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Strategic voting when participation is costly

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Abstract

We study a general multiparty model of plurality rule elections with costly participation, and prove that strategic voting –that is, situations in which some voters abandon their most preferred alternative and vote strategically for the serious contender they dislike less– may emerge in equilibrium; just like when participation is costless/compulsory (Palfrey 1988). This qualifies opposite claims made in more confined setups (e.g. Arzumanyan and Polborn 2017), and establishes that the Duverger’s psychological effect is present in a much larger set of cases than currently believed.

JEL classification codes: D71, D72.

Keywords: Multiparty elections; plurality rule; costly voting; Duverger’s law; strategic voting.

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1 Introduction

Duverger's (1951) law postulates that in plurality rule elections it is highly unlikely that all voters will vote sincerely for their top-ranked alternative. Indeed, when one is endowed with a unique vote, one should want to make the most of it. Since one's vote is relevant in the determination of the outcome only if this vote breaks or generates a tie for the first place, there is often a question between voting for the alternative one likes most and voting for the serious contender one dislikes less. This tension will lead supporters of less popular alternatives to behave "strategically." That is, to abandon their top choices and back, among the serious contenders (i.e. among the alternatives that have substantial chances of winning the election), the one that gives them the highest utility. Duverger concluded that this line of reasoning –usually referred to as Duverger's "psychological" effect or factor (Cox 1997)– will lead to *strategic voting equilibria* in which a small set of serious contenders –typically, two– is supported sincerely by individuals that like them more than any other candidate, and strategically by individuals whose top choice does not enjoy serious election prospects.

Formal analysis of strategic voting under plurality rule has confirmed the existence of strategic voting equilibria only when voting is costless or compulsory (Riker 1982; Palfrey 1988; Myerson and Weber 1993; Cox 1997; Fey 1997). That is, only when full participation is guaranteed. In many relevant real life contexts though (e.g. U.S. presidential elections) participation is voluntary and costly for the voters. Hence, it is of utmost importance to investigate whether the described psychological effect still exists in these settings. So far, the literature has mainly focused on studying costly voting in the framework of two-party elections (see, for example, Palfrey and Rosenthal 1983, 1985; Ledyard 1984; Levine and Palfrey 2007; Krasa and Polborn 2009; Goeree and Grosser 2012; Herrera, Morelli and Palfrey 2014; Krishna and Morgan 2015; Tyson

2016) and, surprizingly, very little is known regarding the effect of introducing voting costs in multiparty settings on the shaping of strategic voting incentives: Do strategic voting equilibria exist even when participation is voluntary and costly? That is, when turnout is partial, are there voters that switch, and abandon their top-ranked alternatives and vote for the serious contender they dislike less?

To our knowledge, the only paper that tries to give a first answer to these questions is Arzumanyan and Polborn (2017). This paper studies multiparty elections under plurality rule with costly participation, and finds that strategic voting cannot take place in equilibrium. In particular, it demonstrates that in equilibrium all voted parties tie (in expectation) and all voters vote sincerely (i.e. all individuals that decide not to abstain, vote for their top-ranked alternative) suggesting that Duverger’s psychological effect is not present when voting is costly. To arrive to this conclusion, Arzumanyan and Polborn (2017) examine a setup with three alternatives and general ordinal preferences (i.e. individuals were allowed to have any strict preference ordering over the three alternatives) but a very special structure of cardinal preferences and voting costs (i.e. all individuals enjoy 1, $\lambda \in (0, 1)$ and 0 units of utility by the election of their top-ranked, their second-best and their bottom-ranked alternative respectively; and voting costs are homogeneous). It is true that when voting is costless or compulsory, cardinal utilities are not really central in the shaping of equilibrium behavior, since the probabilities of ties and their ratios are by far the most relevant determinant factor of voters’ behavior (e.g. Palfrey 1988). When voting is voluntary and costly, though, cardinal utilities and voting costs are as much relevant as the probabilities of ties in determining who turns out to vote and, subsequently, whether strategic voting takes place or not. Hence, it is imperative that we try to study multiparty elections with costly voting, in a more general framework: i.e. allowing for a larger variety in voters’ cardinal characteristics (namely, utility levels and participation costs), in order to get new and robust insights regarding the persistence of strategic voting.

In this paper we undertake this task and employ a rather general model regarding cardinal preferences and voting costs. We consider any arbitrary finite class of voters’ cardinal preference types and variable voting costs –in the tradition of Palfrey and Rosenthal (1985)– and we prove that strategic voting equilibria indeed exist in multiparty elections, even when voting is voluntary and costly. That is, we show that Duverger’s psychological effect survives in additional settings of applied interest by establishing that, when voting costs are heterogeneous and the space of cardinal preferences is rich, we always have equilibria in large elections such that the top-ranked alternative of a significant share of the non-abstaining voters does not coincide with any of the alternatives that are expected to receive a positive vote-share. To this end, we prove existence of Duvergerian equilibria, that is, two-candidate equilibria in which a voter either abstains or votes –sincerely or strategically– for one of the two candidates that are expected to be voted by the rest of the voters.

But why does strategic voting arise in this general framework and not when voting costs and utility levels are homogeneous? When, for example, there are three alternatives –say A , B and C – but only two of them are expected to receive a positive vote-share –say A and B – and voting costs are homogeneous, then, in large elections, only voters who have the highest stakes may turn out to vote (that is, voters whose utility difference between the two alternatives is at least as high as that of any other voter). Moreover, if one assumes that all voters with ordinal preferences $A \succ C \succ B$ and $B \succ C \succ A$ care necessarily more about the election’s outcome, than every voter with ordinal preferences $C \succ A \succ B$ and $C \succ B \succ A$ (as do Arzumanyan and Polborn 2017), then one, essentially, rules out that voters whose top-ranked alternative is C will turn out to vote in large elections. On the contrary, when one allows for richer spaces of cardinal preferences and/or heterogeneous participation costs, then this arguably knife-edge reasoning breaks down and, in large elections, a substantial share of the voting population may vote strategically for the serious contender they dislike less.

In what follows we first present the model (section 2) and then we proceed with the formal analysis (section 3).

2 The model

Let us assume that a society, $K = \{1, 2, \dots, k\}$, that is composed of $k \in \mathbb{N}^0$ individuals, has to make a policy choice from the set $M = \{1, 2, \dots, m\}$ with $m > 2$. The preferences of each individual are given by a vector of real numbers $v_i = (v_i^1, v_i^2, \dots, v_i^m) \in V \subset [0, 1]^m$ which is interpreted in the following way: The utility that individual $i \in K$ derives from the implementation of policy $h \in M$ is v_i^h . The type-space, V , is a finite subset of $[0, 1]^m$ with following properties: a) ordinal preferences are strict (i.e. for every $v = (v^1, v^2, \dots, v^m) \in V$ we have $v^h \neq v^q$ for every $h \neq q$), and b) ordinal preferences are nonidentical (i.e. there exist $v, \tilde{v} \in V$ and $h, q \in M$ such that $v^h > v^q$ and $\tilde{v}^q > \tilde{v}^h$).

Each voter's preference vector is not publicly observed and is considered to be the result of i.i.d. draws from a distribution F with support V , and a strictly positive probability mass function $f : V \rightarrow (0, 1)$. That is, F is public information, while the specific parameter draw for a given individual is her private information. Following, Myerson (2000), Bouton and Castanheira (2012), Herrera, Morelli and Palfrey (2014) and many others, we assume that k is a random draw from a Poisson distribution with parameter $n > 0$:

$$k \sim \frac{e^{-n}(n)^k}{k!}.$$

Each individual, $i \in K$, is also characterized by a cost, $c_i \geq 0$, that she has to pay in case she decides to vote, which is also her private information. These costs

are results of i.i.d. draws from a differentiable distribution function, $g : [0, c] \rightarrow [0, 1]$, with strictly positive density on $[0, c]$, for some $c \geq 1$.¹ An individual, $i \in K$, decides $s_i \in S = \{a, 1, 2, \dots, m\}$: she decides if she wishes to abstain ($s_i = a$) or to vote for a specific policy ($s_i \in M$). If a voter, i , decides to vote for a policy, she incurs the cost c_i but avoids it if she decides to abstain. The voting system is the plurality rule: the alternative that gets more votes than any other alternative wins the election (ties are broken with equiprobable draws). Hence, the utility of an individual, $i \in K$, in action profile $s = (s_i, s_{-i}) \in \{a, 1, 2, \dots, m\}^k$, is given by:

$$u_i(s_i, s_{-i} : v_i, c_i) = \frac{\sum_{j \in M^s} v_i^j}{\#M^s} - c_i \mathbf{1}_{\{s_i \neq a\}},$$

where $M^s \subseteq M$ is the set of plurality winners in strategy profile s with cardinality $\#M^s \in M$, and $\mathbf{1}_{\{s_i \neq a\}} = 1$ if $s_i \neq a$, and $\mathbf{1}_{\{s_i \neq a\}} = 0$ otherwise.

Since there is incomplete information about certain aspects of the game and instrumental decisions are taken simultaneously by all the players, the most suitable equilibrium concept is Bayesian Nash Equilibrium (BNE). In such games the focus is on ex-ante symmetric behavior. That is, we try to find a function $\sigma : V \times [0, c] \rightarrow S$ that, along with its ensuing beliefs, constitutes a BNE.

3 Duvergerian equilibria

In a Duvergerian equilibrium there are exactly two policies that are expected to receive positive vote-shares, and a substantial share of voters engage in strategic voting. Our

¹This restriction only simplifies the argument regarding the existence of an equilibrium, and does not have any other substantial implication. Once the equilibrium analysis for large societies is presented, it will be evident that all main results go through for any possible $c > 0$. This is also noted in Herrera, Morelli and Palfrey (2014) which we follow closely.

main contribution is to argue that such equilibria exist in multiparty elections, even when voting is costly. The structure of our argument is as follows: a) first we show that, in a restriction of our game to two policy alternatives and abstention, an equilibrium with partial participation, always exists (Subsection 3.1); and b) then we prove that the identified equilibrium of the restricted game remains an equilibrium of the unrestricted version of our game, for sufficiently large societies (Subsection 3.2).

3.1 Restricted game

By the fact that individual preferences are nonidentical, we can assume without loss of generality that there exist $v, \tilde{v} \in V$ such that $v^1 > v^2$ and $\tilde{v}^2 > \tilde{v}^1$. We will first study the restriction of the game to $\{1, 2\} \subset M$. That is, the restricted version of the game in which players are allowed to choose any action from $\{a, 1, 2\}$. In this limited version of the game, the whole vector v_i that characterizes the preferences of individual i on M contains a lot of redundant information: the only important part for the choices of individual i is the pair (v_i^1, v_i^2) . Since F is common information, every individual i has well-defined beliefs regarding the value of $v_j^1 - v_j^2$, for every fellow citizen j . We formally assume that i believes that the relevant difference of utilities for every other individual j , $y_j = v_j^1 - v_j^2$, is a random draw from $F_{1,2} : [-1, 1] \rightarrow [0, 1]$ with probability mass function, $f_{1,2}$ that takes positive values in $Y_{1,2} \subset [-1, 1]$. Notice that, given that preferences are strict, there exists $\varepsilon > 0$ such that $F_{1,2}(\varepsilon) - F_{1,2}(-\varepsilon) = 0$ (alternatively, there exists $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon] \cap Y_{1,2} = \emptyset$). Whenever ε appears in the subsequent analysis, it is assumed to have this property.

If $\hat{\sigma}_n$ is a threshold BNE of this restricted game for a given $n > 0$ —that is, if for every $y \in [-1, 1]$ there exists $w_n(y)$ such that when $c_i > w_n(v_i^1 - v_i^2)$, then i prefers to abstain and otherwise votes for the alternative, 1 or 2, that she prefers— we have

that a voter, i , with utility difference $y_i > 0$ is expected to vote for 1 with probability $g(w_n(y_i))$ and to abstain with the remaining probability; and that a voter, i , with utility difference $y_i < 0$ is expected to vote for 2 with probability $g(w_n(y_i))$ and to abstain with the remaining probability. Therefore, each voter believes that a random fellow citizen will vote for 1 with probability $p_1^n = \sum_{y \in Y_{1,2} \cap [0,1]} g(w_n(y)) f_{1,2}(y)$ and will vote for 2 with probability $p_2^n = \sum_{y \in Y_{1,2} \cap [-1,0]} g(w_n(y)) f_{1,2}(y)$. That is, i considers that the number of fellow citizens that will vote for 1 is a draw from a Poisson distribution with parameter $n \times p_1^n$ and that the number of fellow citizens that will vote for 2 is a draw from a Poisson distribution with parameter $n \times p_2^n$.

Hence, for an individual with $y_i \in [-1, 1]$ the expected utility difference from voting for her preferred alternative $h(y_i) = \begin{cases} 1 & \text{if } y_i \geq 0 \\ 2 & \text{if } y_i < 0 \end{cases}$ compared to abstaining, in the threshold equilibrium $\hat{\sigma}_n$ of our restricted game, is:

$$\begin{aligned} P_{h(y_i)}(y_i, c_i, w_n) &= \sum_{k \in \mathbb{N}^0} \frac{e^{-n \times p_{h(y_i)}^n} (n \times p_{h(y_i)}^n)^k}{k!} \frac{e^{-n \times p_{\hat{h}(y_i)}^n} (n \times p_{\hat{h}(y_i)}^n)^k}{k!} \frac{|y_i|}{2} + \\ &+ \sum_{k \in \mathbb{N}^0} \frac{e^{-n \times p_{h(y_i)}^n} (n \times p_{h(y_i)}^n)^k}{k!} \frac{e^{-n \times p_{\hat{h}(y_i)}^n} (n \times p_{\hat{h}(y_i)}^n)^{k+1}}{(k+1)!} \frac{|y_i|}{2} - c_i = \\ &= \frac{|y_i|}{2} \sum_{k \in \mathbb{N}^0} \frac{e^{-n \times p_{h(y_i)}^n} (n \times p_{h(y_i)}^n)^k}{k!} \frac{e^{-n \times p_{\hat{h}(y_i)}^n} (n \times p_{\hat{h}(y_i)}^n)^k}{k!} \left(1 + \frac{n \times p_{\hat{h}(y_i)}^n}{k+1}\right) - c_i \end{aligned}$$

$$\text{where } \hat{h}(y_i) = \begin{cases} 1 & \text{if } h(y_i) = 2 \\ 2 & \text{if } h(y_i) = 1 \end{cases}.$$

Since, $c > 1$, a threshold equilibrium w_n is such that for every $y \in [-1, 1]$ we have $P_{h(y)}(y, w_n(y), w_n) = 0$. Moreover, for $y \in \{-1, 1\}$ we have:

$$w_n(y) = \frac{1}{2} \sum_{k \in \mathbb{N}^0} \frac{e^{-n \times p_{h(y)}^n} (n \times p_{h(y)}^n)^k}{k!} \frac{e^{-n \times p_{\hat{h}(y)}^n} (n \times p_{\hat{h}(y)}^n)^k}{k!} \left(1 + \frac{n \times p_{\hat{h}(y)}^n}{k+1}\right)$$

which implies that for every $y_i \geq 0$, $w_n(y_i) = w_n(1)y_i$ and for every $y_i < 0$, $w_n(y_i) = -w_n(-1)y_i$, and thus:

$$p_1^n = \sum_{y \in Y_{1,2} \cap [0,1]} g(w_n(1)y) f_{1,2}(y) \text{ and } p_2^n = \sum_{y \in Y_{1,2} \cap [-1,0]} g(-w_n(-1)y) f_{1,2}(y).$$

In other words, the issue of existence of a threshold equilibrium is reduced to the issue of existence of a pair $(w_n(1), w_n(-1)) \in [0, c]^2$ such that, for each $y \in \{-1, 1\}$:

$$\xi^{h(y)}(w_n(1), w_n(-1)) = \frac{1}{2} \sum_{k \in \mathbb{N}^0} \frac{e^{-n \times p_{h(y)}^n} (n \times p_{h(y)}^n)^k}{k!} \frac{e^{-n \times p_{h(y)}^n} (n \times p_{h(y)}^n)^k}{k!} \left(1 + \frac{n \times p_{h(y)}^n}{k+1}\right) = w_n(y).$$

Notice that $\Xi(x, z) = \{\xi^1(x, z), \xi^2(x, z)\}$, is a continuous mapping from $[0, c]^2$ into itself so it is guaranteed to have a fixed point by the Brouwer fixed point theorem for every $n > 0$. This observation allows us to state our first lemma.

Lemma 1 *The restriction of the game to $\{1, 2\} \subset M$ admits a threshold BNE for every $n > 0$.*

3.2 Unrestricted game

We now seek to establish that such a BNE of the reduced game is also a BNE of the unrestricted game when the population is large. To this end, it suffices to establish that there exists \hat{n} such that for every $n > \hat{n}$, an individual i of any type prefers to behave according to $\hat{\sigma}_n$ when the other players are expected to behave according to $\hat{\sigma}_n$, even if i can now vote for any of the m alternatives.

To achieve this we first need to show that in a threshold BNE of our restricted game, $\hat{\sigma}_n$, as the size of the society grows, the expected number of votes for each of the two alternatives becomes arbitrarily large.

Lemma 2 *Every threshold BNE of the restricted game is such that: $\lim_{n \rightarrow +\infty} n \times p_1^n = +\infty$ and $\lim_{n \rightarrow +\infty} n \times p_2^n = +\infty$.*

Proof. As a first step it is helpful to observe that $\lim_{n \rightarrow +\infty} w_n(1) = \lim_{n \rightarrow +\infty} w_n(-1) = 0$. Indeed, if there exists a sequence $\{n_t\}_{t=1}^{\infty}$ and $C \in \mathbb{R}_{++}$ such that $w_{n_t}(1) > C$ for every $t \in \mathbb{N}_+$, then $\lim_{t \rightarrow +\infty} \xi^1(w_{n_t}(1), w_{n_t}(-1)) = 0 < C < w_{n_t}(1)$ leading to a contradiction with the equilibrium condition $\xi^1(w_{n_t}(1), w_{n_t}(-1)) = w_{n_t}(1)$.

It is fairly easy to see that if there exists a sequence $\{n_t\}_{t=1}^{\infty}$ and $C, C' \in \mathbb{R}_{++}$ such that $n_t \times p_1^{n_t} < C$ and $n_t \times p_2^{n_t} < C'$ for every $t \in \mathbb{N}_+$, then there exists $C'' \in \mathbb{R}_{++}$ such that:

$$\sum_{k \in \mathbb{N}^0} \frac{e^{-n_t \times p_1^{n_t}} (n_t \times p_1^{n_t})^k}{k!} \frac{e^{-n_t \times p_2^{n_t}} (n_t \times p_2^{n_t})^k}{k!} > C''$$

for every $t \in \mathbb{N}_+$. Hence, by the equilibrium condition $\xi^1(w_{n_t}(1), w_{n_t}(-1)) = w_{n_t}(1)$, we also have $w_{n_t}(1) > \frac{1}{2}C''$ for every $t \in \mathbb{N}_+$. The last inequality implies that $\lim_{t \rightarrow +\infty} n_t \times p_1^{n_t} = +\infty$ which clearly contradicts the assumption that $n_t \times p_1^{n_t} < C$ for every $t \in \mathbb{N}_+$.

If there exists a sequence $\{n_t\}_{t=1}^{\infty}$ and $C \in \mathbb{R}_{++}$ such that $n_t \times p_1^{n_t} < C$ for every $t \in \mathbb{N}_+$ but there is no $C' \in \mathbb{R}_{++}$ such that $n_t \times p_2^{n_t} < C'$ for every $t \in \mathbb{N}_+$, then there exists a subsequence $\{\hat{n}_t\}_{t=1}^{\infty}$ such that $n_t \times p_1^{n_t} < C$ for every $t \in \mathbb{N}_+$ and $n_t \times p_2^{n_t}$ is monotonically increasing and diverging in $t \in \mathbb{N}_+$. Hence, for t sufficiently large, it must be the case that $p_1^{\hat{n}_t} < p_2^{\hat{n}_t}$ and, by $\xi^1(w_{n_t}(1), w_{n_t}(-1)) = w_{n_t}(1)$ and $\xi^2(w_{n_t}(1), w_{n_t}(-1)) = w_{n_t}(-1)$, it must also be the case that $w_{\hat{n}_t}(-1) < w_{\hat{n}_t}(1)$. Therefore:

$$\frac{\hat{n}_t \times p_2^{\hat{n}_t}}{\hat{n}_t \times p_1^{\hat{n}_t}} = \frac{\sum_{y \in Y_{1,2} \cap [-1,0]} g(-w_{\hat{n}_t}(-1)y) f_{1,2}(y)}{\sum_{y \in Y_{1,2} \cap [0,1]} g(w_{\hat{n}_t}(1)y) f_{1,2}(y)} \leq \frac{\nu_2 g(w_{\hat{n}_t}(-1))}{f_{\min}^1 g(w_{\hat{n}_t}(1)\varepsilon)} \leq \frac{\nu_2 g(w_{\hat{n}_t}(1))}{f_{\min}^1 g(w_{\hat{n}_t}(1)\varepsilon)},$$

for t sufficiently large, where ν_2 is the cardinality of $Y_{1,2} \cap [-1, 0]$, f_{\min}^1 is the smallest probability mass that corresponds to an element of $Y_{1,2} \cap [0, 1]$; and $\varepsilon \in \mathbb{R}_{++}$ is such that $[-\varepsilon, \varepsilon] \cap Y_{1,2} = \emptyset$.

Now define a differentiable function $\phi : \mathbb{R}_{++} \rightarrow [0, c]$ such that $\phi(\hat{n}_t) = w_{\hat{n}_t}(1)$ for every $t \in \mathbb{N}_+$. We have that $\lim_{x \rightarrow +\infty} \frac{\nu_2 g(\phi(x))}{f_{\min}^1 g(\phi(x)\varepsilon)} = \lim_{x \rightarrow +\infty} \frac{\nu_2 g'(\phi(x))\phi'(x)}{f_{\min}^1 g'(\phi(x)\varepsilon)\phi'(x)\varepsilon} = \frac{\nu_2 g'(0)}{f_{\min}^1 g'(0)\varepsilon} = \frac{\nu_2}{f_{\min}^1 \varepsilon} \in \mathbb{R}_{++}$, which contradicts our assumption that $\lim_{t \rightarrow +\infty} \frac{\hat{n}_t \times p_2^{\hat{n}_t}}{\hat{n}_t \times p_1^{\hat{n}_t}} = +\infty$.

The above arguments are sufficient to establish $\lim_{n \rightarrow +\infty} n \times p_1^n = +\infty$ and $\lim_{n \rightarrow +\infty} n \times p_2^n = +\infty$.² ■

Consider now an individual, i , who believes that the other voters will behave according to $\hat{\sigma}_n$ in this unrestricted game. For such an individual the expected utility difference from voting for alternative $h \in \{3, 4, \dots, m\}$ compared to abstaining is at most as large as:

$$\sum_{(k,k') \in \{(0,0), (0,1), (1,0), (1,1)\}} \frac{e^{-n \times p_1^n} (n \times p_1^n)^k}{k!} \frac{e^{-n \times p_2^n} (n \times p_2^n)^{k'}}{k'!} - c_i.$$

This is so because when all other voters are expected to vote either for 1 or 2, then one, by voting for a third alternative, can: a) affect the outcome only if each of alternatives 1 and 2 is voted by at most one other voter, and b) gain at most one unit of utility by doing so (because $v_i \in [0, 1]^m$). We observe that:

$$\sum_{(k,k') \in \{(0,0), (0,1), (1,0), (1,1)\}} \frac{e^{-n \times p_1^n} (n \times p_1^n)^k}{k!} \frac{e^{-n \times p_2^n} (n \times p_2^n)^{k'}}{k'!} - c_i \leq 4\tilde{\tau} - c_i$$

²Notice that despite the fact that the expected number of voters diverges, the ratio of the expected number of voters over the expected number of players that decide to abstain converges to zero. This commonly happens in costly voting models unless there is aggregate uncertainty regarding voters' preferences (see Myatt 2012).

where:

$$\tilde{\tau} \in \max T$$

and:

$$T = \left\{ \frac{e^{-n \times p_1^n} (n \times p_1^n)^k}{k!} \frac{e^{-n \times p_2^n} (n \times p_2^n)^{k'}}{k'} \mid (k, k') \in \{(0, 0), (0, 1), (1, 0), (1, 1)\} \right\}.$$

That is, any individual i with $y_i \in Y_{1,2} \cap [0, 1]$, conditional on not abstaining, prefers to vote for 1 rather than any other alternative she likes more than alternative 2 if:

$$\begin{aligned} & \frac{y_i}{2} \sum_{k \in \mathbb{N}^0} \frac{e^{-n \times p_1^n} (n \times p_1^n)^k}{k!} \frac{e^{-n \times p_2^n} (n \times p_2^n)^k}{k!} \left(1 + \frac{n \times p_2^n}{k+1}\right) > 4\tilde{\tau} \Rightarrow \\ \Rightarrow & y_i \frac{\frac{e^{-n \times p_1^n} (n \times p_1^n)^2}{2!} \frac{e^{-n \times p_2^n} (n \times p_2^n)^2}{2!} \left(1 + \frac{n \times p_2^n}{3}\right)}{8\tilde{\tau}} + y_i \frac{\sum_{k \in \mathbb{N}^0 - \{2\}} \frac{e^{-n \times p_1^n} (n \times p_1^n)^k}{k!} \frac{e^{-n \times p_2^n} (n \times p_2^n)^k}{k!} \left(1 + \frac{n \times p_2^n}{k+1}\right)}{8\tilde{\tau}} > 1. \end{aligned}$$

From Lemma 2 we have that $\lim_{n \rightarrow +\infty} \left(1 + \frac{n \times p_2^n}{3}\right) = +\infty$. Moreover, by Lemma 2 and by the fact that $y_i > \varepsilon$ for every $y_i \in Y_{1,2} \cap [0, 1]$, it follows that:

$$\lim_{n \rightarrow +\infty} y_i \frac{\frac{e^{-n \times p_1^n} (n \times p_1^n)^2}{2!} \frac{e^{-n \times p_2^n} (n \times p_2^n)^2}{2!}}{8\tau} = +\infty$$

for every $\tau \in T$ and $y_i \in Y_{1,2} \cap [0, 1]$. This confirms that for n sufficiently large, the threshold equilibrium of the restricted game is an equilibrium even when voters are free to vote among any of the $m > 2$ alternatives. We summarize the above findings in the lemma that follows.

Lemma 3 *There exists \hat{n} such that for every $n > \hat{n}$, the threshold BNE of the restricted game, $\hat{\sigma}_n$, is a BNE of the unrestricted game too.*

Finally, we notice that in this BNE the ratio of the expected number of voters of any pair of preference types (with identical ordinal preference regarding the two voted policies), converges to the ratio of the "stakes" of these voters' types in the given two-policy election, weighted by the share of these types in the overall population. Take for instance two types characterized by y and y' such that $y > y' > 0$. The limit of the ratio of the expected number of voters of these types, when $n \rightarrow +\infty$, is equal to:

$$\lim_{w_n(1) \rightarrow 0} \frac{g(w_n(1)y)f_{1,2}(y)}{g(w_n(1)y')f_{1,2}(y')} = \frac{yf_{1,2}(y)}{y'f_{1,2}(y')} \in \mathbb{R}_{++}.$$

Since this holds also for pairs of types such that voters of the first type like policy 1 more than any other policy, and voters of the second type like some third policy more than policy 1, it follows that the ratio of sincere voters over strategic voters does not diverge when $n \rightarrow +\infty$. This constitutes the last step of our overall argument.

Lemma 4 *In the identified BNE of the unrestricted game the share of strategic voters does not vanish as the society grows arbitrarily large.*

All the above establish that Duvergerian dynamics are not restricted to the case in which voting is costless, and allow us to state a general theorem.

Theorem 1 *When elections are held according to the plurality rule in large societies, Duvergerian equilibria –i.e. two-party equilibria which involve a substantial level of strategic voting– exist both when voting is costless/compulsory, and when voting is voluntary and costly.*

What appears to be the crucial determinant factor for the existence of Duvergerian equilibria –both when voting is costless and when voting is costly– is the strictness of preferences. Indeed, when a voter is indifferent between alternatives h and q and strictly prefers some other alternative d , then even if all other voters are expected to vote either h or q , this voter has incentives to vote d for sufficiently small voting costs. Hence, the existence result is equally general in both cases and subsequently the Duvergerian prediction is proved to be relevant in a quite wide range of frameworks of plurality rule elections.

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