflation Regime ($x \in \mathcal{X}^L$): This is the case where the mean anticipated inflation rate is close enough to zero, being either positive or negative, so that there exist positive as well as negative flexible rates and anticipated inflation rates:

$$\dot{w}_{x0}^{N*} < \dot{P}_{x0}^{e*} < 0 < \dot{P}_{x1}^{e*} < \dot{w}_{x1}^{N*} \tag{61}$$

It follows that $\mathcal{R}_x = \{n, r\}$.

Given that $r \in \mathcal{R}_x$, which is true for all three anticipated inflation regimes (i.e., for $x \in \mathcal{X}^H \cup \mathcal{X}^M \cup \mathcal{X}^L$), the measure of the extent of DRWR is defined as follows:

$$\varrho_x^r \equiv \frac{\Pr(\delta = 1 | R = r, x)}{\Pr\left(\dot{w}^{N*} < \dot{P}^{e*} | R = r, x\right)} \tag{62}$$

effect of the changes in all factors affecting $\dot{w}^{N*}$, other than expected inflation. Furthermore, in all anticipated inflation regimes the support of the anticipated inflation distribution is a subset of the support of the flexible distribution, i.e., $\left[\dot{P}_{x0}^{e*}, \dot{P}_{x1}^{e*}\right] \subset \left[\dot{w}_{x0}^{N*}, \dot{w}_{x1}^{N*}\right]$, which is consistent with $\dot{P}^{e*}$ and $\dot{\tau}$ being non-negatively correlated. (NB: Using US data, Christofides and Mamuneas (2003) find these to be independent.)

[34] Since $\mathcal{W}_{nx}^{c*} = \{0\}$ lies to the left of $\mathcal{W}_{nx}^{N*} = \left[\dot{w}_{x0}^{N*}, \dot{w}_{x1}^{N*}\right]$ and $\mathcal{W}_{rx}^{c*} = \left[\dot{P}_{x0}^{e*}, \dot{P}_{x1}^{e*}\right] \subset \mathcal{W}_{nx}^{N*}$.

[35] Since, as also in the case of the LOW Anticipated Inflation Regime discussed below, $\mathcal{W}_{nx}^{c*} \subset \mathcal{W}_{nx}^{N*}$ and $\mathcal{W}_{rx}^{c*} \subset \mathcal{W}_{rx}^{N*}$.

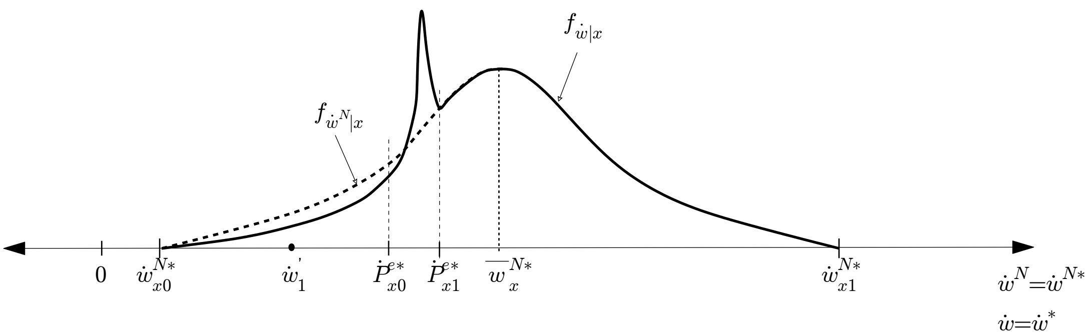
i.e., it is the proportion of unrealised anticipated real wage cuts among the wage adjustments with characteristics x that are candidates to be constrained by DRWR. For the MODERATE and LOW anticipated inflation regimes (i.e., for $x \in \mathcal{X}^M \cup \mathcal{X}^L$), the measure of the extent of DNWR is defined as follows:

$$\varrho_x^n \equiv \frac{\Pr(\delta = 1 | R = n, x)}{\Pr(\dot{w}^{N*} < 0 | R = n, x)} \tag{63}$$

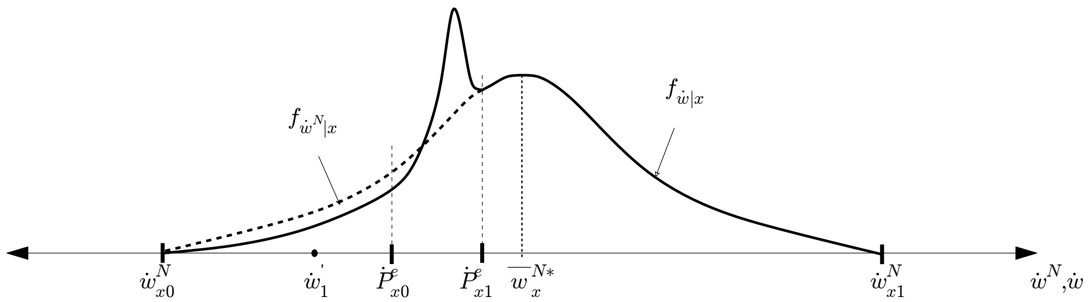
i.e., it is the proportion of unrealised nominal wage cuts among the wage adjustments with characteristics x that are candidates to be constrained by DNWR.

A.2 Results

In Figures 6-8 we depict the graphs of the PDFs of the factual and counterfactual distributions for each of the three anticipated inflation regimes, for the cases of error-free and error-contaminated data. Propositions 6-8 give the corresponding identifying relationships:



(a) Error-free data



(b) Error-contaminated data

Figure 6: Factual Vs counterfactual distributions, HIGH anticipated inflation regime

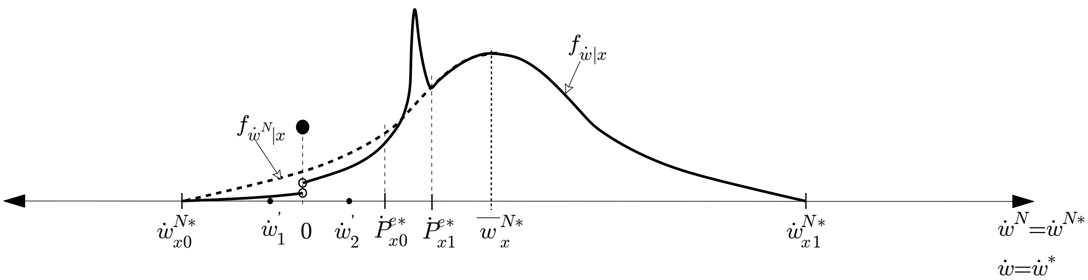
Proposition 6. *(HIGH Anticipated Inflation regime)*

(a) Let $x \in \mathcal{X}^H$ and $f_{\dot{\varepsilon}}$ degenerate (error-free data). Furthermore, consider point $\dot{w}_1' \in (\dot{w}_{x0}, \dot{w}^c_{rx0}) = \left(\dot{w}^{N*}_{x0}, \dot{P}^{e*}_{x0}\right)$ – see example in Figure 6a. From Proposition 4 we get the following relationship:

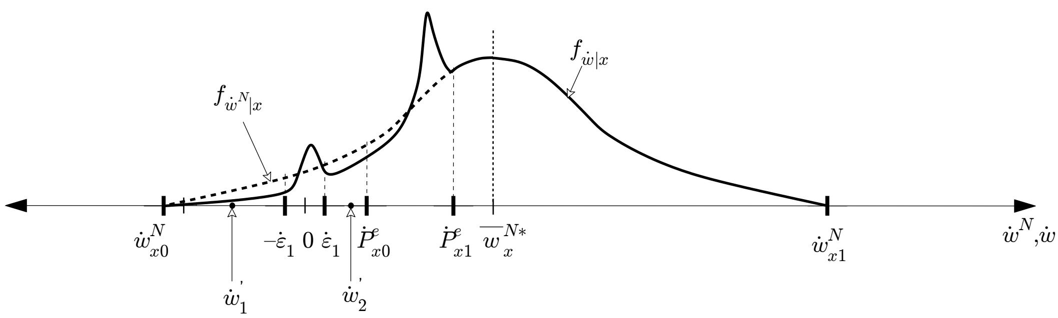
$$F_{\dot{w}|x}\left(\dot{w}_1'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) = -\varrho^{rr}_x F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) \tag{64}$$

(b) Let $x \in \mathcal{X}^H$ and $f_{\dot{\varepsilon}}$ non-degenerate (error-contaminated data). Furthermore, consider point $\dot{w}_1' \in (\dot{w}_{x0}, \dot{w}^c_{rx0}) = \left(\dot{w}^N_{x0}, \dot{P}^e_{x0}\right)$ where $\mathcal{W}^c_{rx} = \left[\dot{P}^e_{x0}, \dot{P}^e_{x1}\right] = \left[\dot{P}^{e*}_{x0} - \dot{\varepsilon}_1, \dot{P}^{e*}_{x1} + \dot{\varepsilon}_1\right]$ is the support of $f_{\dot{P}^{e*}|\dot{w}^{N*},R=r,x}$, i.e., of $f_{\dot{P}^{e*}|R=r,x}$[36] – see example in Figure 6b. From Proposition 4 we get the following relationship:

$$F_{\dot{w}|x}\left(\dot{w}_1'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) = -\varrho^{rr}_x F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) \tag{65}$$



(a) Error-free data



(b) Error-contaminated data

Figure 7: Factual Vs counterfactual distributions, MODERATE anticipated inflation regime

[36] See Lemma 3(b-2).

Proposition 7. *(MODERATE Anticipated Inflation regime)*

(a) Let $x \in \mathcal{X}^M$ and $f_{\dot{\varepsilon}}$ degenerate (error-free data). Furthermore, consider points $\dot{w}_1' \in (\dot{w}_{x0}, \dot{w}_{nx0}^c) = \left(\dot{w}_{x0}^{N*}, 0\right)$ and $\dot{w}_2' \in (\dot{w}_{nx1}^c, \dot{w}_{rx0}^c) = \left(0, \dot{P}_{x0}^{e*}\right)$ – see example in Figure 7a. From Proposition 4 we get the following relationships:

$$
\begin{aligned}
F_{\dot{w}|x}\left(\dot{w}_1'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) &= -\left(\varrho_x^{nn} + \varrho_x^{rr}\right) F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) && (66)\\
F_{\dot{w}|x}\left(\dot{w}_2'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_2'|x\right) &= -\varrho_x^{rr} F_{\dot{w}^N|x}\left(\dot{w}_2'|x\right) && (67)
\end{aligned}
$$

Furthermore, from Proposition 2 we get the following:

$$
\Pr\left(\dot{w} = 0|x\right) = \varrho_x^{nn} F_{\dot{w}^N|x}\left(0|x\right) \tag{68}
$$

(b) Let $x \in \mathcal{X}^M$ and $f_{\dot{\varepsilon}}$ non-degenerate (error-contaminated data). Furthermore, consider points $\dot{w}_1' \in (\dot{w}_{x0}, \dot{w}_{nx0}^c) = \left(\dot{w}_{x0}^{N}, -\dot{\varepsilon}_1\right)$ and $\dot{w}_2' \in (\dot{w}_{nx1}^c, \dot{w}_{rx0}^c) = \left(\dot{\varepsilon}_1, \dot{P}_{x0}^{e}\right)$ assuming that $\dot{P}_{x0}^{e} > \dot{\varepsilon}_1 \Leftrightarrow \dot{P}_{x0}^{e*} > 2\dot{\varepsilon}_1$ – see example in Figure 7b. From Proposition 4 we get the following relationships:

$$
\begin{aligned}
F_{\dot{w}|x}\left(\dot{w}_1'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) &= -\left(\varrho_x^{nn} + \varrho_x^{rr}\right) F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) && (69)\\
F_{\dot{w}|x}\left(\dot{w}_2'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_2'|x\right) &= -\varrho_x^{rr} F_{\dot{w}^N|x}\left(\dot{w}_2'|x\right) && (70)
\end{aligned}
$$

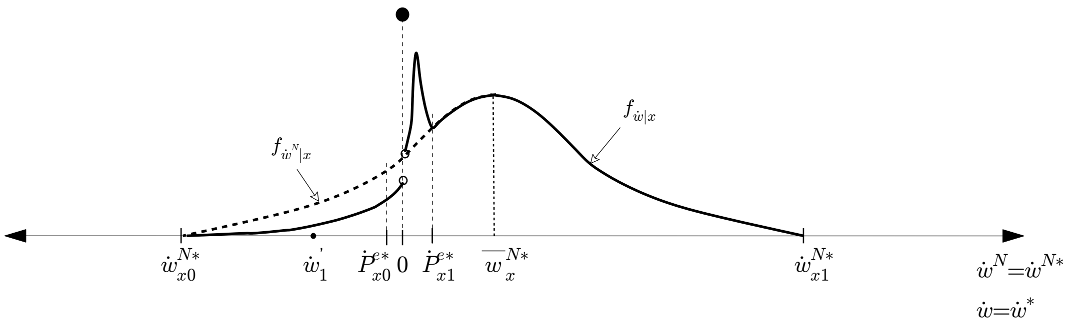
Proposition 8. *(LOW Anticipated Inflation regime)*

(a) Let $x \in \mathcal{X}^L$ and $f_{\dot{\varepsilon}}$ degenerate (error-free data). Furthermore, consider point $\dot{w}_1' \in (\dot{w}_{x0}, \dot{w}_{rx0}^c) = \left(\dot{w}_{x0}^{N*}, \dot{P}_{x0}^{e*}\right)$ – see example in Figure 8a. From Proposition 3 we get the following relationship:

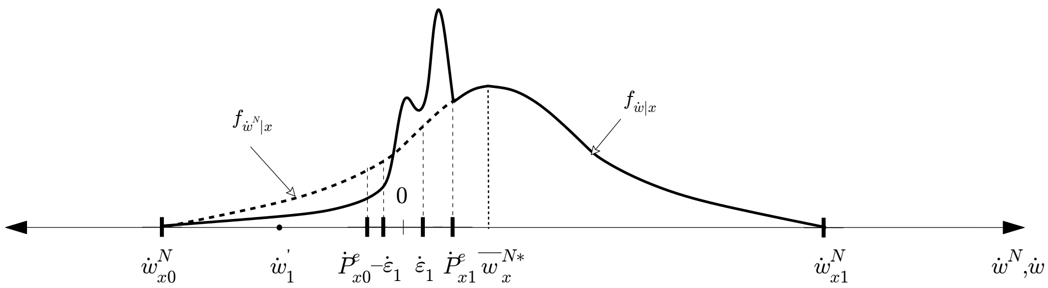
$$
F_{\dot{w}|x}\left(\dot{w}_1'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) = -\left(\varrho_x^{nn} + \varrho_x^{rr}\right) F_{\dot{w}^N|x}\left(\dot{w}_1'|x\right) \tag{71}
$$

Furthermore, from Proposition 2 we get the following:

$$
\Pr\left(\dot{w} = 0|x\right) = \varrho_x^{nn} F_{\dot{w}^N|x}\left(0|x\right) \tag{72}
$$



(a) Error-free data



(b) Error-contaminated data

Figure 8: Factual Vs counterfactual distributions, LOW anticipated inflation regime

(b) Let $x \in \mathcal{X}^L$ and $f_{\dot{\varepsilon}}$ non-degenerate (error-contaminated data). Furthermore, consider point $\dot{w}_1^{'} \in (\dot{w}_{x0}, \dot{w}_{rx0}^c) = \left(\dot{w}_{x0}^N, \dot{P}_{x0}^e\right)$ – see example in Figure 8b. From Proposition 3 we get the following relationship:

$$F_{\dot{w}|x}\left(\dot{w}_1^{'}|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_1^{'}|x\right) \quad = \quad -\left(\varrho_x^{nn} + \varrho_x^{rr}\right) F_{\dot{w}^N|x}\left(\dot{w}_1^{'}|x\right) \tag{73}$$

A.3 Discussion of results in relation to Dickens et al. (2007)

We observe that for the case of error-free data there exist enough identifying relationships under Assumptions 1-5 to point identify the scaled rigidity measures that are relevant for each anticipated inflation regime. In the case of the MODERATE regime there are more than enough relationships. This would require knowledge of the value of the factual and counterfactual CDFs at certain points of their support, and also of $\dot{P}_{x0}^{e*}$ or a lower bound of it, and of $\dot{P}_{x1}^{e*}$ or an upper bounds of it.[37]

These results (error-free case) are directly comparable to the results reported in Dick-

[37] If more than one bounds are known, then any statements about $\dot{P}_{x0}^{e*}$ and $\dot{P}_{x1}^{e*}$ in Propositions 6-8 apply for their respective lower and upper bounds.

ens et al. (2007). First we note that setting $\overline{\varrho}_x^R = 1$, as it is assumed there, then ϱ_x^{RR} reduces to $\Pr(R|x)$, which is their adopted rigidity measure.

Furthermore, in their case the identification of $\Pr(R = n|x)$, i.e., of ϱ_x^{nn}, for the MODERATE regime is based on (68).[38] On the other hand, the identification of $\Pr(R = r|x)$, i.e., of ϱ_x^{rr}, for the HIGH and MODERATE regimes is based on the following relationship:[39]

$$F_{\dot{w}^N|x}\left(\overline{P}_{rx}^{e*}|x\right) - F_{\dot{w}|x}\left(\overline{P}_{rx}^{e*}|x\right) = \varrho_x^{rr}\frac{F_{\dot{w}^N|x}\left(\overline{P}_{rx}^{e*}|x\right)}{2} \tag{74}$$

which can be derived from (50) setting $\dot{w}' = \overline{P}_{rx}^{e*} \equiv E\left(\dot{P}^{e*}|R = r, x\right)$.[40] This result would require, *in addition* to Assumptions 1-5, to assume conditional independence of the flexible rate $\dot{w}^{N*}$ and anticipated inflation rate $\dot{P}^{e*}$ given R and x, and symmetry of the anticipated inflation distribution $f_{\dot{P}^{e*}|R=r,x}$.

In the case of error-contaminated data there exist enough identifying relationships to point identify the relevant scaled rigidity measures only for the HIGH and MODERATE anticipated inflation regimes. In the latter case this would require small measurement error sizes, specifically $\dot{\varepsilon}_1 < \dot{P}_{x0}^e \Leftrightarrow \dot{\varepsilon}_1 > \frac{\dot{P}_{x0}^{e*}}{2}$. In the case of the LOW anticipated inflation regime there is only one identifying relationship while two unknown measures, ϱ_x^{nn} and ϱ_x^{rr}, which is only sufficient to identify their sum. If $\dot{\varepsilon}_1 < \overline{P}_{rx}^{e*}$ then (74) could be the second identifying relationships, leading to point identification of both scaled measures, but only at the cost of making the additional assumptions which underlie this result.[41]

In contrast, for the case of error-contaminated data, Dickens et al. (2007) propose cleaning them from measurement error in a preliminary stage, and then proceeding as if these were error-free. Compared to our approach, this would require additional assump-

[38]Using that $\Pr(\dot{w} = 0|x)$ and $f_{\dot{w}^N|x}$ are the same as $\Pr(\dot{w}^* = 0|x)$ and $f_{\dot{w}^{N*}|x}$, respectively, due to working with error-free data, then rearranging (68) gives $\varrho_x^{nn} = \frac{\Pr(\dot{w}=0|x)}{F_{\dot{w}^N|x}(0|x)} = \Pr(R = n|x)$, which is the formula they use.

[39]Using that $f_{\dot{w}|x}$ and $f_{\dot{w}^N|x}$ are the same as $f_{\dot{w}^*|x}$ and $f_{\dot{w}^{N*}|x}$, respectively, due to working with error-free data, then rearranging (74) gives $\varrho_x^{rr} = \frac{2\left[F_{\dot{w}^N|x}\left(\overline{P}_{rx}^{e*}|x\right) - F_{\dot{w}|x}\left(\overline{P}_{rx}^{e*}|x\right)\right]}{F_{\dot{w}^N|x}\left(\overline{P}_{rx}^{e*}|x\right)} = \Pr(R = r|x)$, which is the formula they use.

[40]I.e., $\overline{P}_{rx}^{e*}$ is the mean anticipated inflation rate given $R = r$ and x. Under our assumptions it is also true that $\overline{P}_{rx}^{e*} = E\left(\dot{P}^e|R = r, x\right)$ (see Lemma 3(b-2)).

[41]The same could also be applied in the case of the MODERATE anticipated inflation regime to provide an additional relationship when $\dot{\varepsilon}_1$ is large, i.e., $\dot{\varepsilon}_1 > \dot{P}_{x0}^e \Leftrightarrow \dot{\varepsilon}_1 > \frac{\dot{P}_{x0}^{e*}}{2}$.

tions to implement this preliminary stage. Furthermore the data produced in such a way are, in fact, estimates of the error-free data and are therefore contaminated with estimation error whose magnitude will depend on the validity of those additional assumptions.

B Proofs

B.1 Proof of Lemma 1

Starting from the RHS of the definition of ϱ_x^R in (12) we can write:

$$\varrho_x^R = \frac{\Pr\left(\delta = 1, \dot{w}^{N*} < \dot{w}^{c*} | R, x\right)}{\Pr\left(\dot{w}^{N*} < \dot{w}^{c*} | R, x\right)} \tag{75}$$

where the numerator can be further expanded as follows:

$$\Pr\left(\delta = 1, \dot{w}^{N*} < \dot{w}^{c*} | R, x\right) = \iint\limits_{(\varpi, v): \varpi < v} f_{\dot{w}^{N*}, \dot{w}^{c*}, \delta | R, x}\left(\varpi, v, 1 | R, x\right) dv d\varpi \tag{76}$$

$$= \iint\limits_{(\varpi, v): \varpi < v} \begin{array}{c} \rho_x^R\left(\varpi, v\right) \times \\ \times f_{\dot{w}^{N*}, \dot{w}^{c*} | R, x}\left(\varpi, v | R, x\right) dv d\varpi \end{array} \tag{77}$$

Given the limits of integration, and using (4), we can further write:

$$\Pr\left(\delta = 1, \dot{w}^{N*} < \dot{w}^{c*} | R, x\right) = \iint\limits_{(\varpi, v): \varpi < v} \overline{\varrho}_x^R f_{\dot{w}^{N*}, \dot{w}^{c*} | R, x}\left(\varpi, v | R, x\right) dv d\varpi \tag{78}$$

$$= \overline{\varrho}_x^R \Pr\left(\dot{w}^{N*} < \dot{w}^{c*} | R, x\right) \tag{79}$$

Substituting the result into the numerator of (75) and simplifying completes the proof.

B.2 Proof of Lemma 2

Adding $\dot{\varepsilon}$ to both sides of equation (1) preserves the equality. Therefore we can write:

$$\dot{w}^* + \dot{\varepsilon} = \left[\delta \cdot \dot{w}^{c*} + (1 - \delta)\, \dot{w}^{N*}\right] + \dot{\varepsilon} = \delta \cdot \left(\dot{w}^{c*} + \dot{\varepsilon}\right) + (1 - \delta)\left(\dot{w}^{N*} + \dot{\varepsilon}\right) \tag{80}$$

or:

$$\dot{w} \quad = \quad \delta \cdot \dot{w}^{c} + (1-\delta)\, \dot{w}^{N} \tag{81}$$

Also, since $\dot{w}^{c*} \neq \dot{w}^{N*}$ then also $\dot{w}^{c*} + \dot{\varepsilon} \neq \dot{w}^{N*} + \dot{\varepsilon} \Leftrightarrow \dot{w}^{c} \neq \dot{w}^{N}$.

B.3 Proof of Lemma 3

(a) *Conditional independence of $\dot{w}^{N}$ and R given x*: Given the definition of $\dot{w}^{N}$ and that $\dot{\varepsilon}$ is independent of all other variables in the model we can write $f_{\dot{w}^{N}|R,x}(\dot{w}|R,x) =$
$= \int_{-\dot{\varepsilon}_1}^{\dot{\varepsilon}_1} f_{\dot{w}^{N*}|R,x}(\dot{w}-\epsilon|R,x)\, f_{\dot{\varepsilon}}(\epsilon)\, d\epsilon$, therefore from Assumption 2(a):

$$\begin{aligned} f_{\dot{w}^{N}|R,x}(\dot{w}|R,x) &= \int_{-\dot{\varepsilon}_1}^{\dot{\varepsilon}_1} f_{\dot{w}^{N*}|x}(\dot{w}-\epsilon|x)\, f_{\dot{\varepsilon}}(\epsilon)\, d\epsilon & (82)\\ &= f_{\dot{w}^{N}|x}(\dot{w}|x) & (83) \end{aligned}$$

which also gives (17).

Continuity of $f_{\dot{w}^{N}|x}$: By assumption $f_{\dot{w}^{N*}|x}$ and $f_{\dot{\varepsilon}}$ are continuous. From (17) follows that $f_{\dot{w}^{N}|x}$ is the sum of products of continuous functions, therefore itself also continuous.

Symmetry of $f_{\dot{w}^{N}|x}$: We need to show that $f_{\dot{w}^{N}|x}\left(\overline{w}_x^{N*} - c|x\right) = f_{\dot{w}^{N}|x}\left(\overline{w}_x^{N*} + c|x\right)$ where $\overline{w}_x^{N*} \equiv E\left(\dot{w}^{N*}|x\right)$ and therefore, also, $\overline{w}_x^{N*} \equiv E\left(\dot{w}^{N}|x\right)$. Starting from (17) we can write:

$$f_{\dot{w}^{N}|x}\left(\overline{w}_x^{N*} - c|x\right) = \int_{-\dot{\varepsilon}_1}^{\dot{\varepsilon}_1} f_{\dot{w}^{N*}|x}\left(\overline{w}_x^{N*} - c - \epsilon|x\right) f_{\dot{\varepsilon}}(\epsilon)\, d\epsilon \tag{84}$$

We note that $f_{\dot{w}^{N*}|x}\left(\overline{w}_x^{N*} - c - \epsilon|x\right) = f_{\dot{w}^{N*}|x}\left(\overline{w}_x^{N*} + c + \epsilon|x\right)$ due to the symmetry of $f_{\dot{w}^{N*}|x}$ around $\overline{w}_x^{N*}$, and that $f_{\dot{\varepsilon}}(\epsilon) = f_{\dot{\varepsilon}}(-\epsilon)$ due to the symmetry of $f_{\dot{\varepsilon}}$ around 0. Substituting to the RHS above gives:

$$\begin{aligned} f_{\dot{w}^{N}|x}\left(\overline{w}_x^{N*} - c|x\right) &= \int_{-\dot{\varepsilon}_1}^{\dot{\varepsilon}_1} f_{\dot{w}^{N*}|x}\left(\overline{w}_x^{N*} + c + \epsilon|x\right) f_{\dot{\varepsilon}}(-\epsilon)\, d\epsilon & (85)\\ &= f_{\dot{w}^{N}|x}\left(\overline{w}_x^{N*} + c|x\right) & (86) \end{aligned}$$

Support of $f_{\dot{w}^{N}|x}$: Since $\dot{w}^{N} \equiv \dot{w}^{N*} + \dot{\varepsilon}$, where $\dot{w}^{N*} \in \left[\dot{w}_{x0}^{N*}, \dot{w}_{x1}^{N*}\right]$ (from Assumption

2(a)), it follows that $\dot{w}^N_{x0} \equiv \min\left(\dot{w}^N|x\right) = \min\left(\dot{w}^{N*} + \dot{\varepsilon}|x\right) = \min\left(\dot{w}^{N*}|x\right) + \min\left(\dot{\varepsilon}|x\right) = \dot{w}^{N*}_{x0} - \dot{\varepsilon}_1$. Also, that $\dot{w}^N_{x1} \equiv \max\left(\dot{w}^N|x\right) = \max\left(\dot{w}^{N*} + \dot{\varepsilon}|x\right) = \max\left(\dot{w}^{N*}|x\right) + \max\left(\dot{\varepsilon}|x\right) = \dot{w}^{N*}_{x1} + \dot{\varepsilon}_1$.

Conditional Mean of $f_{\dot{w}^N|x}$*:* $E\left(\dot{w}^N|x\right) = E\left(\dot{w}^{N*} + \dot{\varepsilon}|x\right) = E\left(\dot{w}^{N*}|x\right) + E\left(\dot{\varepsilon}|x\right) = E\left(\dot{w}^{N*}|x\right)$ since $f_{\dot{\varepsilon}}$ has zero mean.

Conditional Variance of $f_{\dot{w}^N|x}$*:* $Var\left(\dot{w}^N|x\right) = Var\left(\dot{w}^{N*} + \dot{\varepsilon}|x\right) = Var\left(\dot{w}^{N*}|x\right) + Var\left(\dot{\varepsilon}|x\right) + 2Cov\left(\dot{w}^{N*}, \dot{\varepsilon}|x\right) = Var\left(\dot{w}^{N*}|x\right) + Var\left(\dot{\varepsilon}\right)$ since, by definition, $\dot{\varepsilon}$ is independent of all other variables in the model.

(b-1) *Derivation of* (20)*:* This follows immediately from that $\dot{w}^c = \dot{w}^{c*}_{Rx1} + \dot{\varepsilon}$ and the definition of and $\dot{\varepsilon}$.

Continuity of $f_{\dot{w}^c|R,x}$*:* By assumption, $f_{\dot{\varepsilon}}$ is continuous.

Support of $f_{\dot{w}^c|R,x}$*:* Since $\dot{w}^c \equiv \dot{w}^{c*} + \dot{\varepsilon}$, where $\dot{w}^{c*} \in [\dot{w}^{c*}_{Rx0}, \dot{w}^{c*}_{Rx1}]$ (from Assumption 3(a)), it follows that $\dot{w}^c_{Rx0} \equiv \min\left(\dot{w}^c|R,x\right) = \min\left(\dot{w}^{c*}_{Rx1} + \dot{\varepsilon}|R,x\right) = \min\left(\dot{w}^{c*}_{Rx1}|R,x\right) + \min\left(\dot{\varepsilon}|R,x\right) = \dot{w}^{c*}_{Rx1} - \dot{\varepsilon}_1$. Also, that $\dot{w}^c_{x1} \equiv \max\left(\dot{w}^c|R,x\right) = \max\left(\dot{w}^{c*}_{Rx1} + \dot{\varepsilon}|R,x\right) = \max\left(\dot{w}^{c*}_{Rx1}|R,x\right) + \max\left(\dot{\varepsilon}|R,x\right) = \dot{w}^{c*}_{Rx1} + \dot{\varepsilon}_1$.

Conditional Mean of $f_{\dot{w}^c|R,x}$*:* $E\left(\dot{w}^c|R,x\right) = E\left(\dot{w}^{c*}_{Rx1} + \dot{\varepsilon}|R,x\right) = \dot{w}^{c*}_{Rx1} + E\left(\dot{\varepsilon}|R,x\right) = \dot{w}^{c*}_{Rx1} + E\left(\dot{\varepsilon}\right) = \dot{w}^{c*}_{Rx1}$ since $f_{\dot{\varepsilon}}$ has zero mean.

Conditional Variance of $f_{\dot{w}^c|R,x}$*:* $Var\left(\dot{w}^c|R,x\right) = Var\left(\dot{w}^{c*}_{Rx1} + \dot{\varepsilon}|R,x\right) = Var\left(\dot{\varepsilon}\right)$

(b-2) *Derivation of* (23)*:* This follows immediately from the definitions of $\dot{w}^c$ and $\dot{w}^N$, and that $\dot{\varepsilon}$ is independent of all other variables in the model.

Continuity of $f_{\dot{w}^c|\dot{w}^N,R,x}$*:* By assumption, $f_{\dot{w}^{c*}|\dot{w}^{N*},R,x}$ and $f_{\dot{\varepsilon}}$ are continuous. From (23) follows that $f_{\dot{w}^c|\dot{w}^N,R,x}$ is the sum of products of continuous functions, therefore itself also continuous.

Support of $f_{\dot{w}^c|\dot{w}^N,R,x}$*:* Since $\dot{w}^c \equiv \dot{w}^{c*} + \dot{\varepsilon}$, where $\dot{w}^{c*} \in [\dot{w}^{c*}_{Rx0}, \dot{w}^{c*}_{Rx1}]$ (from Assumption 3(a)), it follows that $\dot{w}^c_{Rx0} \equiv \min\left(\dot{w}^c|R,x\right) = \min\left(\dot{w}^{c*} + \dot{\varepsilon}|R,x\right) = \min\left(\dot{w}^{c*}|R,x\right) + \min\left(\dot{\varepsilon}|R,x\right) = \dot{w}^{c*}_{Rx0} - \dot{\varepsilon}_1$. Also, that $\dot{w}^c_{x1} \equiv \max\left(\dot{w}^c|R,x\right) = \max\left(\dot{w}^{c*} + \dot{\varepsilon}|R,x\right) = \max\left(\dot{w}^{c*}|R,x\right) + \max\left(\dot{\varepsilon}|R,x\right) = \dot{w}^{c*}_{Rx1} + \dot{\varepsilon}_1$.

Conditional Mean of $f_{\dot{w}^c|\dot{w}^N,R,x}$: $E\left(\dot{w}^c|\dot{w}^N,R,x\right)=E\left(\dot{w}^{c*}+\dot{\varepsilon}|\dot{w}^N,R,x\right)=$
$=E\left(\dot{w}^{c*}|\dot{w}^N,R,x\right)+E\left(\dot{\varepsilon}|\dot{w}^N,R,x\right)=E\left(\dot{w}^{c*}|\dot{w}^N,R,x\right)+E\left(\dot{\varepsilon}\right)=E\left(\dot{w}^{c*}|\dot{w}^N,R,x\right)$ since $f_{\dot{\varepsilon}}$ has zero mean.

Conditional Variance of $f_{\dot{w}^c|\dot{w}^N,R,x}$: $Var\left(\dot{w}^c|\dot{w}^N,R,x\right)=Var\left(\dot{w}^{c*}+\dot{\varepsilon}|\dot{w}^N,R,x\right)=$
$=Var\left(\dot{w}^{c*}|\dot{w}^N,R,x\right)+Var\left(\dot{\varepsilon}|\dot{w}^N,R,x\right)+2Cov\left(\dot{w}^{c*},\dot{\varepsilon}|\dot{w}^N,R,x\right)=Var\left(\dot{w}^{c*}|\dot{w}^N,R,x\right)+Var\left(\dot{\varepsilon}\right)$ since, by definition, $\dot{\varepsilon}$ is independent of all other variables in the model.

(c) *Derivation of* (26)*:* The starting point is the definition of $\rho_x^R\left(\dot{w}^N,\dot{w}^c\right)$ given in (16). Given the definitions of $\dot{w}^N$ and $\dot{w}^c$, and that $\dot{\varepsilon}$ is independence of all other variables in the model, it follows that we can write:

$$\begin{aligned}\rho_x^R\left(\dot{w}^N,\dot{w}^c\right) &= \Pr\left(\delta=1|\dot{w}^N,\dot{w}^c,R,x\right) &(87)\\ &= \int_{-\dot{\varepsilon}_1}^{\dot{\varepsilon}_1}\Pr\left(\delta=1|\dot{w}^{N*}=\dot{w}^N-\epsilon,\dot{w}^{c*}=\dot{w}^c-\epsilon,R,x\right)f_{\dot{\varepsilon}}\left(\epsilon\right)d\epsilon &(88)\\ &= \int_{-\dot{\varepsilon}_1}^{\dot{\varepsilon}_1}\rho_x^{R*}\left(\dot{w}^N-\epsilon,\dot{w}^c-\epsilon\right)f_{\dot{\varepsilon}}\left(\epsilon\right)d\epsilon &(89)\end{aligned}$$

where (89) follows from (3). Also from (4) follows that:

$$\rho_x^{R*}\left(\dot{w}^N-\epsilon,\dot{w}^c-\epsilon\right)=\begin{cases}\overline{\varrho}_x^R & ,\dot{w}^N-\epsilon<\dot{w}^c-\epsilon\\ 0 & ,o/w\end{cases} \qquad (90)$$

or, equivalently:

$$\rho_x^{R*}\left(\dot{w}^N-\epsilon,\dot{w}^c-\epsilon\right)=\begin{cases}\overline{\varrho}_x^R & ,\dot{w}^N<\dot{w}^c\\ 0 & ,o/w\end{cases} \qquad (91)$$

i.e., the value of $\rho_x^{R*}\left(\dot{w}^N-\epsilon,\dot{w}^c-\epsilon\right)$ is determined solely by the relative size of $\dot{w}^N$ and $\dot{w}^c$. Combining (91) with (89) then gives (26).

Derivation of (27)*:* This follows immediately from combining (26) with (14).

B.4 Proof of Lemma 4

This result lies on the fact that, given R and x, the set of wage adjustments that satisfy the condition $\dot{w}^{N*} < \dot{w}^{c*}$, i.e., those that are candidates to be constrained (by R), is equal to the set of those that satisfy $\dot{w}^{N*} + \dot{\varepsilon} < \dot{w}^{c*} + \dot{\varepsilon} \Leftrightarrow \dot{w}^{N} < \dot{w}^{c}$.

Starting from (12) we can write $\varrho_x^R = \frac{\Pr\left(\delta=1, \dot{w}^{N*} < \dot{w}^{c*} | R, x\right)}{\Pr\left(\dot{w}^{N*} < \dot{w}^{c*} | R, x\right)}$. We observe that $\Pr\left(\delta = 1, \dot{w}^{N*} < \dot{w}^{c*} | R, x\right) = \Pr\left(\delta = 1, \dot{w}^{N*} + \dot{\varepsilon} < \dot{w}^{c*} + \dot{\varepsilon} | R, x\right) =$ $= \Pr\left(\delta = 1, \dot{w}^{N} < \dot{w}^{c} | R, x\right)$. Also $\Pr\left(\dot{w}^{N*} < \dot{w}^{c*} | R, x\right) = \Pr\left(\dot{w}^{N*} + \dot{\varepsilon} < \dot{w}^{c*} + \dot{\varepsilon} | R, x\right) =$ $= \Pr\left(\dot{w}^{N} < \dot{w}^{c} | R, x\right)$. Therefore $\varrho_x^R = \frac{\Pr\left(\delta=1, \dot{w}^{N} < \dot{w}^{c} | R, x\right)}{\Pr\left(\dot{w}^{N} < \dot{w}^{c} | R, x\right)} = \Pr\left(\delta = 1 | \dot{w}^{N} < \dot{w}^{c}, R, x\right)$.

Furthermore $\Pr\left(\delta = 1, \dot{w}^{N} < \dot{w}^{c} | R, x\right) = \Pr\left(\delta = 1 | R, x\right)$ since, given R and x, the set of wage adjustments that are constrained ($\delta = 1$) is a subset of those that are candidates to be constrained ($\dot{w}^{N} < \dot{w}^{c}$). This leads to (30).

B.5 Proof of Proposition 1

This follows from Lemmas 2-4.

B.6 Proof of Lemma 5

From (14), given in Lemma 2,[42] follows that the observed rate for any given wage adjustment with characteristics x, which is negotiated under R, takes a given value $\dot{w}$ if either its unconstrained value is equal to $\dot{w}$ *and* the wage adjustment is not constrained ($\dot{w}^{N} = \dot{w}$ and $\delta = 0$), or its constrained value is equal to $\dot{w}$ *and* the wage adjustment is constrained ($\dot{w}^{c} = \dot{w}$ and $\delta = 1$). Accordingly $f_{\dot{w}|R,x}$ may be decomposed as follows:

$$
\begin{aligned}
f_{\dot{w}|R,x}\left(\dot{w}|R,x\right) &= f_{\dot{w}^{N},\delta|R,x}\left(\dot{w},0|R,x\right) + f_{\dot{w}^{c},\delta|R,x}\left(\dot{w},1|R,x\right) && (92)\\
&= \left[f_{\dot{w}^{N}|R,x}\left(\dot{w}|R,x\right) - f_{\dot{w}^{N},\delta|R,x}\left(\dot{w},1|R,x\right)\right] + \\
&\quad + f_{\dot{w}^{c},\delta|R,x}\left(\dot{w},1|R,x\right) && (93)
\end{aligned}
$$

Given the definitions of $L_x^R(\cdot)$ and $G_x^R(\cdot)$ given in (33) and (34), respectively, (32) follows immediately.

[42] NB: This result follows from Assumptions 1 and 5 only.

B.7 Proof of Lemma 6

Proof of (39)*:* Starting from the definition of $L_x^R(\cdot)$ given in (33), and using the Law of Total Probability, we can write:

$$L_x^R(\dot{w}) = -\int_{v\in\mathcal{W}_{Rx}^c} f_{\dot{w}^N,\dot{w}^c,\delta|R,x}(\dot{w},v,1|R,x)\,dv \tag{94}$$
$$= -\int_{v\in\mathcal{W}_{Rx}^c} \rho_x^R(\dot{w},v)\, f_{\dot{w}^N,\dot{w}^c|R,x}(\dot{w},v|R,x)\,dv \tag{95}$$

where $\rho_x^R(\dot{w}^N,\dot{w}^c) \equiv \Pr\left(\delta=1|\dot{w}^N,\dot{w}^c,R,x\right)$ (as defined in (16)). From the definition of $\mathcal{C}_{Rx}$ follows that $\rho_x^R(\dot{w},v)$ is non-zero iff $(\dot{w},v)\in\mathcal{C}_{Rx}$. From the definitions of $\mathcal{L}_{Rx}$ and $\mathcal{G}_{Rx}(\dot{w})$ given in (35) and (38), respectively, follows that $(\dot{w},v)\in\mathcal{C}_{Rx}$ is equivalent to $\dot{w}$ belonging to $\mathcal{L}_{Rx}$ *and*, for any such value, v belonging to $\mathcal{G}_{Rx}(\dot{w})$ – which gives the RHS of (39).

Proof of (40)*:* Starting from the definition of $G_x^R(\cdot)$ given in (34), and using the Law of Total Probability, we can write:

$$G_x^R(\dot{w}) = \int_{\varpi\in\mathcal{W}_{Rx}^N} f_{\dot{w}^N,\dot{w}^c,\delta|R,x}(\varpi,\dot{w},1|R,x)\,d\varpi \tag{96}$$
$$= \int_{\varpi\in\mathcal{W}_{Rx}^N} \rho_x^R(\varpi,\dot{w})\, f_{\dot{w}^N,\dot{w}^c|R,x}(\varpi,\dot{w}|R,x)\,d\varpi \tag{97}$$

From the definition of $\mathcal{C}_{Rx}$ follows that $\rho_x^R(\varpi,\dot{w})$ is non-zero iff $(\varpi,\dot{w})\in\mathcal{C}_{Rx}$. From the definitions of $\mathcal{G}_{Rx}$ and $\mathcal{L}_{Rx}(\dot{w})$ given in (36) and (37), respectively, follows that $(\varpi,\dot{w})\in\mathcal{C}_{Rx}$ is equivalent to $\dot{w}$ belonging to $\mathcal{G}_{Rx}$ *and*, for any such value, ϖ belonging to $\mathcal{L}_{Rx}(\dot{w})$ – which gives the RHS of (40).

B.8 Proof of Corollary 1

(a) This follows immediately from the definitions given in (35) and (36).

(b) From the definition of $\mathcal{R}_x$ follows that if $R\notin\mathcal{R}_x$ then $\mathcal{C}_{Rx}=\emptyset$ and therefore, from part (a), that $\mathcal{L}_{Rx}=\mathcal{G}_{Rx}=\emptyset$. Given that this is the case then from (39) and (40) follows, respectively, that $L_x^\vartheta(\cdot)$ and $G_x^\vartheta(\cdot)$ are everywhere zero.

B.9 Proof of Lemma 7

Substituting (32) into (31), and using that (Law of Total Probability):

$$f_{\dot{w}^N|x}(\dot{w}|x) = \sum_{\vartheta \in R} \Pr(R = \vartheta|x) f_{\dot{w}^N|R,x}(\dot{w}|\vartheta, x) \tag{98}$$

we can write:

$$f_{\dot{w}|x}(\dot{w}|x) = f_{\dot{w}^N|x}(\dot{w}|x) + \sum_{\vartheta \in \mathcal{R}} \Pr(R = \vartheta|x) \left[L_x^{\vartheta}(\dot{w}) + G_x^{\vartheta}(\dot{w}) \right] \tag{99}$$

This simplifies to (41) since $L_x^R(\cdot)$ and $G_x^R(\cdot)$ are everywhere zero if $R \notin \mathcal{R}_x$ (Corollary 1(b)).

B.10 Proof of Lemma 8

Equation (42) follows from that $f_{\dot{w}|x}$ and $f_{\dot{w}^N|x}$ are, respectively, the mixtures of $f_{\dot{w}|R,x}$ and $f_{\dot{w}^N|R,x}$. Equation (43) follows from rearranging terms in (41).

B.11 Proof of Lemma 9

(a) When $\mathcal{W}_{Rx}^c$ is located to the left of $\mathcal{W}_{Rx}^N$ then $\mathcal{C}_{Rx} = \emptyset$. Using Corollary 1(a) then gives our result.

(b-1) Proof of (44): Starting from (39), and focusing on its first line, we note that $\rho_x^R(\dot{w}, v)$ is evaluated at points $(\dot{w}, v) \in \mathcal{C}_{Rx}$, therefore $\dot{w} < v$. From Lemma 3(c)[43] follows that $\rho_x^R(\dot{w}, v)$ is equal to $\overline{\varrho}_x^R$ at each such point; substituting into (39) gives (44).

Proof of (45): Starting from (40), and focusing on its first line, we note that $\rho_x^R(\varpi, \dot{w})$ is evaluated at points $(\varpi, \dot{w}) \in \mathcal{C}_{Rx}$, therefore $\varpi < \dot{w}$. As above follows that $\rho_x^R(\varpi, \dot{w})$ is equal to $\overline{\varrho}_x^R$ at each such point; substituting into (40) gives (45).

Derivation of $\mathcal{L}_{Rx}$ and $\mathcal{G}_{Rx}$: From the discussion in Section 4 and using Figure 3a follows that, in this particular case, $\mathcal{C}_{Rx}$ corresponds to the area of $add'c'a'$, excluding the points

[43] NB: This follows from Assumption 4.

along $dd^{'}$.[44] As can be seen, the projection of $\mathcal{C}_{Rx}$ on $\mathcal{W}_{Rx}^{N} = \left[\dot{w}_{x0}^{N}, \dot{w}_{x1}^{N}\right]$, i.e., $\mathcal{L}_{Rx}$, is the set $\left[\dot{w}_{x0}^{N}, \dot{w}_{Rx1}^{c}\right)$. Also, the projection of $\mathcal{C}_{Rx}$ on $\mathcal{W}_{Rx}^{c} = [\dot{w}_{Rx0}^{c}, \dot{w}_{Rx1}^{c}]$, i.e., $\mathcal{G}_{Rx}$, is the set $[\dot{w}_{Rx0}^{c}, \dot{w}_{Rx1}^{c}]$.

(b-2) Proof of (46): Since $\dot{w}^{c}$ is fixed, rather than variable, it follows that $f_{\dot{w}^{N},\dot{w}^{c}|R,x}$ reduces to $f_{\dot{w}^{N}|R,x}$, therefore also to $f_{\dot{w}^{N}|x}$ (see Lemma 3(a)). Accordingly, (44) reduces to (46).

Proof of (47): Similarly, (45) reduces to:

$$G_{x}^{R}\left(\dot{w}\right) = \begin{cases} \overline{\varrho}_{x}^{R} \int_{\varpi \in \mathcal{L}_{Rx}(\dot{w})} f_{\dot{w}^{N}|x}\left(\varpi|R,x\right) d\varpi & , \dot{w} \in \mathcal{G}_{Rx} \\ 0 & , o/w \end{cases} \tag{100}$$

Furthermore since $\mathcal{G}_{Rx} = \{\dot{w}_{Rx1}^{c}\}$ and $\mathcal{L}_{Rx} = \left[\dot{w}_{x0}^{N}, \dot{w}_{Rx1}^{c}\right)$ (see below), then for $\dot{w} \in \mathcal{G}_{Rx} \Leftrightarrow \dot{w} = \dot{w}_{Rx1}^{c}$ we have $\mathcal{L}_{Rx}\left(\dot{w}\right) = \mathcal{L}_{Rx}\left(\dot{w}_{Rx1}^{c}\right) = \left[\dot{w}_{x0}^{N}, \dot{w}_{Rx1}^{c}\right)$, which is the subset of the support of $f_{\dot{w}^{N}|x}$ that lies to the left of $\dot{w}$. Accordingly, the definite integral in expression above is equal to $F_{\dot{w}^{N}|x}\left(\dot{w}_{Rx1}|x\right)$, which gives (47).

Derivation of $\mathcal{L}_{Rx}$ and $\mathcal{G}_{Rx}$: From the discussion in Section 4 and using Figure 2a follows that, in this particular case, $\mathcal{C}_{Rx}$ corresponds to the line segment AC, excluding point C. As can be seen, the projection of $\mathcal{C}_{Rx}$ on $\mathcal{W}_{Rx}^{N} = \left[\dot{w}_{x0}^{N}, \dot{w}_{x1}^{N}\right]$, i.e., $\mathcal{L}_{Rx}$, is the set $\left[\dot{w}_{x0}^{N}, \dot{w}_{Rx1}^{c}\right)$. Also, the projection of $\mathcal{C}_{Rx}$ on $\mathcal{W}_{Rx}^{c} = \{\dot{w}_{Rx1}^{c}\}$, i.e., $\mathcal{G}_{Rx}$, is the set $\{\dot{w}_{Rx1}^{c}\}$. (NB: This corresponds to the case where $\dot{w}^{c*}$ is fixed and f_{ε} degenerate (error-free data)).

B.12 Proof of Lemma 10

(a) From Lemma 9(a) follows that in this case $f_{\dot{w}|R,x}$ is identical to $f_{\dot{w}^{N}|R,x}$ and therefore have identical supports, i.e., $\mathcal{W}_{Rx} = \mathcal{W}_{Rx}^{N}$. Furthermore, from Lemma 3(a) we have that $f_{\dot{w}^{N}|R,x}$ is identical to $f_{\dot{w}^{N}|x}$ that has support $\mathcal{W}_{x}^{N}$; combining gives $\mathcal{W}_{Rx} = \mathcal{W}_{x}^{N}$.

[44] This refers to the case where $\dot{w}^{c*}$ is variable and f_{ε} non-degenerate (error-contaminated data), which gives $\dot{w}^{c}$ variable. If f_{ε} is instead degenerate (error-free data), which also gives $\dot{w}^{c}$ variable, then $\mathcal{C}_{Rx}$ corresponds to the area of the trapezoid $ACC^{'}A^{'}$, excluding the points along $CC^{'}$. In the case where $\dot{w}^{c*}$ is fixed and f_{ε} non-degenerate, which is another case where $\dot{w}^{c}$ is variable, then $\mathcal{C}_{Rx}$ corresponds to the area of the parallelogram $acc^{'}a^{'}$ excluding the points along $cc^{'}$, depicted in Figure 2a. For the latter two cases the same expressions for $\mathcal{L}_{Rx}$ and $\mathcal{G}_{Rx}$ hold as in the former case.

(b) From (32), (39) and (40) follows that:

$$f_{\dot{w}|R,x}\left(\dot{w}|R,x\right)=\begin{cases}f_{\dot{w}^N|R,x}\left(\dot{w}|R,x\right)+L_x^R\left(\dot{w}\right) & ,\dot{w}\in\mathcal{L}_{Rx}\backslash\mathcal{G}_{Rx}\\ f_{\dot{w}^N|R,x}\left(\dot{w}|R,x\right)+G_x^R\left(\dot{w}\right) & ,\dot{w}\in\mathcal{G}_{Rx}\backslash\mathcal{L}_{Rx}\\ f_{\dot{w}^N|R,x}\left(\dot{w}|R,x\right)+L_x^R\left(\dot{w}\right)+G_x^R\left(\dot{w}\right) & ,\dot{w}\in\mathcal{L}_{Rx}\cap\mathcal{G}_{Rx}\\ f_{\dot{w}^N|R,x}\left(\dot{w}|R,x\right) & ,o/w\end{cases}\tag{101}$$

Using the results from Lemma 3(a) regarding the properties of $f_{\dot{w}^N|R,x}$, and Lemma 9(b-1, b-2) with regard to the nature of $\mathcal{L}_{Rx}$ and $\mathcal{G}_{Rx}$, it follows that the above specialises to the following for both cases of $\dot{w}^c$ variable and fixed (setting $\dot{w}^c_{Rx0}=\dot{w}^c_{Rx1}$ for the latter case):[45]

$$f_{\dot{w}|R,x}\left(\dot{w}|R,x\right)=\begin{cases}f_{\dot{w}^N|x}\left(\dot{w}|x\right)+L_x^R\left(\dot{w}\right) & ,\dot{w}^N_{x0}\leq\dot{w}<\dot{w}^c_{Rx0}\\ f_{\dot{w}^N|x}\left(\dot{w}|x\right)+L_x^R\left(\dot{w}\right)+G_x^R\left(\dot{w}\right) & ,\dot{w}^c_{Rx0}\leq\dot{w}<\dot{w}^c_{Rx1}\\ f_{\dot{w}^N|x}\left(\dot{w}|x\right)+G_x^R\left(\dot{w}\right) & ,\dot{w}=\{\dot{w}^c_{Rx1}\}\\ f_{\dot{w}^N|x}\left(\dot{w}|x\right) & ,\dot{w}^c_{Rx1}<\dot{w}\leq\dot{w}^N_{x1}\end{cases}\tag{102}$$

It follows that $\mathcal{W}_{Rx}\subseteq\mathcal{W}^N_x=\left[\dot{w}^N_{x0},\dot{w}^N_{x1}\right]$.

Furthermore from Line 1 of (102) follows that $\left[\dot{w}^N_{x0},\dot{w}^c_{Rx0}\right)\subset\mathcal{W}_{Rx}$ if $0\leq\overline{\varrho}^R_x<1$, and $\left[\dot{w}^N_{x0},\dot{w}^c_{Rx0}\right)\cap\mathcal{W}_{Rx}=\emptyset$ if $\overline{\varrho}^R_x=1$, as in the latter case $L^R_x\left(\dot{w}\right)=-f_{\dot{w}^N|x}\left(\dot{w}|x\right)$ for $\dot{w}\in\left[\dot{w}^N_{x0},\dot{w}^c_{Rx0}\right)$ – see (44) and (46).

Also from Line 2 (and only for the case of $\dot{w}^c$ variable) that $[\dot{w}^c_{Rx0},\dot{w}^c_{Rx1})\subset\mathcal{W}^N_x$ since, for $\dot{w}\in[\dot{w}^c_{Rx0},\dot{w}^c_{Rx1})$, $\left|L^R_x\left(\dot{w}\right)\right|<f_{\dot{w}^N|x}\left(\dot{w}|x\right)$ and $G^R_x\left(\dot{w}\right)\geq0$ – see (44)-(47).

Furthermore, from Lines 3 and 4 follows that $\left[\dot{w}^c_{Rx1},\dot{w}^N_{x1}\right]\subset\mathcal{W}_{Rx}$ since $G^R_x\left(\dot{w}\right)\geq0$, and $f_{\dot{w}^N|x}\left(\dot{w}|x\right)>0$ when evaluated in that interval given that $\left[\dot{w}^c_{Rx0},\dot{w}^N_{x1}\right]\subset\mathcal{W}^N_x$. We conclude that iff $\overline{\varrho}^R_x=1$ then $\mathcal{W}_{Rx}=\mathcal{W}^N_x\backslash\left[\dot{w}^N_{x0},\dot{w}^c_{Rx0}\right)=\left[\dot{w}^N_{Rx0},\dot{w}^N_{x1}\right]$, otherwise $\mathcal{W}_{Rx}=\mathcal{W}^N_x=\left[\dot{w}^N_{x0},\dot{w}^N_{x1}\right]$.

[45] Then, for $\dot{w}^c$ fixed, line 2 of (102) becomes obsolete as the relevant condition is never satisfied.

B.13 Proof of Lemma 11

(a) From (31) follows that $\mathcal{W}_x = \bigcup_{R\in\mathcal{R}} \mathcal{W}_{Rx}$, assuming that $\Pr(R|x) \neq 0$ for all $R \in \mathcal{R}$. Combining with that $\mathcal{W}_{Rx} \subseteq \mathcal{W}_x^N$ for each $R \in \mathcal{R}$ (from Lemma 10), follows that $\mathcal{W}_x \subseteq \mathcal{W}_x^N$. Moreover $\mathcal{W}_x = \mathcal{W}_x^N$ unless $\mathcal{W}_{Rx} \subset \mathcal{W}_x^N$ for all $R \in \mathcal{R}$.

If $\mathcal{R}_x \subset \mathcal{R}$ then $f_{\dot{w}|R,x}$ is undistorted for $R \in \mathcal{R}\backslash\mathcal{R}_x$, therefore $\mathcal{W}_{Rx} = \mathcal{W}_x^N$ for $R \in \mathcal{R}\backslash\mathcal{R}_x$ (see Lemma 10). It follows that $\mathcal{W}_x = \mathcal{W}_x^N$.

(b) We assume that $R \in \mathcal{R}_x \Leftrightarrow \mathcal{W}_{Rx}^c \subset \mathcal{W}_{Rx}^N$. From 10 follows that $\mathcal{W}_{Rx} = \mathcal{W}_{Rx}^N$ iff $\overline{\varrho}_x^R < 1$. Given that $\mathcal{W}_x = \bigcup_{R\in\mathcal{R}} \mathcal{W}_{Rx}$ it follows that $\mathcal{W}_x$ is equal to $\mathcal{W}_x^N$ if $\overline{\varrho}_x^R < 1$ for at least one case of $R \in \mathcal{R}$.

B.14 Proof of Proposition 2

If f_ε is degenerate then $\dot{w} = \dot{w}_{Rx1}^{c*}$, i.e., fixed, for each $R \in \overline{\mathcal{R}}_x$. Then combining (32) with the expressions for $L_x^R(\dot{w})$ and $G_x^R(\dot{w})$ from Lemma 9(b-2) gives that $f_{\dot{w}|R,x}$, for each $R \in \overline{\mathcal{R}}_x$, exhibits a mass point at $\dot{w} = \dot{w}_{Rx1}^{c*}$ which attracts probability mass equal to $\Pr(\dot{w} = \dot{w}_{Rx1}^{c*}|R,x) = G_x^R(\dot{w}_{Rx1}^{c*}) = \overline{\varrho}_x^R F_{\dot{w}^N|x}(\dot{w}_{Rx1}^{c*}|x)$, where $\overline{\varrho}_x^R = \varrho_x^R$ (NB: $\Pr\left(\dot{w}^N = \dot{w}_{Rx1}^{c*}|R,x\right) = \Pr\left(\dot{w}^N = \dot{w}_{Rx1}^{c*}|x\right) = 0$ due the continuity of $f_{\dot{w}^N|x}$, while $L_x^R(\dot{w}_{Rx1}^{c*}) = 0$.) At the same time $\Pr(\dot{w} = \dot{w}_{Rx1}^{c*}|R,x) = 0$ for all $R \in \mathcal{R}\backslash\overline{\mathcal{R}}_x$ since in those cases $L_x^R(\cdot)$ and $G_x^R(\cdot)$ are continuous in $\dot{w}$ – see Lemma 9(b-1).

Combining these results with (31) then gives that $f_{\dot{w}|x}$ exhibits mass points at each (distinct) value $\dot{w}_{Rx1}^{c*}$, $R \in \overline{\mathcal{R}}_x$, which attract probability mass equal to:

$$\Pr(\dot{w} = \dot{w}_{Rx1}^{c*}|x) = \Pr(R|x)\,\varrho_x^R\,F_{\dot{w}^N|x}(\dot{w}_{Rx1}^{c*}|x) \tag{103}$$

$$= \varrho_x^{RR}\,F_{\dot{w}^N|x}(\dot{w}_{Rx1}^{c*}|x) \tag{104}$$

where $\varrho_x^{RR} \equiv \Pr(R|x)\,\varrho_x^R$.

B.15 Proof of Lemma 12

Staring from (43) we can write:

$$
\begin{aligned}
& F_{\dot{w}|x}\left(\dot{w}|x\right) - F_{\dot{w}^N|x}\left(\dot{w}|x\right) = \\
= & \int_{w_{x0}^N}^{\dot{w}} \left\{ \sum_{\vartheta \in \mathcal{R}_x} \Pr\left(R = \vartheta|x\right) \left[L_x^{\vartheta}\left(\varpi\right) + G_x^{\vartheta}\left(\varpi\right)\right] \right\} d\varpi \qquad (105) \\
= & \sum_{\vartheta \in \mathcal{R}_x} \Pr\left(R = \vartheta|x\right) \left[\int_{w_{x0}^N}^{\dot{w}} L_x^{\vartheta}\left(\varpi\right) d\varpi + \int_{w_{x0}^N}^{\dot{w}} G_x^{\vartheta}\left(v\right) dv\right] \qquad (106)
\end{aligned}
$$

noting that w_{x0}^N (the lower limit of the integral in (105) and 106, and left limit of $\mathcal{W}_x^N$) is less than or equal to w_{x0} (the left limit of $\mathcal{W}_x$) – see Lemma 11.

Using (44) we can write:

$$
\int_{w_{x0}^N}^{\dot{w}} L_x^{\vartheta}\left(\varpi\right) d\varpi = -\overline{\varrho}_x^R A^R\left(\dot{w}\right) \qquad (107)
$$

where:

$$
A_x^R\left(\dot{w}\right) = \int_{w_{x0}^N}^{\min\left(\dot{w}, \dot{w}_{R_1 x1}^c\right)} \int_{v \in \mathcal{G}_{Rx}(\varpi)} f_{\dot{w}^N, \dot{w}^c|R,x}\left(\varpi, v|R, x\right) dv d\varpi \qquad (108)
$$

is the probability mass of $f_{\dot{w}^N, \dot{w}^c|R,x}$ allocated to the points in the locus of $\mathcal{C}_{Rx}$ that lie to the *left* of point $\dot{w}$ (NB: $L_x^{\vartheta}\left(\varpi\right) = 0$ for $\varpi > \max \mathcal{L}_{Rx} = \dot{w}_{R_1 x1}^c$, thus the limits in the first integral above).

Similarly, using (45) we can write:

$$
\int_{w_{x0}^N}^{\dot{w}} G_x^{\vartheta}\left(v\right) dv = \overline{\varrho}_x^R B^R\left(\dot{w}\right) \qquad (109)
$$

where:

$$
B_x^R\left(\dot{w}\right) = \begin{cases} 0 & , \dot{w} < \min \mathcal{G}_{Rx} \\ \int_{w_{x0}^N}^{\min\left(\dot{w}, \dot{w}_{R_1 x1}^c\right)} \int_{\varpi \in \mathcal{L}_{Rx}(v)} f_{\dot{w}^N, \dot{w}^c|R,x}\left(\varpi, v|R, x\right) d\varpi dv & , \dot{w} \geq \min \mathcal{G}_{Rx} \end{cases} \qquad (110)
$$

is the probability mass of $f_{\dot{w}^N,\dot{w}^c|R,x}$ allocated to the points in the locus of $\mathcal{C}_{Rx}$ that lie *below* point $\dot{w}$. (NB: $G_x^{\vartheta}(v) = 0$ for $v < \min \mathcal{G}_{Rx}$ (see (40)), where $\min \mathcal{G}_{Rx} = \dot{w}^c_{R_1 x0}$ if $\dot{w}^c$ is variable and $\min \mathcal{G}_{Rx} = \dot{w}^c_{R_1 x1}$ if $\dot{w}^c$ is fixed, which gives the first line above. Also, $G_x^{\vartheta}(v) = 0$ for $v > \max \mathcal{G}_{Rx}$, where $\max \mathcal{G}_{Rx} = \dot{w}^c_{R_1 x1}$ for both cases of $\dot{w}^c$ being variable and variable, thus the limits for the first integral in the second line above).

Let and $c_x^R(\dot{w}') \equiv A_x^R(\dot{w}) - B_x^R(\dot{w})$. Substituting (107) and (109) into (106), and using that $\varrho_x^R = \overline{\varrho}_x^R$ (from Lemma 1), gives:

$$
\begin{aligned}
F_{\dot{w}|x}(\dot{w}|x) - F_{\dot{w}^N|x}(\dot{w}|x) &= \sum_{\vartheta \in \mathcal{R}_x} \Pr(R = \vartheta|x)\left[-\varrho_x^R c_x^R(\dot{w})\right] && (111)\\
&= -\sum_{\vartheta \in \mathcal{R}_x} \Pr(R = \vartheta|x)\, \varrho_x^R c_x^R(\dot{w}) && (112)\\
&= -\sum_{\vartheta \in \mathcal{R}_x} \varrho_x^{\vartheta\vartheta} c_x^{\vartheta}(\dot{w}) && (113)
\end{aligned}
$$

where $\varrho_x^{\vartheta\vartheta} \equiv \Pr(R = \vartheta|x)\, \varrho_x^{\vartheta}$. Furthermore, from the discussion above follows that $c_x^R(\dot{w})$ is equal to the probability mass of $f_{\dot{w}^N,\dot{w}^c|R,x}$ that is allocated to the points in the locus of $\mathcal{C}_{Rx}$ that lie to the *left* and *above* point $\dot{w}$, which is formally described by(51).

B.16 Proof of Corollary 2

(a) In this case $\dot{w}^c \in \mathcal{W}^c_{Rx} = [\dot{w}^c_{Rx0}, \dot{w}^c_{Rx1}]$, where $\dot{w}^N_{x0} < \dot{w}^c_{Rx0} < \dot{w}^c_{Rx1} < \dot{w}^N_{x1}$.

First we note that for $\dot{w} \geq \dot{w}^c_{Rx1}$ the set $\left\{(\dot{w}^N, \dot{w}^c) \in \mathcal{C}_{Rx} : \dot{w}^N < \dot{w}, \dot{w}^c > \dot{w}\right\}$ is empty since $\dot{w}^c \leq \dot{w}^c_{Rx1}$; this gives Line 3 of (52).

On the other hand for any point $\dot{w}$ such that $\dot{w}^N_{x0} \leq \dot{w} < \dot{w}^c_{Rx1}$, and assuming that $\overline{\varrho}_x^R > 0$, then this set is non-empty and (51) specialises to the following:

$$
\begin{aligned}
c_x^R(\dot{w}) &= \int_{\min \mathcal{L}_{Rx}}^{\dot{w}} \int_{\max(\dot{w}, \min \mathcal{G}_{Rx}(\varpi))}^{\max \mathcal{G}_{Rx}(\varpi)} f_{\dot{w}^N,\dot{w}^c|R,x}(\varpi, v|R, x)\, dv d\varpi && (114)\\
&= \int_{\dot{w}^N_{x0}}^{\dot{w}} \int_{\max(\dot{w}, \min \mathcal{G}_{Rx}(\varpi))}^{\max \mathcal{G}_{Rx}(\varpi)} f_{\dot{w}^N,\dot{w}^c|R,x}(\varpi, v|R, x)\, dv d\varpi && (115)
\end{aligned}
$$

where the proof of $\min \mathcal{L}_{Rx} = \dot{w}^N_{x0}$ is given in Lemma 9.

For the case of points $\dot{w}$ such that $\dot{w}^N_{x0} \leq \dot{w} < \dot{w}^c_{Rx0}$ (the case considered in Line 1 of

(52)), we have that $\max\left(\dot{w}, \min \mathcal{G}_{Rx}\left(\varpi\right)\right) = \min \mathcal{G}_{Rx}\left(\varpi\right)$ since $\min \mathcal{G}_{Rx}\left(\varpi\right) \geq \min \mathcal{G}_{Rx} = \dot{w}^c_{Rxo}$ (see Lemma 9). Accordingly we can write:

$$
\begin{aligned}
c^R_x\left(\dot{w}\right) &= \int_{\dot{w}^N_{x0}}^{\dot{w}} \int_{\min \mathcal{G}_{Rx}(\varpi)}^{\max \mathcal{G}_{Rx}(\varpi)} f_{\dot{w}^N,\dot{w}^c|R,x}\left(\varpi, v|R, x\right) dv d\varpi & (116)\\
&= \int_{\dot{w}^N_{x0}}^{\dot{w}} \int_{v\in\mathcal{G}_{Rx}(\varpi)} f_{\dot{w}^N,\dot{w}^c|R,x}\left(\varpi, v|R, x\right) dv d\varpi & (117)
\end{aligned}
$$

i.e., the points in the locus of $\mathcal{C}_{Rx}$ that lie to the *left* and *above* of point $\dot{w}$ are also the points in the locus of $\mathcal{C}_{Rx}$ that lie to the *left* of point $\dot{w}$. Furthermore from the depictions of $\mathcal{C}_{Rx}$ for the three cases that under our assumptions $\dot{w}^c$ can be variable (i.e., when $\dot{w}^{c*}$ is fixed and the data are error-contaminated (Figure 2a), or when $\dot{w}^{c*}$ is variable and the data are error-free or error-contaminated data (Figure 3a)) it is apparent that these are also the points in the locus of $\mathcal{W}^{Nc}_{Rx}$ that lie to the *left* of point $\dot{w}$. From the latter follows that, for any point $\dot{w}$ such that $\dot{w}^N_{x0} \leq \dot{w} < \dot{w}^c_{Rx0}$, equation (51) simplifies to $c^R_x\left(\dot{w}\right) = \int_{\varpi<\dot{w}} f_{\dot{w}^N|R,x}\left(\varpi|R, x\right) d\varpi = F_{\dot{w}^N|R,x}\left(\dot{w}|R, x\right)$; using further that $F_{\dot{w}^N|R,x}\left(\dot{w}|R, x\right) = F_{\dot{w}^N|x}\left(\dot{w}|x\right)$ (from Lemma 3(a)) gives the result in Line 1 of (52).

On the other hand for $\dot{w}^c_{Rx0} \leq \dot{w} < \dot{w}^c_{Rx1}$ (the case considered in Line 2 of (52)) we have that $\max\left(\dot{w}, \min \mathcal{G}_{Rx}\left(\varpi\right)\right) = \dot{w}$, given that $\min \mathcal{G}_{Rx}\left(\varpi\right) = \dot{w}^c_{Rxo}$. Substituting into the lower limit in the second integral of (115) gives the result in Line 2 of (52).

(b) In this case $\dot{w}^c \in \mathcal{W}^c_{Rx} = \{\dot{w}^c_{Rx1}\}$, where $\dot{w}^N_{x0} < \dot{w}^c_{Rx1} < \dot{w}^N_{x1}$.

First we note that for $\dot{w} \geq \dot{w}^c_{Rx1}$ the set $\left\{\left(\dot{w}^N, \dot{w}^c\right) \in \mathcal{C}_{Rx} : \dot{w}^N < \dot{w}, \dot{w}^c > \dot{w}\right\}$ is empty since $\dot{w}^c = \dot{w}^c_{Rx1}$; this gives Line 2 of (53).

On the other hand for any point $\dot{w}$ such that $\dot{w}^N_{x0} \leq \dot{w} < \dot{w}^c_{Rx1}$, and assuming that $\overline{\varrho}^R_x > 0$, then this set is non-empty and (51) specialises to the following:

$$
\begin{aligned}
c^R_x\left(\dot{w}\right) &= \int_{\min \mathcal{L}_{Rx}}^{\dot{w}} f_{\dot{w}^N|R,x}\left(\varpi|R, x\right) d\varpi & (118)\\
&= \int_{\dot{w}^N_{x0}}^{\dot{w}} f_{\dot{w}^N|R,x}\left(\varpi|R, x\right) d\varpi = & (119)\\
&= F_{\dot{w}^N|x}\left(\dot{w}|x\right) & (120)
\end{aligned}
$$

which gives Line 1 of (53).

B.17 Proof of Proposition 3

First consider $R_1 \in \mathcal{R}_x$. Since $\mathcal{W}^c_{R_1x} \subset \mathcal{W}^N_{R_1x}$ where $\mathcal{W}^N_{R_1x} = \mathcal{W}^N_x = \left[\dot{w}^N_{x0}, \dot{w}^N_{x1}\right]$ (from Lemma 3(a)) it follows that $\dot{w}^N_{x0} < \dot{w}^c_{R_1x1}$. Furthermore since, by assumption, $\mathcal{W}_x = \mathcal{W}^N_x$, i.e., $[\dot{w}_{x0}, \dot{w}_{x1}] = \left[\dot{w}^N_{x0}, \dot{w}^N_{x1}\right]$, then $\dot{w}_{x0} < \dot{w}^c_{R_1x1}$. Accordingly there exists a point $\dot{w}^{'}_1$ such that $\dot{w}_{x0} < \dot{w}^{'}_1 < \dot{w}^c_{R_1x1}$. Given that[46] $\dot{w}^c_{R_1x1} < \dot{w}^c_{R_kx1}$ for all $k > 1$, it follows that $\dot{w}^{'}_1 < \dot{w}^c_{R_kx1}$ for all $k \geq 1$. From (52) and (53) then follows that $c^{R_k}_x\left(\dot{w}^{'}_1\right) \neq 0$ for all $k \geq 1$, and therefore using (50) we can write:

$$F_{\dot{w}|x}\left(\dot{w}^{'}_1|x\right) - F_{\dot{w}^N|x}\left(\dot{w}^{'}_1|x\right) = -\sum_{j=1}^{K_x} \varrho^{R_jR_j}_x c^{R_j}_x\left(\dot{w}^{'}_1\right) \tag{121}$$

Also for each $k = 2, \ldots, K_x$ there exists point $\dot{w}^{'}_k$ such that $\dot{w}^c_{R_{k-1}x1} < \dot{w}^{'}_k < \dot{w}^c_{R_kx1}$ since $\dot{w}^c_{R_kx1} < \dot{w}^c_{R_{k-1}x1}$ (see Footnote 46). From this follows that, given k, $\dot{w}^{'}_k < \dot{w}^c_{R_\kappa x1}$ for all $\kappa \geq k$. From (52) and (53) then follows that $c^{R_\kappa}_x\left(\dot{w}^{'}_k\right) \neq 0$ for all $\kappa \geq k$, and using (50) we can write:

$$F_{\dot{w}|x}\left(\dot{w}^{'}_k|x\right) - F_{\dot{w}^N|x}\left(\dot{w}^{'}_k|x\right) = -\sum_{j=k}^{K_x} \varrho^{R_jR_j}_x c^{R_j}_x\left(\dot{w}^{'}_k\right) \quad , \quad 1 < k \leq K_x \tag{122}$$

B.18 Proof of Proposition 4

First we note that the restrictions set by Proposition 4 also imply the restrictions that underlie the results of Proposition 3. In particular, for $k = 2, \ldots, K_x$ and $\{R_{k-1}, R_k\} \subseteq \mathcal{R}_x$, if $\dot{w}^c_{R_{k-1}x1} < \dot{w}^c_{R_kx0}$ then also $\dot{w}^c_{R_kx1} \neq \dot{w}^c_{R_{k-1}x1}$ since $\dot{w}^c_{R_kx0} \leq \dot{w}^c_{R_kx1}$. Also for the same reason, $\dot{w}_{x0} < \dot{w}^{'}_1 < \dot{w}^c_{R_1x0}$ implies that $\dot{w}_{x0} < \dot{w}^{'}_1 < \dot{w}^c_{R_1x1}$, and $\dot{w}^c_{R_{k-1}x1} < \dot{w}^{'}_k < \dot{w}^c_{R_kx0}$ implies that $\dot{w}^c_{R_{k-1}x1} < \dot{w}^{'}_k < \dot{w}^c_{R_kx1}$ for $2 \leq k \leq K_x$. Accordingly equations (54) given in Proposition 3 also hold for points $\dot{w}^{'}_1, \ldots, \dot{w}^{'}_{K_x}$ defined in Proposition 4, i.e., for $k = 1, \ldots, K_x$:

$$F_{\dot{w}|x}\left(\dot{w}^{'}_k|x\right) - F_{\dot{w}^N|x}\left(\dot{w}^{'}_k|x\right) = -\sum_{j=k}^{K_x} \varrho^{R_jR_j}_x c^{R_j}_x\left(\dot{w}^{'}_k\right) \tag{123}$$

From Corollary 2 holds that $c^R_x\left(\dot{w}^{'}\right) = F_{\dot{w}^N|x}\left(\dot{w}^{'}|x\right)$ for R s.t. $\dot{w}^{'} < \dot{w}^c_{Rx0}$. Setting

[46] This follows from that $\dot{w}^c_{R_kx1} \leq \dot{w}^c_{R_{k-1}x1}$ (by construction of the index k) and $\dot{w}^c_{R_kx1} \neq \dot{w}^c_{R_{k-1}x1}$ (by assumption).

$\dot{w}' = \dot{w}_k'$ then $c_x^{R_j}\left(\dot{w}_k'\right) = F_{\dot{w}^N|x}\left(\dot{w}_k'|x\right)$ for R_j, $j \geq k$, since (by assumption) $\dot{w}_k' < \dot{w}_{R_k x0}^c \leq \dot{w}_{R_k x1}^c < \dot{w}_{R_j x0}^c$ for $j > k$. Therefore (123) becomes:

$$F_{\dot{w}|x}\left(\dot{w}_k'|x\right) - F_{\dot{w}^N|x}\left(\dot{w}_k'|x\right) = -\sum_{j=k}^{K_x} \varrho_x^{R_j R_j} F_{\dot{w}^N|x}\left(\dot{w}_k'|x\right) \tag{124}$$

$$= -\left(\sum_{j=k}^{K_x} \varrho_x^{R_j R_j}\right) F_{\dot{w}^N|x}\left(\dot{w}_k'|x\right) \tag{125}$$

which is the generic form of the equations given in (55).

B.19 Proof of Lemma 13

From Lemma 3(a) we have that $f_{\dot{w}^N|x}$ is symmetric. Therefore for points $\dot{w}$ and $\left(2m_x^N - \dot{w}\right)$ – which are symmetrically located on either side of the median m_x^N of $f_{\dot{w}^N|x}$ – we can write:

$$F_{\dot{w}^N|x}\left(\dot{w}|x\right) = 1 - F_{\dot{w}^N|x}\left(2m_x^N - \dot{w}|x\right) \tag{126}$$

From (43) and Lemma 9 follows that $f_{\dot{w}|x}$ may be distorted only to the left of and including point $\dot{w}_{R_{K_x}x1}^c$; this is because $L_x^R\left(\dot{w}\right)$ and $G_x^R\left(\dot{w}\right)$, for $R = R_k \in \mathcal{R}_x$, may be non-zero only if $\dot{w} < \dot{w}_{Rx1}^c$, where $\dot{w}_{R_{K_x}x1}^c \geq \dot{w}_{R_k x1}^c$ among all $R_k \in \mathcal{R}_x$. Accordingly:

$$F_{\dot{w}|x}\left(\dot{w}|x\right) = F_{\dot{w}^N|x}\left(\dot{w}|x\right) \ , \ \dot{w} \geq \dot{w}_{R_{K_x}x1}^c \tag{127}$$

Furthermore, since the of relocation of probability mass in the distorted part of $f_{\dot{w}|x}$ (i.e., to the left of point $\dot{w}_{R_{K_x}x1}^c$) is only directed to the right it follows that:

$$F_{\dot{w}|x}\left(\dot{w}|x\right) \leq F_{\dot{w}^N|x}\left(\dot{w}|x\right) \ , \ \dot{w} < \dot{w}_{R_{K_x}x1}^c \tag{128}$$

Let $q_{\alpha x}$ and $q_{\alpha x}^N$ be the α'th percentiles of $f_{\dot{w}|x}$ and $f_{\dot{w}^N|x}$, respectively (thus also $m_x^N =$

q^N_{50x}); then from (127) and (128) follows that:

$$q^N_{\alpha x} < q_{\alpha x} \ , \ q_{\alpha x} < \dot{w}^c_{R_{K_x} x1} \tag{129}$$

$$q^N_{\alpha x} = q_{\alpha x} \ , \ q_{\alpha x} \geq \dot{w}^c_{R_{K_x} x1} \tag{130}$$

Let $\dot{w}^c_{R_{K_x} x1} < m_x$ where $m_x \equiv q_{50x}$ is the median of $f_{\dot{w}|x}$, and consider a particular value of α such that $q_{\alpha x} > \dot{w}^c_{R_{K_x} x1}$; then from (130) follows that $q_{\alpha x} = q^N_{\alpha x}$, which therefore must also be true for $\alpha = 50$, i.e., $m_x = m^N_x$. Furthermore, consider point $\dot{w}$ such that $\dot{w} < \dot{w}^c_{R_{K_x} x1} < m_x$, i.e., $\dot{w}$ lies in the distorted part of (the lower tail of) $f_{\dot{w}|x}$ while its symmetrically located point $\left(2m^N_x - \dot{w}\right)$ in the (undistorted) upper tail. Combining (126) and (127) gives $F_{\dot{w}^N|x}\left(\dot{w}|x\right) = 1 - F_{\dot{w}|x}\left(2m^N_x - \dot{w}|x\right)$. Furthermore since $m_x = m^N_x$ this becomes $F_{\dot{w}^N|x}\left(\dot{w}|x\right) = 1 - F_{\dot{w}|x}\left(2m_x - \dot{w}|x\right)$.

B.20 Proof of Proposition 5

Given that $\varrho^R_x = \Pr\left(\delta = 1|\dot{w}^N < \dot{w}^{c*}, R, x\right)$, $R \in \mathcal{R}_x$, are probabilities, the following must hold:

$$\varrho^R_x \geq 0 \tag{131}$$

$$\varrho^R_x \leq 1 \tag{132}$$

Similarly, for $\pi^R_x \equiv \Pr\left(R|x\right)$, $R \in \mathcal{R}$:

$$\pi^R_x \geq 0 \tag{133}$$

$$\pi^R_x \leq 1 \tag{134}$$

In this case, also:

$$\sum_{\vartheta \in \mathcal{R}} \pi^\vartheta_x = 1 \tag{135}$$

Combining (132) with (49) gives that ϱ_x^{RR} is a lower bound for ϱ_x^R, $R \in \mathcal{R}_x$:

$$\varrho_x^R \geq \varrho_x^{RR} \geq 0 \ , \quad R \in \mathcal{R}_x \tag{136}$$

Similarly, combining (134) with (49) gives that ϱ_x^{RR} is a also a lower bound for π_x^R, $R \in \mathcal{R}_x$:

$$\pi_x^R \geq \varrho_x^{RR} \geq 0 \ , \quad R \in \mathcal{R}_x \tag{137}$$

Noting that (137) adds no information about the values of $\pi_x^R \in [0,1]$ for $R \in \mathcal{R} \backslash \mathcal{R}_x$, when combined with (133) allows to update the known lower bounds of π_x^R as follows:

$$\pi_x^R \geq \begin{cases} \varrho_x^{RR} \geq 0 & , \ R \in \mathcal{R}_x \\ 0 & , \ R \in \mathcal{R} \backslash \mathcal{R}_x \end{cases} \tag{138}$$

Furthermore, from (135) follows that:

$$\pi_x^R = 1 - \sum_{\vartheta \in \mathcal{R} \backslash \{R\}} \pi_x^\vartheta \tag{139}$$

Substituting π_x^ϑ, $\vartheta \in \mathcal{R} \backslash \{R\}$, on the RHS above with their lower bounds from (138) allows to update the upper bound of each π_x^R, $R \in \mathcal{R}$, as follows:

$$\pi_x^R \leq 1 - \sum_{\vartheta \in \mathcal{R}_x \backslash \{R\}} \varrho_x^{\vartheta\vartheta} \ \leq 1 \ , \quad R \in \mathcal{R} \tag{140}$$

Combining (138) and (140) then gives (58).

Also combining (140) with (49) gives:

$$\varrho_x^R \geq \frac{\varrho_x^{RR}}{1 - \sum_{\vartheta \in \mathcal{R}_x \backslash \{R\}} \varrho_x^{\vartheta\vartheta}} \geq \varrho_x^{RR} \geq 0 \ , \quad R \in \mathcal{R}_x \tag{141}$$

which updates the lower bounds of ϱ_x^R given in (136). Combined with (132) they give (57).

B.21 Proof of Proposition 6

In this case $K_x = 1$ and $\mathcal{R}_x = \{r\}$ (see definition of this regime), therefore the unknown parameter be identified is ϱ_x^r.

(a) *Proof of* (64)*:* In this case $\mathcal{R}_x \subset \mathcal{R}$. Accordingly, from Lemma 11(a) follows that $\mathcal{W}_x = \mathcal{W}_x^N$, thus $\dot{w}_{x0} = \dot{w}_{x0}^N$. Given that f_ε is degenerate it follows that $\dot{w}_{x0} = \dot{w}_{x0}^{N*}$.

Furthermore, by assumption, if $R = r$ then $\dot{w}_{rx0}^{c*} = \dot{P}_{x0}^{e*}$. Given that f_ε is degenerate it follows that $\dot{w}_{rx0}^{c} = \dot{P}_{x0}^{e*}$.

From the definition of the HIGH regime we have $0 < \dot{w}_{x0}^{N*} < \dot{P}_{x0}^{e*}$. Writing $\mathcal{R}_x = \{R_1\}$ where $R_1 = r$, then $\dot{w}_{x0} < \dot{w}_{rx0}^c \Leftrightarrow \dot{w}_{x0} < \dot{w}_{R_1x0}^c$ thus the conditions for Proposition 4 are fulfilled. Accordingly for $\dot{w}_1' \in \left(\dot{w}_{x0}, \dot{w}_{R_1x0}^c\right) = \left(\dot{w}_{x0}, \dot{w}_{rx0}^c\right) = \left(\dot{w}_{x0}^{N*}, \dot{P}_{x0}^{e*}\right)$ equation (64) follows immediately from equation (55).

(b) *Proof of* (65)*:* In this case $\mathcal{R}_x \subset \mathcal{R}$ therefore $\mathcal{W}_x = \mathcal{W}_x^N$, thus $\dot{w}_{x0} = \dot{w}_{x0}^N$. Given that f_ε is degenerate it follows that $\dot{w}_{x0} = \dot{w}_{x0}^{N*} - \dot{\varepsilon}_1$.

Furthermore that $\dot{w}_{rx0}^c = \dot{w}_{rx0}^{c*} - \dot{\varepsilon}_1 = \dot{P}_{x0}^{e*} - \dot{\varepsilon}_1$.

From the definition of the HIGH regime we have $0 < \dot{w}_{x0}^{N*} < \dot{P}_{x0}^{e*}$. Writing $\mathcal{R}_x = \{R_1\}$ where $R_1 = r$, then $\dot{w}_{x0} < \dot{w}_{rx0}^c \Leftrightarrow \dot{w}_{x0} < \dot{w}_{R_1x0}^c$ thus the conditions for Proposition 4 are fulfilled.

Then for $\dot{w}_1' \in \left(\dot{w}_{x0}, \dot{w}_{rx0}^c\right) = \left(\dot{w}_{x0}^{N}, \dot{P}_{x0}^{e}\right)$ equation (65) follows immediately from equation (55).

B.22 Proof of Proposition 7

In this case $K_x = 2$ and $\mathcal{R}_x = \{n, r\}$ (see definition of this regime), therefore the unknown parameters to be identified are ϱ_x^n and ϱ_x^r.

(a) *Proof of* (66) *and* (67)*:* In this case $\mathcal{R}_x \subset \mathcal{R}$ therefore $\mathcal{W}_x = \mathcal{W}_x^N$, thus $\dot{w}_{x0} = \dot{w}_{x0}^N$. Given that f_ε is degenerate it follows that $\dot{w}_{x0} = \dot{w}_{x0}^{N*}$.

Furthermore, by assumption, if $R = n$ then $\dot{w}_{nx1}^{c*} = \dot{w}_{nx0}^{c*} = 0$, and if $R = r$ then $\dot{w}_{rx0}^{c*} = \dot{P}_{x0}^{e*}$ and $\dot{w}_{rx1}^{c*} = \dot{P}_{x1}^{e*}$. Given that f_ε is degenerate it follows that $\dot{w}_{nx0}^c = \dot{w}_{nx1}^c = 0$, $\dot{w}_{rx0}^c = \dot{P}_{x0}^{e*}$and $\dot{w}_{rx1}^c = \dot{P}_{x1}^{e*}$.

From the definition of the MODERATE regime we have $\dot{w}_{x0}^{N*} < 0 < \dot{P}_{x0}^{e*} < \dot{P}_{x1}^{e*}$.

It follows that $\dot{w}_{nx1}^{c} < \dot{w}_{rx1}^{c}$, therefore we can write $\mathcal{R}_x = \{R_1, R_2\}$ where $R_1 = n$ and $R_2 = r$.

Furthermore, $\dot{w}_{x0} < \dot{w}_{nx0}^{c} \Leftrightarrow \dot{w}_{x0} < \dot{w}_{R_1x0}^{c}$ and $\dot{w}_{nx1}^{c} < \dot{w}_{rx0}^{c} \Leftrightarrow \dot{w}_{R_1x1}^{c} < \dot{w}_{R_2x0}^{c}$ therefore the conditions for Proposition 4 are fulfilled. Accordingly for $\dot{w}_1' \in \left(\dot{w}_{x0}, \dot{w}_{R_1x0}^{c}\right) = \left(\dot{w}_{x0}, \dot{w}_{nx0}^{c}\right) = \left(\dot{w}_{x0}^{N*}, 0\right)$ and $\dot{w}_2' \in \left(\dot{w}_{R_1x1}^{c}, \dot{w}_{R_2x0}^{c}\right) = \left(\dot{w}_{nx1}^{c}, \dot{w}_{rx0}^{c}\right) = \left(0, \dot{P}_{x0}^{e*}\right)$, equations (66) and (67) follow immediately from equation (55).

Proof of (68): By assumption, if $R = n$ then $\dot{w}^{c*} = 0$, i.e., fixed, whereas if $R = r$ then $\dot{w}^{c*} = \dot{P}^{e*}$ which is variable, therefore $\overline{\mathcal{R}}_x = \{n\}$. Furthermore, if $R = n$ then $\dot{w}_{Rx1}^{c*} = \dot{w}_{nx1}^{c*} = 0$. Substituting into (48) in Proposition 2 gives (68).

(b) *Proof of* (69) *and* (70): In this case $\mathcal{R}_x \subset \mathcal{R}$ therefore $\mathcal{W}_x = \mathcal{W}_x^N$, thus $\dot{w}_{x0} = \dot{w}_{x0}^{N}$. Given that f_ε is degenerate it follows that $\dot{w}_{x0} = \dot{w}_{x0}^{N*} - \dot{\varepsilon}_1$.

Furthermore that $\dot{w}_{nx0}^{c} = \dot{w}_{nx0}^{c*} - \dot{\varepsilon}_1 = -\dot{\varepsilon}_1$, $\dot{w}_{nx1}^{c} = \dot{w}_{nx1}^{c*} + \dot{\varepsilon}_1 = \dot{\varepsilon}_1$, $\dot{w}_{rx0}^{c} = \dot{w}_{rx0}^{c*} - \dot{\varepsilon}_1 = \dot{P}_{x0}^{e*} - \dot{\varepsilon}_1$ and $\dot{w}_{rx1}^{c} = \dot{w}_{rx1}^{c*} + \dot{\varepsilon}_1 = \dot{P}_{x1}^{e*} + \dot{\varepsilon}_1$.

Given that $\dot{w}_{x0}^{N*} < 0 < \dot{P}_{x0}^{e*} < \dot{P}_{x1}^{e*}$ (from the definition of the MODERATE regime) it follows that $\dot{w}_{nx1}^{c} < \dot{w}_{rx1}^{c}$, therefore we can write $\mathcal{R}_x = \{R_1, R_2\}$ where $R_1 = n$ and $R_2 = r$.

Furthermore, $\dot{w}_{x0} < \dot{w}_{nx0}^{c} \Leftrightarrow \dot{w}_{x0} < \dot{w}_{R_1x0}^{c}$ and $\dot{w}_{nx1}^{c} < \dot{w}_{rx0}^{c} \Leftrightarrow \dot{w}_{R_1x1}^{c} < \dot{w}_{R_2x0}^{c}$ if $\dot{P}_{x0}^{e} > \dot{\varepsilon}_1 \Leftrightarrow \dot{P}_{x0}^{e*} > 2\dot{\varepsilon}_1$ therefore the conditions for Proposition 4 are fulfilled. Accordingly for $\dot{w}_1' \in \left(\dot{w}_{x0}, \dot{w}_{R_1x0}^{c}\right) = \left(\dot{w}_{x0}, \dot{w}_{nx0}^{c}\right) = \left(\dot{w}_{x0}^{N}, -\dot{\varepsilon}_1\right)$ and $\dot{w}_2' \in \left(\dot{w}_{R_1x1}^{c}, \dot{w}_{R_2x0}^{c}\right) = \left(\dot{w}_{nx1}^{c}, \dot{w}_{rx0}^{c}\right) = \left(\dot{\varepsilon}_1, \dot{P}_{x0}^{e}\right)$, equations (69) and (70) follow immediately from equation (55).

B.23 Proof of Proposition 8

In this case $K_x = 2$ and $\mathcal{R}_x = \{n, r\}$ (see definition of this regime), therefore the unknown parameters to be identified are ϱ_x^n and ϱ_x^r.

(a) *Proof of* (71): In this case $\mathcal{R}_x \subset \mathcal{R}$ therefore $\mathcal{W}_x = \mathcal{W}_x^N$, thus $\dot{w}_{x0} = \dot{w}_{x0}^{N}$. Given that f_ε is degenerate it follows that $\dot{w}_{x0} = \dot{w}_{x0}^{N*}$.

Furthermore, by assumption, if $R = n$ then $\dot{w}_{nx1}^{c*} = \dot{w}_{nx0}^{c*} = 0$, and if $R = r$ then

$\dot{w}^{c*}_{rx0} = \dot{P}^{e*}_{x0}$ and $\dot{w}^{c*}_{rx1} = \dot{P}^{e*}_{x1}$. Given that f_ε is degenerate it follows that $\dot{w}^{c}_{nx0} = \dot{w}^{c}_{nx1} = 0$, $\dot{w}^{c}_{rx0} = \dot{P}^{e*}_{x0}$and $\dot{w}^{c}_{rx1} = \dot{P}^{e*}_{x1}$.

From the definition of the LOW regime we have $\dot{w}^{N*}_{x0} < \dot{P}^{e*}_{x0} < 0 < \dot{P}^{e*}_{x1}$.

It follows that $\dot{w}^{c}_{nx1} < \dot{w}^{c}_{rx1}$, therefore we can write $\mathcal{R}_x = \{R_1, R_2\}$ where $R_1 = n$ and $R_2 = r$.

Furthermore, $\dot{w}_{x0} < \dot{w}^{c}_{nx0} \Leftrightarrow \dot{w}_{x0} < \dot{w}^{c}_{R_1x0}$ but $\dot{w}^{c}_{nx1} \nless \dot{w}^{c}_{rx0} \Leftrightarrow \dot{w}^{c}_{R_1x1} \nless \dot{w}^{c}_{R_2x0}$ therefore the conditions for Proposition 4 are not fulfilled.

However, for $\dot{w}_1' \in \left(\dot{w}^{N*}_{x0}, \dot{P}^{e*}_{x0}\right)$ holds that $\dot{w}_1' < \dot{w}^{c}_{nx0} \Leftrightarrow \dot{w}_1' < \dot{w}^{c}_{R_1x0}$ and $\dot{w}_1' < \dot{w}^{c}_{rx0} \Leftrightarrow \dot{w}_1' < \dot{w}^{c}_{R_2x0}$ therefore from Lemma 12 and Corollary 2 follows equation (71).

Proof of (72)*:* By assumption, if $R = n$ then $\dot{w}^{c*} = 0$, i.e., fixed, whereas if $R = r$ then $\dot{w}^{c*} = \dot{P}^{e*}$ which is variable, therefore $\overline{\mathcal{R}}_x = \{n\}$. Furthermore, if $R = n$ then $\dot{w}^{c*}_{Rx1} = \dot{w}^{c*}_{nx1} = 0$. Substituting into (48) in Proposition 2 gives (72).

(b) *Proof of* (73)*:* In this case $\mathcal{R}_x \subset \mathcal{R}$ therefore $\mathcal{W}_x = \mathcal{W}^N_x$, thus $\dot{w}_{x0} = \dot{w}^N_{x0}$. Given that f_ε is degenerate it follows that $\dot{w}_{x0} = \dot{w}^{N*}_{x0} - \dot{\varepsilon}_1$.

Furthermore that $\dot{w}^{c}_{nx0} = \dot{w}^{c*}_{nx0} - \dot{\varepsilon}_1 = -\dot{\varepsilon}_1$, $\dot{w}^{c}_{nx1} = \dot{w}^{c*}_{nx1} + \dot{\varepsilon}_1 = \dot{\varepsilon}_1$, $\dot{w}^{c}_{rx0} = \dot{w}^{c*}_{rx0} - \dot{\varepsilon}_1 = \dot{P}^{e*}_{x0} - \dot{\varepsilon}_1$ and $\dot{w}^{c}_{rx1} = \dot{w}^{c*}_{rx1} + \dot{\varepsilon}_1 = \dot{P}^{e*}_{x1} + \dot{\varepsilon}_1$.

Given that $\dot{w}^{N*}_{x0} < \dot{P}^{e*}_{x0} < 0 < \dot{P}^{e*}_{x1}$ (from the definition of the LOW regime) it follows that $\dot{\varepsilon}_1 < \dot{P}^{e*}_{x1} + \dot{\varepsilon}_1 \Leftrightarrow \dot{w}^{c}_{nx1} < \dot{w}^{c}_{rx1}$, therefore we can write $\mathcal{R}_x = \{R_1, R_2\}$ where $R_1 = n$ and $R_2 = r$.

Furthermore, $\dot{w}_{x0} < \dot{w}^{c}_{nx0} \Leftrightarrow \dot{w}_{x0} < \dot{w}^{c}_{R_1x0}$ but $\dot{w}^{c}_{nx1} \nless \dot{w}^{c}_{rx0} \Leftrightarrow \dot{w}^{c}_{R_1x1} \nless \dot{w}^{c}_{R_2x0}$ therefore the conditions for Proposition 4 are not fulfilled.

However, for $\dot{w}_1' \in \left(\dot{w}^{N}_{x0}, \dot{P}^{e}_{x0}\right)$ holds that $\dot{w}_1' < \dot{w}^{c}_{nx0} \Leftrightarrow \dot{w}_1' < \dot{w}^{c}_{R_1x0}$ and $\dot{w}_1' < \dot{w}^{c}_{rx0} \Leftrightarrow \dot{w}_1' < \dot{w}^{c}_{R_2x0}$ therefore from Lemma 12 and Corollary 2 follows equation (73).