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Working Paper 04-2018

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Implementation by vote-buying mechanisms*

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March 15, 2018

Abstract

A vote-buying mechanism is such that each agent buys a quantity of votes x to cast for an alternative of her choosing, at a cost $c(x)$, and the outcome is determined by the total number of votes cast for each alternative. In the context of binary decisions, we prove that the choice rules that can be implemented by vote-buying mechanisms in large societies are parameterized by a positive parameter ρ, which measures the importance of individual preference intensities on the social choice: The limit with $\rho = 0$ is majority rule, $\rho = 1$ is utilitarianism, and $\rho \longrightarrow \infty$ is the Rawlsian maximin rule. We show that any vote-buying mechanism with limit cost elasticity $\lim_{x \to 0} \frac{c'(x)x}{c(x)} = 1 + 1/\rho$ implements the choice rule defined by ρ. The utilitarian efficiency of quadratic voting (Lalley and Weyl, 2016) follows as a special case.

Keywords: implementation; mechanism design; vote-buying; social welfare; utilitarianism; quadratic voting.

JEL classification: D72, D71, D61.

*We thank Hari Govindan, Tilman Börgers, and participants at the Great Lakes Political Economy Theory conference and seminars at U. Maastricht, Michigan State U. and U. Illinois for comments.
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1 Introduction

Consider a binary collective choice problem: a society must choose one of two alternatives. Which alternative is socially preferable depends on the value system we have in mind. According to Dahl (1989), each citizen's vote must be weighed equally. Specifically, political philosophers such as Locke (1689) and Spitz (1984), argue that society should follow majoritarianism: it should choose the alternative preferred by a majority of voters, disregarding the intensity of individual preferences. In contrast, the maximin principle (Rawls 1971) declares that the socially preferred alternative is the one that maximizes the utility of the individual who is most affected by the social choice. Majoritarianism and Rawlsian maximin are the two extremes of a class composed of a continuum of intermediate value systems, including utilitarianism (Bentham 1789, Stuart Mill 1863). Each value system in this class takes intensity of individual preferences into account to a different degree.

Economic theory has developed mechanisms to make social decisions that follow some of these value systems. In particular, majoritarianism (axiomatized by May 1952) is implemented by a majority voting rule; Rawlsian outcomes can be implemented by vote-trading (Casella, Llorente-Saguer and Palfrey 2012); and utilitarianism by "quadratic voting" (Lalley and Weyl 2016).[1]

To the extent that different societies embrace different value systems, each society needs a mechanism tailored to its own values. We address this need: for each value system in a large axiomatized class of such systems, we propose a mechanism that chooses the socially preferred alternative with a probability converging to one, as the society becomes large. The class of mechanisms that we study are "vote-buying" mechanisms: each agent can express her intensity of preference by acquiring any quantity of votes x for either alternative, at a pre-announced monetary amount $c(x)$ that is evenly redistributed to the

[1] We say that a mechanism implements a value system if its equilibrium outcome is socially preferred according to the given value system.

rest of the players, and the social choice is determined by the total number of votes cast for each alternative.

Besides designing a vote-buying mechanism for every value system in our class, we also prove that vote-buying mechanisms only implement value systems within this class. These results establish a two-way mapping between a simple class of transferable utility mechanisms and an intuitive class of value systems, which range from the majority rule all the way to the Rawlsian optimum and which differ in the weight that they assign to individuals' preference intensities.

To gain an intuition over our results, consider the following formalization. Suppose that each subject i would trade v_i units of real wealth to change the social choice from a random coin toss to A with certainty; that is, the valuation v_i measures how intensely subject i cares that society chooses A and not B (agents who prefer B have a negative valuation). Then, a possible value system for a given $\rho \in \mathbb{R}_{++}$, is to declare that alternative A is socially preferable if $\sum_{i=1}^{n} sgn(v_i)|v_i|^\rho > 0$, and alternative B if $\sum_{i=1}^{n} sgn(v_i)|v_i|^\rho < 0$; where $sgn(v_i)$ is the sign (positive or negative) of the valuation v_i. The class, indexed by $\rho \in \mathbb{R}_{++}$, of all such value systems is characterized by a collection of appealing axioms (Bergson 1936, Roberts 1986, Moulin 1988, Eguia and Xefteris 2018).[2]

The majoritarian principle is the lower limit of this class, $\rho = 0$. Utilitarianism corresponds to parameter $\rho = 1$: it declares alternative A socially preferred if $\sum_{i=1}^{n} v_i > 0$. At the higher limit of the class, the alternative socially preferred given $\rho = \infty$ is the alternative preferred by the agent whose valuation has the highest absolute value. Throughout the class of value systems, the social preference according to a small ρ is highly influenced by the number of agents who support each alternative, and less so by their intensity, while if ρ is large the social preference better reflects the preferences of the individuals whose well-being is greatly affected by the decision.

A mechanism asymptotically implements a given value system if the probability that

[2] The axioms are: anonymity, neutrality, monotonicity, continuity, separability, and scale-invariance.

the mechanism chooses the socially preferred alternative –according to this value system– is arbitrarily close to one in arbitrarily large societies. For each value system in our class, we find a vote-buying mechanism that asymptotically implements it. Further, we characterize the class of social choice correspondences that are asymptotically implementable by vote-buying mechanisms: we show that any vote-buying mechanism with limit cost elasticity $\lim_{x\to 0}\frac{c'(x)x}{c(x)} = 1 + 1/\rho$ asymptotically implements the social choice correspondence that asymptotically follows the value system with intensity parameter ρ; and any correspondence that does not follow any of these value systems is not asymptotically implementable.[3]

We stress that the vote-buying mechanisms that we consider are robust in the sense that at the time she designs the mechanism, the designer does not need to know the particular features of the society, such as the number of individuals, the exact distribution of types from which individual preferences are drawn, or the importance of the choice under consideration. Hence, we interpret the proposed vote-buying mechanisms as *institutions* which implement in large societies the choice rule corresponding to society's value system, regardless of changes in distributional parameters.

Literature Review

Our work builds on Lalley and Weyl (2016) and on the literature on quadratic voting that has developed around it, including limit and heuristic approximations (Goeere and Zhang 2017; Lalley and Weyl 2018), and the special issues 1 and 2 of Volume 172 of the journal *Public Choice,* edited by Weyl and Posner (2017), in their entirety.[4] Like this literature, we propose vote-buying mechanisms to implement social choice correspondences in binary collective choice problems. Unlike it, we look beyond utilitarianism: we let

[3]We bring attention to a limitation of this implementation result: the asymptotic optimality of vote-buying mechanisms hinges on the assumption that agents are risk-neutral. If agents are risk averse, wealthier agents acquire more votes, and the equilibrium outcome fails to respect the axiom of anonymity. We discuss this limitation and a solution in Section 4.

[4]Of particular interest to us are the entries on robustness to collusion (Weyl 2017), agenda-setting (Patty and Penn 2017) and turnout (Kaplov and Kominers 2017).

the mechanism designer embrace any of a large class of value systems axiomatized by Roberts (1980) and Moulin (1988), and we offer a mechanism that asymptotically chooses the alternative that is socially preferred according to the designer's value system.

Our work, like all the literature on vote-buying mechanisms, has deeper roots in classic mechanism design. The VCG mechanism (Vickrey 1961, Clark 1971 and Groves 1973) satisfies utilitarian efficiency, but is not budget-balanced. We want a budget-balanced mechanism. The mechanisms by Arrow (1979) and AGV (D'Aspremont and Gerard-Varet 1979) are budget-balanced and attain utilitarian efficiency by requiring each agent to pay the expected externality of her choices, but to calculate this expected externality, the designer must know population parameters such as the distribution from which individual preferences are drawn. The designer we have in mind does not have this information. Put differently: the AGV mechanism works when it is designed specifically for a particular society with known population parameters at a specific point in time; whereas, we propose mechanisms that work for societies that differ in their population parameters, so that each mechanism is robust as the values of exogenous parameters change across societies in space or time.

Related approaches to gauge intensity of preferences through voting involve majority voting with heterogenous turnout costs, or vote trading in a competitive market for votes. Majority rule with heterogeneous turnout costs asymptotically implements utilitarianism (Krishna and Morgan 2015).[5] Whereas, a competitive equilibrium in a decentralized market for votes is very similar to our special case with parameter $\rho = \infty$: the cost of votes is linear, and the agent who cares most about the decision buys most votes (Dekel, Jackson and Wolinski 2008; Casella, Llorente-Saguer and Palfrey 2012).

While our results generalize Lalley and Weyl's (2016) finding that quadratic voting asymptotically attains utilitarian efficiency, the two models are not nested: to obtain simpler and shorter proofs, we make assumptions on the payoff function that are sub-

[5] See as well Krishna and Morgan (2011).

stantially similar, but technically distinct. A greater conceptual difference between Lalley and Weyl (2016)'s approach and ours is that they study the properties of a particular mechanism; whereas, our theory is an exercise in Bayesian implementation (Jackson 1991): for any desired social choice correspondence, we seek a mechanism such that in any equilibrium, in any society, the outcome coincides with the desired social choice, for any realization of preferences.[6]

2 The Formal Framework

Summary. A set of agents must make a binary social choice. The decision is made via a vote-buying mechanism: agents purchase votes, and the alternative with the most votes is chosen. We characterize the set of social choice correspondences that are asymptotically implementable by these vote buying mechanisms.

Sequence of societies. Let N^n denote a society with $n \in \mathbb{N}\backslash\{1\}$ agents. We consider a sequence of societies $\{N^n\}_{n=2}^{\infty}$, where, for each $n \in \mathbb{N}\backslash\{1\}$, $N^{n+1} \equiv N^n \cup \{n+1\}$. For each $n \in \mathbb{N}\backslash\{1\}$, and for any arbitrary variable z, let $z_{N^n} \equiv (z_1, ..., z_n)$. We will establish results for sufficiently large societies.

Social choice set. For each $n \in \mathbb{N}\backslash\{1\}$, society N^n must make a binary choice over $\{A, B\}$. Let the social decision $d^n \in \{A, B\}$ denote the alternative chosen by society N^n.

Wealth. Each agent $i \in \mathbb{N}\backslash\{1\}$ is endowed with initial wealth $w_i^I \in \mathbb{R}_+$. Let $w_i \in \mathbb{R}$ denote the final wealth of agent i in nominal terms, after the social decision is made. Let

[6] Lalley and Weyl (2016) provide a more extensive discourse of quadratic voting, its precedents and related literature, its heuristic intuition, and potential challenges to its roll-out in real world applications; which broadly applies to all vote-buying mechanisms. We refer the interested reader to their insightful discussion, which we do not replicate here.

$p \in \mathbb{R}_{++}$ be a price parameter, an exogenously given characteristic of the society. Let $\frac{w_i^I}{p}$ and $\frac{w_i}{p}$ denote the real initial wealth and real final wealth of agent i.

Assumption 1. *Agents face no budget constraints (final wealth can be negative).*

Equivalently, we could assume that the initial wealth of each agent is sufficiently large relative to the importance of the social choice that no undominated expense related to the social choice would hit the budget constraint.[7]

Outcomes. The set of outcomes in society N^n is $\{A, B\} \times \mathbb{R}^n$, where the first component is the social decision d^n, and the second is the vector $\frac{w_{N^n}}{p}$ of real final wealth. Let $\mathcal{M}(\{A, B\} \times \mathbb{R}^n)$ denote the set of probability measures over the set of outcomes, allowing us to consider stochastic outcomes.

Individual preferences. For each $n \in \mathbb{N} \backslash \{1\}$, each agent $i \in N^n$ has a (complete, transitive) preference order $\succsim_i$ over $\mathcal{M}(\{A, B\} \times \mathbb{R}^n)$. Let $\sim_i$ be the associated indifference relation and note that $\succsim_{N^n} \equiv (\succsim_1, ..., \succsim_n)$ denotes the preference profile.

We assume that for each $i \in N^n$, the preference order $\succsim_i$ is continuous and it satisfies independence over decomposition of lotteries, so that it can be represented by a continuous utility function in expected utility form (von Neumann and Morgernstern 1944). Further, we assume that each $i \in N^n$ cares only about the social decision $d^n \in \{A, B\}$, and about her final real wealth $\frac{w_i}{p}$, (and not about the wealth of other agents), that $\succsim_i$ is separable (Debreu 1960) and strictly monotonic on real final wealth, and that agent i is risk neutral with respect to final real wealth. For convenience, we provide formal statements of these standard assumptions in the Appendix (Assumptions 2-6).

Together, these assumptions on preferences imply that $\succsim_i$ is representable by an additively separable, quasilinear expected utility function (Fishburn 1970), in which the

[7] This assumption is not restrictive, because –as we will show– if the society is large, the amount voters spend to purchase votes converges to zero, so for any given positive budget constraint, in a sufficiently large society, the constraint would not bind.

first term of the summation is the expected utility from the social decision, and the second term is the expected final wealth.

Distribution of attitudes toward the alternatives. Let $[-1,1]$ denote the set of possible attitudes toward choice A, from least favorable (-1) to most favorable (+1). Let $\mathcal{F}$ be the set of all cumulative distributions over $[-1,1]$ that are continuously differentiable, have strictly positive density over the domain, and no mass at any point. Let $F \in \mathcal{F}$ denote one such arbitrary distribution, and let f denote its density. Let $\mathcal{F}^* \equiv \{F \in \mathcal{F} : \; F(x) = 1 - F(1-x) \text{ for any } x \in [-1,1]\} \subset \mathcal{F}$ denote the set of neutral distributions. F is neutral if the strategic environment is identical up to relabeling alternatives.

Let $\bar{\theta}$ be a random variable that follows distribution F. Assume that for any $n \in \mathbb{N}\backslash\{1\}$, for each $i \in N^n$, attitude θ_i is an independent draw of $\bar{\theta}$. Assume that for each $n \in \mathbb{N}\backslash\{1\}$ and for each $i \in N^n$, θ_i is privately observed.

Valuation of the Social Choice. Let $\gamma \in \mathbb{R}_{++}$ be a parameter that represents the importance to society of the social choice under consideration. This is a society-wide parameter, observed by all agents.

Assumption 7. For any $n \in \mathbb{N}\backslash\{1\}$, for any $i \in N^n$ and for any $w_i \in \mathbb{R}$, $\left(A, \frac{w_i}{p} - \gamma\theta_i\right) \sim_i \left(B, \frac{w_i}{p} + \gamma\theta_i\right)$.

That is, each agent is indifferent between an outcome with the most preferred social choice and a real wealth loss of $\gamma\theta_i$ and an outcome with the least preferred social choice and a real wealth gain of $\gamma\theta_i$. Put differently, the social choice is worth $2\gamma\theta_i$ units of real wealth to agent i. We say $\gamma\theta_i$ is the "valuation" of alternative A for agent i, and we refer to $\gamma\theta$ as a "valuation profile." The valuation of alternative B is then $-\gamma\theta_i$.

Actions. For any $n \in \mathbb{N}\backslash\{1\}$, each agent $i \in N^n$ chooses an action $a_i \in \mathbb{R}$. Strictly positive actions are interpreted as in favor of A, and strictly negative ones, as against A (or, equivalently, in favor of B).

Vote-buying mechanisms. A vote-buying mechanism is defined by a cost function $c : \mathbb{R} \longrightarrow \mathbb{R}_+$, such that for any $n \in \mathbb{N}\backslash\{1\}$, and for any $x \in \mathbb{R}$, any agent $i \in N^n$ who chooses action $a_i = x$ pays a cost $c(x)$. All payments are redistributed equally among all other agents, so given a vector of actions $a_{N^n} \in \mathbb{R}^n$, each agent $i \in N^n$ obtains a net nominal wealth transfer $-c(a_i) + \sum\limits_{j \in N^n \backslash \{i\}} \frac{c(a_j)}{n-1}$.

Let C denote a class of admissible mechanisms. A perfect execution of a mechanism $c \in C$ would entail society choosing $d^n = A$ if $\sum\limits_{j \in N^n} a_j > 0$ and $d^n = B$ if $\sum\limits_{j \in N^n} a_j < 0$. However, we assume that the execution of any mechanism entails some element of uncertainty, so that the mapping from actions to outcomes is stochastic: while the probability that $d^n = A$ is increasing in $\sum\limits_{j \in N^n} a_j$, it is not a step function.

Formally, we assume that there exists an outcome function $G : \mathbb{R} \longrightarrow [0, 1]$ such that for any $n \in \mathbb{N}\backslash\{1\}$ and any $a_{N^n} \in \mathbb{R}^n$, the probability that $d^n = A$ is $G\left(\sum\limits_{j \in N^n} a_j\right)$. Let $\mathcal{G}$ be the class of strictly increasing, twice continuously differentiable functions from $\mathbb{R} \longrightarrow [0, 1]$ such that for any $\tilde{G} \in \mathcal{G}$ with density $\tilde{g}$ and derivative of the density $\tilde{g}'$:

i) $\tilde{G}(x) - \frac{1}{2} = \frac{1}{2} - \tilde{G}(-x)$ for any $x \in \mathbb{R}_{++}$;

ii) $\lim\limits_{x \longrightarrow -\infty} \tilde{G}(x) = 0$ and $\lim\limits_{x \longrightarrow -\infty} \tilde{g}(x) = 0$;

iii) $\exists \hat{\varepsilon} \in \mathbb{R}_{++}$ such that $\lim_{x \to \infty} \frac{\tilde{g}'(x+\varepsilon)}{\tilde{g}(x)} \in \mathbb{R}\ \forall \varepsilon \in (-\hat{\varepsilon}, \hat{\varepsilon})$.

Condition (i) is neutrality. Condition (ii) is a responsiveness condition: if the vote margin is sufficiently large, the outcome is the one with the vote advantage with probability arbitrarily close to one. Condition iii) requires the tails of the density not to drop to zero too steeply. The set $\mathcal{G}$ contains, among others, all Student-t distributions.

We assume that $G \in \mathcal{G}$, but G is not known to the mechanism designer, and hence we will propose mechanisms whose results are robust for any $G \in \mathcal{G}$, including those that are arbitrarily close to a step function with discontinuity at zero, as in Figure 1.

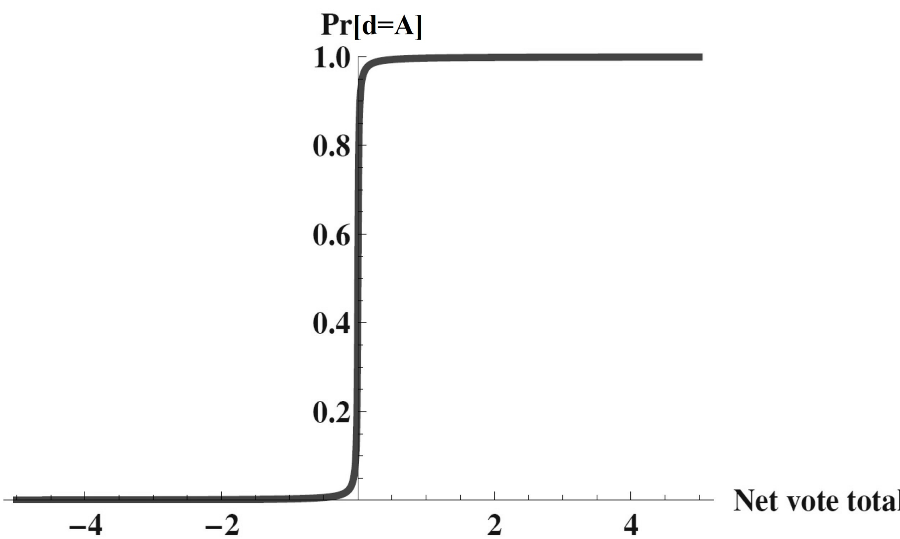

Figure 1: An outcome function G.

Strategies. Each agent i in society N^n with size $n \in \mathbb{N}\backslash\{1\}$, with price index $p \in \mathbb{R}_{++}$, with wealth distribution $w^I_{N^n} \in \mathbb{R}^n_+$, facing a social decision of importance $\gamma \in \mathbb{R}_{++}$ to be decided according to mechanism $c \in C$ under uncertainty $G \in \mathcal{G}$, and taking into account that the ex-ante distribution of attitudes toward the decision is given by distribution $F \in \mathcal{F}$, chooses an action $a_i \in \mathbb{R}$ as a function of the realization $\theta_i \in [-1, 1]$ of her own attitude toward the decision. We assume actions are taken simultaneously, that n, p, $w^I_{N^n}$, γ, F, c and G are common knowledge, and that the θ_i is private knowledge to agent i. Therefore, for any given tuple $(n, p, w^I_{N^n}, \gamma, F, c, G)$, a pure strategy is a mapping $s : [-1, 1] \longrightarrow \mathbb{R}$. Let S be the set of all feasible pure strategies. For each $s \in S$ and each $\theta \in [-1, 1]$, let $s(\theta) \in \mathbb{R}$ be the action taken given θ according to strategy s, always given n, p, $w^I_{N^n}$, γ, F, c and G. For each $s \in S$, for each $n \in \mathbb{N}\backslash\{1\}$, and for each $i \in N^n$, let $s_i = s$ denote that agent i chooses strategy s.

Definition 1 *We say that a strategy s is neutral if $s(-\theta) = -s(\theta)$ for any $\theta \in [-1, 1]$. We say s is monotone if $\frac{\partial s}{\partial \theta} \geq 0$.*

Utilities. Given a society N^n with $(n, p, w^I_{N^n}, \gamma, F, G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{G}$ and given a mechanism $c \in C$, for any agent $i \in N^n$ with preference order $\succsim_i$ over $\mathcal{M}(\{A, B\} \times \mathbb{R}^n)$, we can compute the expected utility of agent i as a function of her attitude θ_i, her strategy s_i and the strategy profile of every other player s_{-i}. Let $EU_i : [-1, 1] \times S^n \longrightarrow \mathbb{R}$ denote the expected utility of agent i. Then, given $(n, p, w^I_{N^n}, \gamma, F, c, G)$, for any $\theta_i \in [-1, 1]$ and $s_{N^n} \in S^n$, $EU_i[\theta_i, s_{N^n}]$ is equal to expected utility from the social decision ($\gamma\theta_i \Pr[d^n = A] - \gamma\theta_i \Pr[d^n = B]$), plus the expected wealth transfer in real terms. The precise expression of $EU_i[\theta_i, s_{N^n}]$ is

$$\gamma\theta_i \left(2 \int\limits_{\theta_{-i} \in [-1,1]^{n-1}} \left(\prod_{j \in N^n \backslash \{i\}} f(\theta_j) \right) G \left(s_i(\theta_i) + \sum_{j \in N^n \backslash \{i\}} s_j(\theta_j) \right) d\theta_{-i} - 1 \right)$$
$$+\frac{1}{p} \left(w^I_i - c(s_i(\theta_i)) + \frac{1}{n-1} \sum_{j \in N^n \backslash \{i\}} \int\limits_{\theta_j \in [-1,1]} f(\theta_j) c(s_j(\theta_j)) d\theta_j \right). \tag{1}$$

Game. For each tuple $(n, p, w^I_{N^n}, \gamma, F, c, G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, let $\Gamma^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ denote the game played by the n players in society N^n, with strategy set S for each agent, and expected utility given by EU_i in Expression (1) for each $n \in \mathbb{N}\backslash\{1\}$ and each $i \in N^n$.

Equilibrium. For any tuple $(n, p, w^I_{N^n}, \gamma, F, c, G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, let $BNE^{(n,p,w^I_{N^n},\gamma,F,c,G)} \subseteq S^n$ denote the set of pure Bayes Nash equilibria of game $\Gamma^{(n,p,w^I_{N^n},\gamma,F,c,G)}$. We are interested in the subset of symmetric pure Bayes Nash equilibria, in which each player plays the same pure, monotone strategy $s \in S$. Let $E^{(n,p,\gamma,F,c,G)} \subseteq S$ denote the set of pure and monotone strategies that constitute a symmetric Bayes Nash equilibrium of game $\Gamma^{(n,p,w^I_{N^n},\gamma,F,c,G)}$.[8]

[8] We drop the superindex $w^I_{N^n}$ because the equilibria, as we show below, will not depend on $w^I_{N^n}$.

Social preferences. For each $n \in \mathbb{N}\backslash\{1\}$, let R^n denote a complete and transitive relation over $\mathbb{R}^n$, interpreted as preference over valuation profiles: for any $\gamma \in \mathbb{R}_{++}$ and for any $\theta_{N^n}, \tilde{\theta}_{N^n} \in [-1,1]^n$, we interpret $(\gamma\theta_{N^n})\, R^n(\gamma\tilde{\theta}_{N^n})$ to mean that according to preference R^n, valuation profile $\gamma\theta_{N^n}$ is preferable to valuation profile $\gamma\tilde{\theta}_{N^n}$. We can interpret this preference as a preference held by the mechanism designer, or as an abstract preference relation over valuation profiles. Let $R \equiv \{R^n\}_{n=2}^{\infty}$ denote an infinite sequence of such preferences over valuation profiles, and let $\mathcal{R}$ denote the set of all such sequences. For each $n \in \mathbb{N}\backslash\{1\}$, define as well the strict preference P^n by $\gamma\theta_{N^n}P^n(\gamma\tilde{\theta}_{N^n}) \iff \neg(\gamma\tilde{\theta}_{N^n})R^n\left(\gamma\theta_{N^n}\right)$, where $\neg$ denotes the negation of a logical statement.

A sequence R of preferences over valuation profiles determines a social preference over $\{A, B\}$ as a function of n, γ and θ_{N^n} : for any $\gamma \in \mathbb{R}_{++}$, and for any $n \in \mathbb{N}\backslash\{1\}$, since the valuation profile of A is $\gamma\theta_{N^n}$ and the valuation profile of B is $-\gamma\theta_{N^n}$, we say that alternative A is socially weakly preferred to B if and only if $\left(\gamma\theta_{N^n}\right) R^n \left(-\gamma\theta_{N^n}\right)$, and socially strictly preferred if $\left(\gamma\theta_{N^n}\right) P^n \left(-\gamma\theta_{N^n}\right)$.

Welfare representation. If the preference relation R^n over valuation profiles is continuous, then it can be represented by a continuous function (Debreu 1954). We refer to this utility representation as a "welfare" function and we represent it by the mapping $W : \mathbb{R}_{++} \times \bigcup_{n=2}^{\infty} [-1,1]^n \longrightarrow \mathbb{R}$. For any $R \in \mathcal{R}$, we say that the welfare function W represents R if for any $n \in \mathbb{N}\backslash\{1\}$, any $\gamma \in \mathbb{R}_{++}$ and any $\theta_{N^n}, \tilde{\theta}_{N^n} \in [-1,1]^n$, $W(\gamma, \theta_{N^n}) \geq W(\gamma, \tilde{\theta}_{N^n})$ if and only if $(\gamma\theta_{N^n})\, R^n(\gamma\tilde{\theta}_{N^n})$.

Let $sgn : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ be the sign function, defined by $sgn(x) = -1$ if $x < 0$, $sgn(x) = 0$ and $sgn(x) = 1$ if $x > 0$. For each $\rho \in \mathbb{R}_{++}$, define the Bergson welfare function W_ρ (Burk 1936) by

$$W_\rho(\gamma, \theta_{N^n}) \equiv \sum_{i \in N^n} sgn(\theta_i)|\gamma\theta_i|^\rho.$$

Value system. A value system is a collection of normative axioms over the set $\mathcal{R}$ of all sequences of preferences over valuation profiles. For each sequence $R \equiv \{R^n\}_{n=2}^{\infty} \in \mathcal{R}$, and for each $n \in \mathbb{N}\backslash\{1\}$, we say that preference R^n follows a given value system if R^n satisfies the system's axioms. Denote by $\mathcal{V}$ the value system composed of the axioms of continuity, anonymity, neutrality, monotonicity, separability, and scale invariance (defined formally in the Appendix).

Any $R \in \mathcal{R}$ is representable by a Bergson welfare function W_ρ for some $\rho \in \mathbb{R}_{++}$ if and only if R^n follows value system $\mathcal{V}$ for each $n \in \mathbb{N}\backslash\{1\}$ (Eguia and Xefteris 2018, based on Roberts 1980 and Moulin 1988).[9]

Therefore, for each $n \in \mathbb{N}\backslash\{1\}$, the class of preferences over valuation profiles characterized by value system $\mathcal{V}$ is parameterized by a single parameter $\rho \in \mathbb{R}_{++}$. This parameter ρ measures how much the preference over valuation profiles responds to intensity of individual preferences over alternatives. Each value $\rho \in \mathbb{R}_{++}$ can be interpreted as a distinct normative axiom on preferences over valuations. Under this interpretation, for each $\rho \in \mathbb{R}_{++}$, the collection of axioms $\{\mathcal{V}, \rho\}$ is a primitive that fully characterizes a specific $R \in \mathcal{R}$: for each $n \in \mathbb{R}^n$, and for each $\rho \in \mathbb{R}_{++}$, let R^n_ρ denote the preference relation over valuation profiles that follows value system $\{\mathcal{V}, \rho\}$. Preference R^n_ρ is represented by the Bergson welfare function W_ρ. Define $R_\rho \equiv \{R^n_\rho\}_{n=2}^{\infty}$.

Optimality. Given a sequence of preference profiles $R \in \mathcal{R}$, and given a society size $n \in \mathbb{N}\backslash\{1\}$, a social decision $d^n \in \{A, B\}$ is optimal according to R for society N^n if the decision d^n is the socially preferred alternative according to R^n; that is, if $(\gamma\theta_{N^n})P^n(-\gamma\theta_{N^n})$, then d^n is optimal according to R if and only if $d^n = A$; whereas, if

[9] Roberts's (1980) Theorem 6 axiomatically characterizes the set of welfare functionals (mappings from strictly positive utility profiles to social preferences over alternatives) that are representable by Bergson welfare function. Moulin's (1988) Theorem 2.6 applies Roberts' axioms to characterize the set of preferences over strictly positive utility profiles that are representable by Bergson welfare functions. It is only a small corollary to extend Roberts' and Moulin's result to allow for negative valuations.

$(-\gamma\theta_{N^n})P^n(\gamma\theta_{N^n})$ then d^n is optimal according to R if and only if $d^n = B$. Equivalently, if W is a welfare function that represents R, then a social decision is optimal if the decision is A if $W(\gamma, \theta_{N^n}) > W(\gamma, -\theta_{N^n})$ and B if $W(\gamma, -\theta_{N^n}) > W(\gamma, \theta_{N^n})$. In particular, we define $\rho-$optimality as optimal according to parameter ρ.

Definition 2 *For each $\rho \in \mathbb{R}_{++}$, for each $n \in \mathbb{N}\backslash\{1\}$ and for each $\theta_{N^n} \in [-1, 1]^n$, we say that a social decision $d^n \in \{A, B\}$ is $\rho-$optimal if it is optimal according to R_ρ.*

Equivalently (by definition of R_ρ), a decision is $\rho-$optimal if it follows value system $(\mathcal{V}, \rho)$, or again equivalently, if it maximizes the Bergson welfare function W_ρ.

Without further ado, we anticipate a first result. For any $\rho \in \mathbb{R}_{++}$, we find a mechanism that is asymptotically $\rho-$optimal over all neutral attitude distributions. All the proofs are in the appendix.

Proposition 1 *For any $\rho \in \mathbb{R}_{++}$, and for any $F \in \mathcal{F}^*$, given the vote-buying mechanism c defined by $c(a) = |a|^{\frac{1+\rho}{\rho}}$ for any $a \in \mathbb{R}$, and given any sequence of neutral equilibria $\{s^n\}_{n=1}^{\infty}$, the probability that the social decision is $\rho-$optimal converges to one as $n \longrightarrow \infty$.*

We present Proposition 1 only as an illustrative example of a possibility result of vote-buying mechanisms. Three questions arise from this partial result.

First: what about societies with non-neutral, functional forms of the cumulative distribution F from which attitudes are drawn? Particularly challenging are distributions in which, with probability converging to one, the $\rho-$optimal decision is A but a majority of voters have a negative valuation, or the $\rho-$optimal decision is B but a majority of voters have a positive valuation. These distributions are substantively important, and likely to arise in any social decision involving concentrated gains (as in Figure 2), such as any targeted spending, paid with general taxation; or concentrated losses (for instance, consumer-friendly industry regulations, or NIMBY projects). We will show that the result is largely robust to a generalization to these non-neutral cumulative distribution functions.

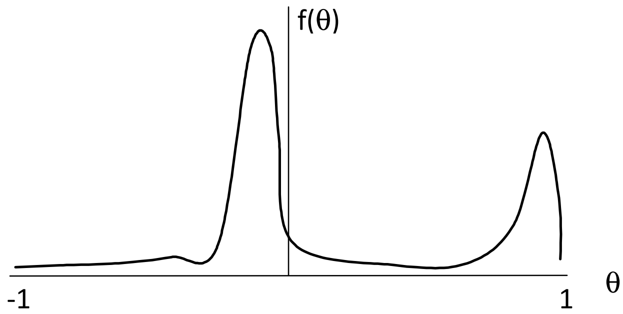

Figure 2: A density function f with concentrated gains.

Reinterpreting Proposition 1 in the language of implementation theory leads to an additional two dual questions (and answers). The implementation problem starts with a desired mapping from the realization of valuations for any society, to the subset of alternatives that are deemed desirable for this society and these valuations. This mapping is a social choice correspondence. A mechanism implements this social choice correspondence if all its equilibrium outcomes are in the social choice correspondence. So Proposition 1 says that for any positive real number ρ, any sequence of social choice correspondences that consist of selecting the maximizer of a Bergson welfare function W_ρ is asymptotically implemented across societies with neutral distributions over attitudes, by a vote buying mechanism with cost function $c(a) = |a|^{1+\frac{1}{\rho}}$.

We may wonder: which other social choice correspondences are asymptotically implementable by vote-buying mechanisms, and how? Conversely, for each vote-buying mechanism, which social choice correspondence is implemented by this mechanism?

To formalize and answer these queries, we precisely define the set of vote-buying mechanisms under consideration, social choice correspondences, and asymptotic implementability. We then align the set of social choice correspondences with the set of vote-buying mechanisms, by characterizing for each social choice correspondence the subset of mechanisms that asymptotically implement it; and by characterizing for each mechanism the subset of social choice correspondences that it asymptotically implements.

Admissible vote-buying mechanisms. We specify the set of admissible cost functions C. Let $\hat{C}$ be the set of continuously differentiable functions defined over $\mathbb{R}$ that are twice continuously differentiable over $\mathbb{R}\backslash\{0\}$. For any $c \in \hat{C}$, define the elasticity of c as $\eta_c(x) \equiv \frac{xc'(x)}{c(x)}$ for any $x \in \mathbb{R}\backslash\{0\}$. Assume that $C \equiv \{\ c \in \hat{C} : c(0) = 0,\ c'(0) = 0,$ $\lim\limits_{x\longrightarrow 0} \eta_c(x) \in (1, \infty)$, $c'(x) > 0$ for any $x' \in \mathbb{R}_{++}$, $\lim\limits_{x\longrightarrow\infty} c(x) = \infty$, and $c(x) = c(-x)$ for any $x \in \mathbb{R}\}$. The intuition on C is that, in addition to continuity and differentiability, an admissible cost functions has the following properties:

i) a zero action, interpreted as abstention, is free;

ii) to encourage positive participation, the marginal cost of taking a positive action (interpreted as acquiring a quantity of votes) at zero is zero, so for any strictly positive willingness to pay, some strictly positive quantity of votes can be acquired at that price;

iii) but the elasticity of the cost function near zero is greater than one (so c is strictly convex) near zero, and thus the marginal cost of votes becomes immediately positive;

iv) and while elsewhere the cost function need not be convex, this marginal cost is always positive for all positive quantities;

v) and very high quantities of votes are prohibitively expensive; and

vi) neutrality: votes for A cost the same as votes against A.

All power functions with exponent greater than one (and their sums), among other functions, are included in the set C.

Social Choice correspondences. For any $n \in \mathbb{N}$, a social choice correspondence $SC^n : \mathbb{R}_{++} \times [-1, 1]^n \rightrightarrows \{A, B\}$ maps a pair (γ, θ_{N^n}) into the subset of normatively desirable social decisions $SC(\gamma, \theta_{N^n})$. Let $SC \equiv \{SC^n\}_{n=1}^{\infty}$ denote a sequence of social choice correspondences, and let $\mathcal{SC}$ denote the set of all possible such sequences.

For each $\rho \in \mathbb{R}_{++}$, and for each $n \in \mathbb{N}$, define the Bergson choice correspondence SC^n_ρ

by

$$SC^n_\rho(\gamma,\theta_{N^n}) \equiv \begin{cases} B \text{ if } \sum\limits_{i\in N^n} sgn(\theta_i)|\gamma\theta_i|^\rho < 0 \\ \{A,B\} \text{ if } \sum\limits_{i\in N^n} sgn(\theta_i)|\gamma\theta_i|^\rho = 0 \\ A \text{ if } \sum\limits_{i\in N^n} sgn(\theta_i)|\gamma\theta_i|^\rho > 0. \end{cases}.$$

Note that SC^n_ρ is the social choice correspondence that chooses the alternative(s) that are socially preferred given the Bergson preference over valuation profiles R^n_ρ (which is represented by the Bergson welfare function W_ρ). Define the sequence of Bergson social choice correspondences $SC_\rho \equiv \{SC^n_\rho\}^\infty_{n=2}$. Define $SC_\rho \equiv \{SC^n_\rho\}^\infty_{n=1}$ and $\mathcal{SC}_\mathcal{V} \equiv \bigcup\limits_{\rho\in\mathbb{R}_{++}} SC_\rho$. In other words, SC_ρ is the sequences of correspondences of optimal choices according to value system $\{\mathcal{V},\rho\}$, and $\mathcal{SC}_\mathcal{V}$ is the set of all such sequences of optimal correspondences that satisfy value system $\mathcal{V}$ (one sequence for each intensity parameter ρ).

For any $n \in \mathbb{N}$, for each $J \in \{A, B, \{A, B\}\}$, for any $\gamma \in \mathbb{R}_{++}$, and for any social choice correspondence $SC^n : \mathbb{R}_{++} \times [-1,1]^n \rightrightarrows \{A, B\}$, define

$$\Theta^\gamma_J(SC^n) \equiv \{\theta_{N^n} \in [-1,1]^n : SC^n(\gamma,\theta_{N^n}) = J\},$$

and let $(\Theta^\gamma_J(SC^n))^c \equiv ([-1,1]^n)\setminus\Theta^\gamma_J(SC^n)$ denote the complement of $\Theta^\gamma_J(SC^n)$. Note $\Theta^\gamma_J(SC^n) \subseteq [-1,1]^n$ is the set of attitude profiles for which social choice correspondence SC^n declares J the normatively desirable alternative(s).

Convergence of Social Choice correspondences.

We say that two sequences of social choice correspondences SC and $\widetilde{SC}$ converge to each other if the probability that they select the same outcome converges to one, as $n \longrightarrow \infty$. We say a property holds generically if it holds in an open dense subset of the set under consideration. To formally define convergence of SC and $\widetilde{SC}$ to each other generically over $\mathcal{F}$, we need to define more structure on $\mathcal{F}$.

Let $C[-1,1]$ denote the set of all continuous functions over $[-1,1]$ and let d_∞ be the sup-metric over $C[-1,1]$, so that for any $\varphi, \hat{\varphi} \in C[-1,1]$, $d_\infty(\varphi, \hat{\varphi}) \equiv \sup\limits_{\theta \in [-1,1]} \{|\varphi(\theta) - \hat{\varphi}(\theta)|\}$. We consider the metric space $(\mathcal{F}, d_{\infty,\infty})$ with distance function $d_{\infty,\infty} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{R}_+$ defined by $d_{\infty,\infty}(F, \hat{F}) \equiv d_\infty(F, \hat{F}) + d_\infty(f, \hat{f})$.[10] A subset $\mathcal{F}^D \subset \mathcal{F}$ is dense in $\mathcal{F}$ if the closure of $\mathcal{F}^D$ is equal to $\mathcal{F}$ (so any cumulative distribution $F \in \mathcal{F} \backslash \mathcal{F}^D$ is the limit of a sequence of distributions in $\mathcal{F}^D$). We can now precisely define the desired convergence notion.

Definition 3 *For any $F \in \mathcal{F}$ and any $\mathcal{SC}, \widetilde{SC} \in \mathcal{SC}$, we say that SC and $\widetilde{SC}$ converge to each other with respect to F if $\lim\limits_{n \longrightarrow \infty} \Pr\left[SC(\gamma, \bar{\theta}_{N^n}) \neq \widetilde{SC}(\gamma, \bar{\theta}_{N^n})\right] = 0$. We say that SC and $\widetilde{SC}$ converge to each other generically if they converge to each other for any F in an open dense set $\mathcal{F}^D \subseteq \mathcal{F}$.*

Implementability.

We say that a vote buying mechanism c asymptotically implements a sequence of social choice correspondences SC over a given subdomain of possible distribution functions from which attitudes are drawn if two conditions hold: i) an equilibrium (symmetric, monotonic and pure) exists for any large society; and ii), the probability that the social decision coincides with the alternative chosen by SC converges to one. The formal definition is as follows.

Definition 4 *For any $\tilde{\mathcal{F}} \subseteq \mathcal{F}$, a vote-buying mechanism $c \in C$ asymptotically implements a sequence of social choice correspondences SC over $\tilde{\mathcal{F}}$ in symmetric, monotone and pure equilibria if for any $(p, \{w_k^I\}_{k=1}^\infty, \gamma, F, G) \in \mathbb{R}_{++} \times \mathbb{R}_+^\infty \times \mathbb{R}_{++} \times \tilde{\mathcal{F}} \times \mathcal{G}$,*

i) there exists $\hat{n} \in \mathbb{N}$ such that the set of equilibria $E^{(n,p,\gamma,F,c,G)}$ is non-empty for any $n \geq \hat{n}$; and

[10] This is the standard distance to metricize the set of continuously differentiable functions; we follow Ok (2007); see Chapter C, Example 2[3].

ii) for any $\varepsilon \in (0,1)$ and for any sequence of strategies $\{s^t\}_{t=2}^{\infty}$ such that $s^t \in E^{(t,F,\gamma,p,c,G)}$ for each $t \in \mathbb{N}\backslash\{1\}$, there exists $n_{\varepsilon,\gamma,F,p,G} \in \mathbb{N}$ such that for any $n > n_{\varepsilon,\gamma,F,p,G}$,

$$\left(\begin{array}{c} \int\limits_{\theta_{N^n} \in \Theta_A^{\gamma}(SC^n)} \left(\prod_{i=1}^{n} f(\theta_i)\right) G\left(\sum_{i\in 1}^{n} s^n(\theta_i)\right) d\theta_{N^n} \\ + \int\limits_{\theta_{N^n} \in \Theta_B^{\gamma}(SC^n)} \left(\prod_{i=1}^{n} f(\theta_i)\right) \left(1 - G\left(\sum_{i\in 1}^{n} s^n(\theta_i)\right)\right) d\theta_{N^n} \end{array} \right) > 1 - \varepsilon.$$

We say that a sequence of social choice correspondences SC is asymptotically implementable if there exists a mechanism $c \in C$ that asymptotically implements SC in symmetric, monotone and pure equilibria.

Since our implementation results are always asymptotic, and always in symmetric, monotone and pure equilibria, if a mechanism c implements asymptotically SC over $\hat{\mathcal{F}}$ in symmetric, monotone and pure equilibria, we say simply that c "implements SC over $\hat{\mathcal{F}}$."

This implementation notion requires that, if the society is sufficiently large, the outcome in every equilibrium of the game induced by the mechanism must be the outcome desired by the social choice rule with probability arbitrarily close to one, for any distribution parameters. Depending on the domain of distributions $\hat{\mathcal{F}}$ under consideration, such robustness across societies may not be attainable. We then seek, as a second best, a mechanism that works for most societies in the domain under consideration.

We define generic asymptotic implementability accordingly.

Definition 5 *A vote-buying mechanism $c \in C$ asymptotically implements a sequence of social choice correspondences SC generically in symmetric, monotone and pure equilibria if there exists an open $\mathcal{F}^D$ dense in $\mathcal{F}$ such that c implements SC over $\mathcal{F}^D$.*

We say that a sequence of social choice correspondences SC is generically asymptotically implementable if there exists a mechanism $c \in C$ that generically asymptotically implements SC in symmetric, monotone and pure equilibria.

Once again, if a mechanism c asymptotically implements a sequence of social choice correspondences SC generically in symmetric, monotone and pure equilibria, we say simply that c "implements SC generically." For any $\rho \in \mathbb{R}_{++}$, we say that a mechanism c implements value system $(\mathcal{V}, \rho)$ if it implements SC_ρ, .

3 Main Result

We provide a characterization of the set of sequences of social choice correspondences that are implementable by vote-buying mechanisms, generically over all possible distribution functions from which attitudes are drawn. We also provide, for each sequence of social choice correspondences that is generically implementable, a class of vote-buying mechanisms that generically implements it. In particular, we show that the set of social choice correspondences implemented by any given vote-buying mechanism are entirely determined by the elasticity $\eta_c(a)$ of the mechanism, evaluated at the limit with zero acquisition of votes.

Theorem 2 *A sequence SC of social choice correspondences is generically implementable by a vote-buying mechanism in C if and only if there exists $\rho \in \mathbb{R}_{++}$ such that SC and SC_ρ converge to each other generically, in which case, any vote-buying mechanism $c \in C$ such that $\lim_{a \longrightarrow 0^+} \eta_c(a) = \frac{1+\rho}{\rho}$ generically implements SC.*

That is, only sequences of social choice correspondences that converge toward choosing optimally according to value system $(\mathcal{V}, \rho)$ for some $\rho \in \mathbb{R}_{++}$ are generically implementable by vote-buying mechanisms, and specifically, any vote-buying mechanism with limit elasticity $\kappa \in (1, \infty)$ generically implements value system $(\mathcal{V}, \frac{1}{\kappa-1})$.

Since, for any $\kappa \in (1, \infty)$ the vote-buying mechanism with power cost function $c(a) = |a|^\kappa$ has $\lim_{a \longrightarrow 0^+} \eta_c(a) = \kappa$, we obtain as a corollary that $c(a) = |a|^\kappa$ implements value system $(\mathcal{V}, \frac{1}{\kappa-1})$; quadratic voting is the special case with $\kappa = 2$ and value system $(\mathcal{V}, 1)$, i.e. utilitarianism. Goeree and Zhang (2017) and Lalley and Weyl (2018) provide a

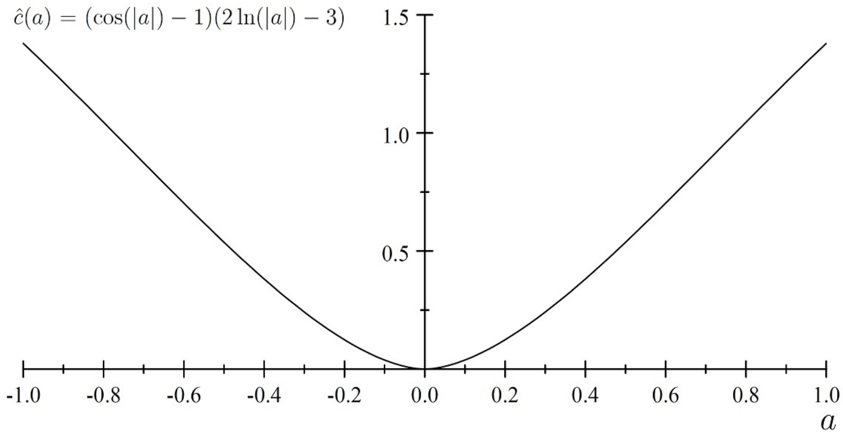

Figure 3: A non-polynomial mechanism $\hat{c}$ that implements utilitarianism.

heuristic intuition for this special case: if agents (incorrectly) assume that their marginal benefit of acquiring votes is constant in the quantity of votes acquired, then agents infer that their marginal benefit of acquiring votes is linear in their attitude. Given a mechanism $c(a)$ with derivative $c'(a)$ that is linear in a, agents equate perceived marginal benefit and marginal cost by acquiring votes in proportion to their attitude, which leads to utilitarian efficiency.

This heuristic intuition is useful as far as other power cost mechanisms are concerned, but beyond these functions, it does not generalize well: what matters for asymptotic implementation is the limit elasticity $\lim_{a \longrightarrow 0^+} \eta_c(a)$ of the cost function $c(a)$, and not the shape of the derivative $c'(a)$. Consider, for example, the mechanism $\hat{c} \in C$ depicted in Figure 3, and defined by $\hat{c}(a) = (\cos(|a|) - 1)(2\ln(|a|) - 3)$ for any $a \in [-1, 1]$ (and increasing arbitrarily for higher quantities).

Notice that $\hat{c}(a)$ and $c(a) = |a|^2$ are generically unequal. In fact, $\lim_{a \longrightarrow 0^+} \frac{\hat{c}(a)}{c(a)} = \lim_{a \longrightarrow 0^+} \frac{\hat{c}'(a)}{c'(a)} = +\infty$, ($c$ converges to zero arbitrarily faster than $\hat{c}$). The marginal cost $\hat{c}'(a)$ is a (cumbersome) trigonometric function, suggesting that if the heuristic intuition based on the marginal cost were correct, mechanism $\hat{c}$ would implement a social choice correspondence that maximized some trigonometric welfare function. But this is not the case: It is easy to check that $\lim\limits_{a \longrightarrow 0^+} \eta_{\hat{c}}(a) = 2$, so $\hat{c}$ implements utilitarianism as well. Put differently: quadratic voting implements utilitarianism not because its marginal cost is linear, but rather, because its limit elasticity at zero is 2, and any other mechanism with limit elasticity of 2 also implements utilitarianism.

To grasp the intuition why the limit elasticity is the significant element here, we sketch the most relevant steps of the proof. In line with the heuristic intuition we find that in a sequence of equilibria, the ratio of the marginal costs corresponding, for instance, to two distinct types of alternative A supporters, must converge to the ratio of the attitudes of these types. That is, for every $(\theta, \hat{\theta}) \in (0,1]^2$, we get:

$$\lim_{n\to\infty} \tfrac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \tfrac{\theta}{\hat{\theta}} \Rightarrow \lim_{n\to\infty} \ln \tfrac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \ln \tfrac{\theta}{\hat{\theta}}.$$

Moreover, we can show that the function $J : \mathbb{R}^2_{++} \to \mathbb{R}$ given by:

$$J(x,y) = \begin{cases} \frac{yc''(y)}{c'(y)} & \text{if } x = y \\ \frac{\ln \frac{c'(x)}{c'(y)}}{\ln \frac{x}{y}} & \text{if } x \neq y \end{cases}$$

converges to $\lim\limits_{a \longrightarrow 0^+} \eta_c(a) - 1$ as $(x,y) \to (0,0)$. Hence,

$$\lim_{n\to\infty} \frac{\ln \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))}}{\ln \frac{s^n(\theta)}{s^n(\hat{\theta})}} = \lim_{a \longrightarrow 0^+} \eta_{\hat{c}}(a) - 1 \Longrightarrow \lim_{n\to\infty} \ln \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \lim_{n\to\infty} \ln \left(\frac{s^n(\theta)}{s^n(\hat{\theta})}\right)^{\lim\limits_{a \longrightarrow 0^+} \eta_c(a)-1},$$

and thus substituting the left hand side according to $\lim_{n\to\infty} \ln \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \ln \frac{\theta}{\hat{\theta}}$, we get

$$\ln \frac{\theta}{\hat{\theta}} = \lim_{n\to\infty} \ln \left(\frac{s^n(\theta)}{s^n(\hat{\theta})}\right)^{\lim\limits_{a \longrightarrow 0^+} \eta_c(a)-1} \Rightarrow \lim_{n\to\infty} \frac{s^n(\theta)}{s^n(\hat{\theta})} = \left(\frac{\theta}{\hat{\theta}}\right)^{\frac{1}{\lim\limits_{a \longrightarrow 0^+} \eta_c(a)-1}}.$$

That is, the equilibrium vote acquisitions become proportional to the ratio of the attitudes raised to a power that depends on the limit cost elasticity; and this leads to the implementation of the value system $\left(\mathcal{V}, \frac{1}{\lim\limits_{a \longrightarrow 0^+} \eta_c(a)-1}\right)$.

4 Discussion

Given a binary collective choice problem, a mechanism implements a value system if for any ex ante distribution and any ex post realization of individual preferences, and for any equilibrium induced by the mechanism, the probability that the social decision is socially preferred according to the value system converges to one in the size of the society.

A particular class of value systems, characterized by a set of normative axioms, is indexed by a parameter ρ that measures the degree of caring about intensity of preference. At one end of this class, majoritarianism assigns equal importance to each individual ordinal preference, entirely disregarding intensity. At the opposite end of the class, the maximin notion cares maximally about intensity and equates welfare with the utility of the agent with the most intense preference. Utilitarianism is an interior principle, caring for all agents' preferences in linear proportion to their intensity.

For any value system in this axiomatized class, we find a vote-buying mechanism that implements it. In particular, for each value system with attention ρ to intensity of individual preferences, any vote-buying mechanism given by a cost function with limit elasticity $\frac{1+\rho}{\rho}$ at zero (for example, a power function $c(a) = |a|^{\frac{1+\rho}{\rho}}$), generically implements the value system parameterized by ρ.[11]

We characterize the set of social choice correspondences that are generically implementable by a vote-buying mechanism in sufficiently large societies: a sequence of social choice correspondences is generically implementable if and only if it asymptotically follows a given value system in our axiomatized class (Theorem 2).

The standard relative majority voting rule that assigns one vote to each person for free is equivalent to the limit $\rho = 0$ of our range of parameters: as the limit cost elasticity $\lim_{a \longrightarrow 0^+} \eta_{\hat{c}}(a) = \frac{1+\rho}{\rho}$ diverges to $\lim_{\rho \to 0} \frac{1+\rho}{\rho} = +\infty$, the marginal cost of an extra vote becomes arbitrarily larger compared to the average one, so everyone converges toward acquiring

[11] Note that these vote-buying mechanisms are "bounded" in the sense of Jackson (1992), but they are not "strategically simple" in the sense of Börgers and Li (2017). Nor are they robust to coalitional deviations (Bierbrauer and Hellwig 2016).

the same amount of votes.[12]

A decentralized, competitive market for votes, similar to the ones proposed for instance by Dekel, Jackson and Wolinski (2008) and Casella, Llorente-Saguer and Palfrey (2012), implements the opposite extreme, $\rho = \infty$: as the limit cost elasticity converges to $\lim_{\rho\to\infty} \frac{1+\rho}{\rho} = 1$, the marginal cost of an extra vote becomes identical to the average one –as in a competitive market– and the agent or agents with most intense preferences purchase most votes and determine the social decision.

Casella, Llorente-Saguer and Palfrey (2012) interpret the outcome with a market for votes as a social welfare loss, because they judge welfare according to an utilitarian perspective. We interpret the finding differently: the outcome is optimal according to a welfare notion in which we care overwhelmingly more about the agent with the most intense preference. For that welfare criterion, a market for votes with linear pricing, be it a centralized one as in our mechanism, or a decentralized one as in Casella, Llorente-Saguer and Palfrey (2012), is optimal. If that is not the welfare criterion we have in mind, then we should not choose linear pricing for votes. Rather, we should choose the pricing that corresponds to our welfare notion. For utilitarian welfare, corresponding to a parameter value of $\rho = 1$, quadratic pricing is optimal (Lalley and Weyl 2016). For any other welfare notion corresponding to parameter value $\rho \in \mathbb{R}_{++}$, an optimal pricing of votes is any c with limit elasticity $\frac{1+\rho}{\rho}$ at zero.

We address two important substantive limitations.

Wealth inequality.

A common criticism of vote-buying mechanisms that rely on linear or quadratic pricing is that in practice they would favor the rich, effectively disenfranchising the poor. In our theory, as in previous theories of vote-buying mechanisms, agents are risk neutral and preferences over wealth are separable, so the utility representation is quasilinear and

[12] For instance, if $c(a) = |a|^\infty$, any quantity of votes smaller than one is free, while any quantity of votes above one is infinitely expensive, leading all players to acquire exactly one vote.

there are no wealth effects: agents' actions are independent of their wealth.

Concerns about the effects of wealth inequality arise if we assume that agents are risk averse, so that their utility over wealth is concave. If so, for any given preference intensity over the social choice, a wealthier agent would acquire more votes than an agent with the same intensity of preference and lesser wealth. If the planner cares only about the social decision, and not about wealth redistribution, the optimality of the mechanisms we have studied is lost: since the cost function conditions only on the number of votes, the preferences of wealthier votes are overweighed, so the axiom of anonymity is violated. Optimality with respect to a value system that includes anonymity can be restored by allowing for vote-buying mechanisms such that the cost function conditions on wealth and on the number of votes acquired (this result is available from the authors).

Multiple alternatives.

We have identified mechanisms to make binary social decision. If the set of alternatives under consideration contains multiple alternatives, the welfare properties of these vote-buying mechanisms are weakened. As in elections with multiple candidates, coordination can result in only two alternatives being competitive, so agents purchase and cast votes for only these two. These two alternatives may be any pair, and not necessarily the best two. Our result, in this case, only implies that the least desirable alternative will be defeated with probability converging to one. This limitation is not intrinsic to vote-buying mechanism; it is a feature shared by standard voting practices in which each agent has one vote.

We have shown that binary social choice correspondences that choose the optimal alternative according to a value system representable by a Bergson welfare function with parameter ρ, can be generically implemented in large societies by a vote-buying mechanism with any cost function whose elasticity converges to $\frac{1+\rho}{\rho}$ at zero votes.

5 Appendix

5.1 Assumptions and Axioms

Recall that for any $n \in \mathbb{N}\backslash\{1\}$ and for any $i \in N^n$, $\succsim_i$ is a complete and transitive relation over the set of probability measures $\mathcal{M}(\{A, B\} \times \mathbb{R}^n)$. For any $n \in \mathbb{N}\backslash\{1\}$, and for any $i \in N^n$, we assume the following on $\succsim_i$.

Assumption 2. *The preference relation $\succsim_i$ is continuous and satisfies independence over decomposition of lotteries.*

From Assumption 2, it follows that each preference relation $\succsim_{N^n}$ is representable by a continuous utility function in expected utility form (von Neumann and Morgernstern 1944).

Assumption 3. *F*or any $\left(d^n, \frac{w_{N^n}}{p}\right) \in \{A, B\} \times \mathbb{R}^n$ and $\left(d^n, \frac{w'_{N^n}}{p}\right) \in \{A, B\} \times \mathbb{R}^n$ such that $w_i = w'_i$, agent i is indifferent between $\left(d^n, \frac{w_{N^n}}{p}\right)$ and $\left(d^n, \frac{w'_{N^n}}{p}\right)$.

Assumption 3 means that each agent cares only about the social decision, and about her own real final wealth. With a slight abuse of notation we can then refer $\succsim_i$ to a preference order over $\Delta(\{A, B\} \times \mathbb{R})$.

Assumption 4. *F*or any $\left(d^n, \frac{w_{N^n}}{p}\right) \in \{A, B\} \times \mathbb{R}^n$ and $\left(d^n, \frac{w'_{N^n}}{p}\right) \in \{A, B\} \times \mathbb{R}^n$ such that $w_i > w'_i$, agent i strictly prefers $\left(d^n, \frac{w_{N^n}}{p}\right)$ to $\left(d^n, \frac{w'_{N^n}}{p}\right)$.

Assumption 4 means that each agent has strictly monotonically increasing preferences over final real wealth.

Let L denote a simple lottery over $\{A, B\} \times \mathbb{R}^n$ (a probability measure that assigns strictly positive probability only to finitely many outcomes).

Let a 50-50 lottery be a probability distribution over $\{A, B\} \times \mathbb{R}^n$ that assigns probability 0.5 to exactly two outcomes. For any probability measure $\mu \in \mathcal{M}(\{A, B\} \times \mathbb{R}^n)$, and in particular for any simple lottery L, let μ_d and $\mu_{w_i/p}$ be the marginal probability measures over $\{A, B\}$ and over $\mathbb{R}$ (respectively), derived from μ.

Assumption 5. (Fishburn's *(1970) Separability*) *F*or any two 50-50 lotteries L, L' such that $L_d = L'_d$ and $L_{w_i/p} = L'_{w_i/p}$, $L \sim_i L'$.

This means that preferences over lotteries are driven only by the marginal probability distributions, and not by their correlation. This is a separability condition, because it implies that the preferences over lotteries in one dimension do not change with changes in the other dimension.

Assumptions 1-3 and 5 jointly imply that the preferences of agent i can be represented by an additively separable function of the outcome and the final real wealth of i in expected utility form (Fishburn 1970, Theorem 11.1).

Assumption 6. Agent i is risk neutral with respect to final real wealth.

Note that for each $n \in \mathbb{N}\backslash\{1\}$, for each probability measure μ over $\mathcal{M}(\{A, B\} \times \mathbb{R}^n)$ and for each $J \in \{A, B\}$, $\mu_d(\{J\}) = \Pr[d^n = J]$, while for each $x \in \mathbb{R}$, $\mu_{w_i/p}(\{x\}) =$

$\Pr\left[\frac{w_i}{p} = x\right]$. For any $x \in \mathbb{R}$ such that $\mu_{w_i/p}$ is differentiable at x, let $\mu'_{w_i/p}(x) \equiv \frac{\partial}{\partial}\mu_{w_i/p}(x)$ denote the derivative of $\mu_{w_i/p}$ at x, so that $\mu'_{w_i/p}$ is the density function associated to the marginal probability measure $\mu_{w_i/p}$. Hence, for any interval $I \subset \mathbb{R}$,

$$\Pr\left[\frac{w_i}{p} \in I\right] = \sum_{\frac{w_i}{p} \in I : \mu\left(\left\{\frac{w_i}{p}\right\}\right) > 0} \mu\left(\left\{\frac{w_i}{p}\right\}\right) + \int_{x \in I} \mu'_{w_i/p}(\{x\})\, dx,$$

where the first term captures the probability mass points, and the second the integral over the density, wherever defined.

Assumptions 1-6 jointly imply that $\succsim_i$ is representable by an additively separable, quasilinear utility function $\tilde{u}_i$ such that for each $\mu \in \mathcal{M}(\{A, B\} \times \mathbb{R}^n)$,

$$\tilde{u}_i(\mu) = \sum_{J \in \{A,B\}} \mu_d(\{J\}) u^i_d(J) + \sum_{w_i \in \mathbb{R}} \mu_{w_i/p}\left(\left\{\frac{w_i}{p}\right\}\right) \frac{w_i}{p} + \int_{x \in \mathbb{R}} \mu'_{w_i/p}(\{x\})\, x dx$$

where $u^i_d : \{A, B\} \longrightarrow \mathbb{R}$ is a function that represents the preferences over the social choice.

We next list the six axioms on the set $\mathcal{R}$ of sequences of social preferences over valuation profiles that constitute the value system $\mathcal{V}$. For any $n \in \mathbb{N}\backslash\{1\}$, for any $\varepsilon \in \mathbb{R}_{++}$ and for any $x \in \mathbb{R}^n$, let $N_\varepsilon(x)$ the open $\varepsilon-$neighborhood around x.

Axiom 1 (Continuity) *A sequence $R \equiv \{R^n\}_{n=1}^\infty \in \mathcal{R}$ is continuous if for any $n \in \mathbb{N}\backslash\{1\}$, and for any $x, y \in \mathbb{R}^n$ such that $xP^n y$, $\exists \varepsilon \in \mathbb{R}_{++}$ such that $x' P^n y'$ for any $x' \in N_\varepsilon(x)$ and any $y' \in N_\varepsilon(y)$.*

By continuity, sufficiently small changes in individual utilities do not reverse a strict order relation between two net utility profiles.

Axiom 2 (Anonymity) *A sequence $R \equiv \{R^n\}_{n=1}^\infty \in \mathcal{R}$ is anonymous if for any $n \in \mathbb{N}\backslash\{1\}$, and for any $x \in \mathbb{R}^n$ and any $y \in \mathbb{R}^n$ such that x and y are a permutation of each other, $xR^n y$ and $yR^n x$.*

Anonymity guarantees that the social welfare ordering does not care about voters' names, and pays attention only to the set of individual valuations.

Axiom 3 (Neutrality) *A sequence $R \equiv \{R^n\}_{n=1}^\infty \in \mathcal{R}$ is neutral if for any $n \in \mathbb{N}\backslash\{1\}$, and for any $x \in \mathbb{R}^n$ and any $y \in \mathbb{R}^n$, $xR^n y \Longleftrightarrow -yR^n(-x)$.*

Neutrality guarantees that the label of which of the two alternatives is labeled A and which one is B, does not matter.

Axiom 4 (Monotonicity) *A sequence $R \equiv \{R^n\}_{n=1}^{\infty} \in \mathcal{R}$ is monotonic if for any $n \in \mathbb{N}\backslash\{1\}$, $xR^n y$ for any $x \in \mathbb{R}^n$ and any $y \in \mathbb{R}^n$ such that $x - y \in \mathbb{R}^n_+$, and $xP^n y$ for any $x, y \in \mathbb{R}^n$ such that $x - y \in \mathbb{R}^n_{++}$.*

Monotonicity guarantees that the social value of choosing A over B is increasing in individual valuations.

For any $n \in \mathbb{N}\backslash\{1\}$, and for any $M \subseteq \{1, ..., n\}$, and for any $x \in \mathbb{R}^n$, let $x_M \in \mathbb{R}^{|M|}$ define the vector restricted to the subset of agents M, and let $(x_M, x_{N\backslash M})$ be another way to write vector x.

Axiom 5 (Separability) *A sequence $R \equiv \{R^n\}_{n=1}^{\infty} \in \mathcal{R}$ is separable if for any $n \in \mathbb{N}\backslash\{1\}$, for any $M \subseteq \{1, ..., n\}$, and for any $x, y \in \mathbb{R}^n$,*

$$(x_M, x_{N\backslash M})R^n(y_M, x_{N/M}) \Longleftrightarrow (x_M, y_{N\backslash M})R^n(y_M, y_{N/M}).$$

This corresponds to Debreu's (1960) notion of strong separability. It means that can evaluate partial valuation profiles restricted to a subset of agents without taking into account the valuations of other agents.

Axiom 6 (Scale invariance) *A sequence $R \equiv \{R^n\}_{n=1}^{\infty} \in \mathcal{R}$ is scale invariant if for any $n \in \mathbb{N}\backslash\{1\}$, for any $x, y \in \mathbb{R}^n$, and for any $\lambda \in \mathbb{R}_{++}$, $\lambda x R^n \lambda y \Longleftrightarrow x R^n y$.*

Scale invariance guarantees that society applies the same criteria to important decisions (γ high) than to less important ones (γ low).

The value system $\mathcal{V}$ is defined as the collection of axioms 1-6.

5.2 Proofs

In this section, we prove our results. The proofs are long. They proceed in ten steps: Proposition 1 requires to follow steps 1-6 and 10, and Theorem 2 requires steps 1-9.

One - We prove existence of a symmetric equilibrium in pure, monotone strategies for any parameter tuple (Lemma 3), and existence of a symmetric equilibrium in pure, monotone and neutral strategies for any neutral F (Lemma 4).

Two - We prove that net vote acquisitions for A are strictly increasing in attitude θ (Lemm 5), and use this result to write the first order condition of the individual optimization problem (Equation (3)).

Three - We prove that equilibrium vote acquisitions converge to zero (Lemma 7 establishes the result for most attitudes; later Lemma 12 extends this result to all attitudes).

Four - We prove that the ratio of marginal costs converges to the ratio of attitudes (Lemma 9).

Five - We prove that the marginal benefit of acquiring votes converges to zero (Lemma 11), and use this result to prove that the third and fourth steps extend to all attitudes (Lemma 12, Corollary 13).

Six - We prove that the ratio of vote acquisitions converges to a power function of the ratio of attitudes; first we prove it piecewise (Lemma 15) and then over the whole domain (Lemma 16).

Seven - After two technical lemmas (Lemma 17 and Lemma 18) we establish a sufficient condition for a sequence of social choice correspondences to be implementable over a subset of distribution functions that is open and dense over the set over all cumulative distribution functions (Proposition 19).

Eight - We find a necessary condition for such implementation (Proposition 20).

Nine - We show that the necessary condition is sufficient for generic implementability, establishing our main result (Theorem 2).

Ten - We conclude by proving a result on implementation in neutral equilibria, restricted to neutral distributions, from which Proposition 1 follows as an immediate corollary.

Lemma 3 *For any tuple $(n, p, w^I_{N^n}, \gamma, F, c, G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, a pure symmetric monotone equilibrium of game $\Gamma^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ exists.*

Proof. For each tuple $(n, p, w^I_{N^n}, \gamma, F, c, G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, define S^γ as the set of all functions $y : [-1, 1] \longrightarrow [-c^{-1}(2p\gamma), c^{-1}(2p\gamma)]$, and let $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ denote the restricted game played by the n players in society N^n, with strategy set S^γ for each agent, and expected utility given by EU_i in Expression (1) for each $i \in N^n$.

Note that game $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ satisfies the nine conditions for existence of a symmetric, pure monotone equilibrium in Reny's (2011) Theorem 4.5. Conditions G1-G6 in this theorem, as explained by Reny, are standard and applied to a vast class of more general environments that includes our own as a very special case. The three additional conditions are the following:

i) the game must be symmetric. Note that initial wealth enters the utility function (1) as an additive term; therefore, game $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ is strategically equivalent to a game $\Gamma_R^{(n,p,\hat{w}^I_{N^n},\gamma,F,c,G)}$ with $\hat{w}^I_i = 0$ for each $i \in N^n$, and $\Gamma_R^{(n,p,\hat{w}^I_{N^n},\gamma,F,c,G)}$ is symmetric, because each player's preference is drawn from the same distribution F, and G is anonymous, aggregating total contributions.

ii) each player's set of monotone pure best replies is non-empty. Given the actions by other players, each player i maximizes a continuous function over a compact domain, so a maximum exists, and this maximum is a best response. Furthermore, the utility function is supermodular in $|\theta_i|$ and $|a_i|$ (it satisfies increasing differences in $(|\theta_i|, |a_i|)$ and so the set of maximizers is non-decreasing, and thus we can select a monotonically increasing best response.

iii) each player's set of monotone pure best replies is join-closed whenever the other players employ the same monotone pure strategy. A subset of strategies is join-closed if the pointwise supremum of any pair of strategies in the set is also in the set. Since the maximization problem is independently solved for each $|\theta_i|$ to obtain a best response, the pointwise maximum of any pair of strategies is in the set. Since the set of best responses is closed, the pointwise supremum is a pointwise maximum, for each point.

Therefore, game $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ has a symmetric, pure monotone equilibrium. For each $n \in \mathbb{N}\backslash\{1\}$, for any voter $i \in N^n$, for any type $\theta_i \in [-1,1]$, and for any strategy profile s_{-i} for $N^n\backslash\{i\}$, $a_i \notin [-c^{-1}(2p\gamma), c^{-1}(2p\gamma)]$ is a dominated action: it leads to a strictly lower payoff than $a_i = 0$. Thus, we can restrict attention to the restricted game $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ with bounded action space $X \equiv [-c^{-1}(2p\gamma), c^{-1}(2p\gamma)]$, and any equilibrium of $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ is also an equilibrium of $\Gamma^{(n,p,w^I_{N^n},\gamma,F,c,G)}$. It follows that a symmetric pure monotone equilibrium of game $\Gamma^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ exists. ■

Lemma 4 *For any tuple $(n,p,w^I_{N^n},\gamma,F,c,G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}_{++} \times \mathcal{F}^* \times C \times \mathcal{G}$, a symmetric pure neutral monotone equilibrium of game $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ exists.*

Proof. Consider a further restricted game in which each agent $i \in N^n$ privately observes $|\theta_i|$ but not θ_i and then chooses an action $|a_i| \in \mathbb{R}_+$. Subsequently, agent i learns the sign of θ_i and chooses the sign of a_i. Since it is dominated to choose $sgn(a_i) \neq sgn(\theta_i)$, assume that $sgn(a_i) = sgn(\theta_i)$. Let $\hat{S}^\gamma$ as the set of all functions $\tilde{s} : [0,1] \longrightarrow [0, c^{-1}(2p\gamma)]$, and let $\hat{\Gamma}^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ denote the restricted game played by the n players in society N^n, with strategy set $\hat{S}^\gamma$ for each agent, and expected utility given by EU_i in Expression (1) for each $i \in N^n$.

Note that game $\hat{\Gamma}^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ satisfies the nine conditions for existence of a symmetric, pure monotone equilibrium in Reny's (2011) Theorem 4.5. Conditions G1-G6 in this theorem, as explained by Reny, are standard and applied to a vast class of more general environments that includes our own as a very special case. The three additional conditions hold in game $\hat{\Gamma}^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ exactly as in game $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$, as explained in the proof of Lemma 3.

Therefore, game $\hat{\Gamma}^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ has a symmetric, pure monotone equilibrium. This equilibrium is neutral by construction. We next show that this equilibrium is also an equilibrium of the game $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$. Denote s^n the strategy played in a symmetric neutral, monotone, pure equilibrium of game $\hat{\Gamma}^{(n,p,w^I_{N^n},\gamma,F,c,G)}$, and assume that s^n is not an equilibrium strategy of $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$. Then, there exists θ such that any agent i with $\theta_i = \theta$ prefers to deviate to $s_i = s'$ with $s'(\theta) \neq s^n(\theta)$. Since s^n is neutral and $G(x) - \frac{1}{2} = \frac{1}{2} - G(-x)$, the utility for an agent j with $\theta_j = -\theta$ of deviating to play $a_j = -s'(\theta)$ equals the utility for i of deviating to play $a_i = -s'(\theta)$, and thus j would deviate as well. But then, $s^n(|\theta|) = |s^n(\theta)|$ is not a best response in game $\hat{\Gamma}^{(n,p,w^I_{N^n},\gamma,F,c,G)}$, since for $|\theta|$, any agent i prefers to deviate to $|s'(\theta)|$. So we arrive at a contradiction. It must thus be that s^n is also an equilibrium of $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$. ■

Since any equilibrium of $\Gamma_R^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ is also an equilibrium of $\Gamma^{(n,p,w^I_{N^n},\gamma,F,c,G)}$, it follows as a corollary that a symmetric pure neutral monotone equilibrium of game $\Gamma^{(n,p,w^I_{N^n},\gamma,F,c,G)}$ exists as well.

We denote arbitrary real-valued random variables by notation $\bar{v}$ with realization $v \in \mathbb{R}$, expected value $E[\bar{v}] \in \mathbb{R}$ and variance $Var[\bar{v}] \in \mathbb{R}_+$. In particular, $\bar{\theta}$ is a random variable with cumulative distribution F. For each $k \in \mathbb{N}$, let $\bar{\theta}_k$ be another, independent random variable with cumulative distribution function F, and for each $n \in \mathbb{N}\backslash\{1\}$, and for each $k \in \{1, ..., n\}$, consider the random variable $s^n(\bar{\theta}_k)$.

Denote by H^n the cumulative distribution function of the random variable $\sum_{k\in N\backslash\{i\}} s^n(\bar{\theta}_k)$.

By equilibrium symmetry, H^n does not depend on $i \in N^n$. Notice that since it is strictly dominated for any player with valuation zero to incur costs, it follows that $s^n(0) = 0$ for any $n \in \mathbb{N}\backslash\{1\}$, and further, for any $n \in \mathbb{N}\backslash\{1\}$, since s^n is monotonic and the equilibrium is symmetric, either $s^n(1) > 0$ or $s^n(-1) > 0$, because if $s^n(1) = s^n(-1) = 0$, then $s(\theta) = 0$ for any $\theta \in [-1, 1]$, and if so, any agent with $\theta_i \neq 0$ prefers to deviate to invest a positive quantity. Further, the variance of H^n is strictly positive for each $n \in \mathbb{N}\backslash\{1\}$. Note that $Var(H^n) = 0$ implies that the set $\{\theta \in [-1, 1] : s^n(\theta) \neq 0\}$ has measure zero, and thus, $\Pr\left[\sum_{j\in N\backslash\{i\}} s^n(\bar{\theta}_j) = 0\right] = 1$, in which case, any agent $i \in N^n$ with $\theta_i \in [-1, 0) \cup (0, 1]$ prefers to deviate and contribute a positive quantity. Hence, $Var(H^n) > 0$ for each $n \in \mathbb{N}\backslash\{1\}$.

As noted in the proof of Lemma 3, for any $n \in \mathbb{N}\backslash\{1\}$, any strategy $s^n \in E^{(n,p,\gamma,F,c,G)}$ is such that for any $\theta \in [-1, 1]$, $s^n(\theta) \in [-c^{-1}(2p\gamma), c^{-1}(2p\gamma)]$, because choosing $a_i \in \mathbb{R}\backslash[-c^{-1}(2p\gamma), c^{-1}(2p\gamma)]$ costs more than $2p\gamma$ nominal wealth units, or equivalently, more than 2γ real wealth units, which is the maximum real wealth that any agent is willing to pay to change the social decision from her least to her most preferred alternative. Therefore, $H^n(a) = 0$ for any $a < -(n-1)c^{-1}(2p\gamma)$ and $H^n((n-1)c^{-1}(2p\gamma)) = 1$.

Lemma 5 *For any $(n, p, \gamma, F, c, G) \in \mathbb{N}\backslash\{1\} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, for any $s^n \in E^{(n,p,\gamma,F,c,G)}$, $s^n : [0, 1] \longrightarrow \mathbb{R}$ is strictly increasing.*

Proof. Fix $(p, \gamma, F, c, G) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$. Recall $X \equiv [-c^{-1}(2p\gamma), c^{-1}(2p\gamma)]$, and for any $n \in \mathbb{N}\backslash\{1\}$ and any $x \in (n-1)X$, define $\varphi^n(x) \equiv \Pr\left[\sum_{k\in N^n\backslash\{i\}} s^n(\bar{\theta}_k) = x\right]$, and define $h^n : (n-1)X \longrightarrow \mathbb{R}_+$ as the probability density of H^n such that

$$\sum_{x\in(n-1)X} \varphi^n(x) + \int_{x\in(n-1)X} h^n(x)dx = 1.$$

Then, given any equilibrium $s^n \in E^{(n,p,\gamma,F,c,G)}$, the optimization problem of agent $i \in N^n$

with attitude $\theta_i \in [-1,1]$ is

$$\max_{a_i \in X} \gamma\theta_i \left(\sum_{x\in(n-1)X} \varphi^n(x)G(x+a_i) + \int_{x\in(n-1)X} h^n(x)G(x+a_i)dx \right)$$
$$-\gamma\theta_i \left(\sum_{x\in(n-1)X} \varphi^n(x)(1-G(x+a_i)) + \int_{x\in(n-1)X} h^n(x)(1-G(x+a_i))dx \right) - \frac{c(a_i)}{p},$$

or equivalently

$$\max_{a_i \in X} p\gamma\theta_i \left(\sum_{x\in(n-1)X} \varphi^n(x)(2G(x+a_i)-1) + \int_{x\in(n-1)X} h^n(x)(2G(x+a_i)-1)dx \right) - c(a_i).$$

Since G is continuously differentiable and the constraint $a_i \in X$ is not binding, we obtain a solution by the First Order Condition

$$2p\gamma\theta_i \left(\sum_{x\in(n-1)X} \varphi^n(x)g(x+a_i) + \int_{x\in(n-1)X} h^n(x)g(x+a_i)dx \right) = c'(a_i). \tag{2}$$

Note that since g is strictly positive in $\mathbb{R}$, and $\sum_{x\in(n-1)X} \varphi^n(x) + \int_{x\in(n-1)X} h^n(x)dx = 1$, it follows that the summation within the parenthesis on the left-hand side of Equation (2) is strictly positive for any $a_i \in X$, and thus the left hand side is overall strictly increasing in θ_i. Assume $a_i = a \in X$ is a solution to the First Order Condition (2) for agent i with attitude θ_i, and for an arbitrary agent $j \in N^n \backslash \{i\}$, assume $\theta_j \neq \theta_i$; without loss of generality assume $\theta_j > \theta_i$. Then, the left hand side of Equation (2) has a lower value than the left hand side of the analogous First Order Condition to the optimization problem of agent j. Hence, $a_j = a$ cannot solve $j's$ first order condition, so it must be $s^n(\theta_j) \neq s^n(\theta_j)$ and thus for any $\theta, \theta' \in [-1,1]$ such that $\theta \neq \theta'$ we obtain $s^n(\theta) \neq s^n(\theta')$, which, since s^n is weakly increasing, implies s^n is strictly increasing. ■

As an immediate corollary to Lemma 5, H^n does not have a mass point, so for each $n \in N \backslash \{1\}$, we can define the probability density function $h : (n-1)X \longrightarrow \mathbb{R}_+$ so that $\int_{-(n-1)X}^{x} h^n(t)dt = H^n(t)$.

Given any equilibrium $s^n \in E^{(n,p,\gamma,F,c,G)}$, the first order condition for the optimization problem of player $i \in N^n$ with attitude $\theta_i \in [-1,1]$ can be simplified to:

$$2p\gamma\theta_i \int_{x\in(n-1)X} h^n(x)g(x+a_i)dx = c'(a_i). \tag{3}$$

Lemma 7 establishes that vote acquisitions converge to zero. We use the Berry-Esseen theorem (Berry 1941, Esseen 1942), copied here for convenience.

Theorem 6 (Berry-Esseen) *For any $n \in \mathbb{N}$, let $\{\bar{x}_1, ..., \bar{x}_n\}$ be a set of n independent, identically distributed random variables with $E[\bar{x}_k] = 0$, $E[(\bar{x}_k)^2] > 0$ and $E[[|\bar{x}_k|^3] \in \mathbb{R}$ for each $k \in \{1, ..., n\}$; let F_n be the cumulative distribution function of*

$$\frac{\sum_{k=1}^{n} \bar{x}_k}{\sqrt{nE[(\bar{x}_k)^2]}},$$

and let $N[0,1](x)$ be the cumulative distribution of the standard normal distribution function evaluated at x. Then, there exists $\kappa \in \mathbb{R}_{++}$ such that for any $x \in \mathbb{R}$ and for any $n \in \mathbb{N}$,

$$|F_n(x) - N[0,1](x)| \leq \frac{\alpha E[[|\bar{x}_k|^3]}{\sqrt{n}\left(E[(\bar{x}_k)^2]\right)^{\frac{3}{2}}}.$$

Lemma 7 *For any tuple $(p, \{w_i^I\}_{i=1}^{\infty}, \gamma, F, c, G) \in \mathbb{R}_{++} \times \mathbb{R}_{+}^{\infty} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, and any sequence $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,p,\gamma,F,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$, $\lim_{n\to\infty} s^n(\theta) = 0$ for each $\theta \in (-1,1)$.*

Proof. Proof by contradiction. For any tuple $(p, \{w_i^I\}_{i=1}^{\infty}, \gamma, F, c, G) \in \mathbb{R}_{++} \times \mathbb{R}_{+}^{\infty} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, assume that $\{s^n\}_{n=2}^{\infty}$ is a sequence of monotone, symmetric, pure equilibrium strategies of game $\Gamma^{(n,p,w_{N^n}^I,\gamma,F,c,G)}$, and assume (absurd) that there exists $\theta' \in (-1,1)$ such that $\lim_{n\to\infty} s^n(\theta') \neq 0$. Then there exist a $\delta \in \mathbb{R}_{++}$ and an infinite subsequence $\{s^{n(\tau)}\}_{\tau=1}^{\infty}$ of $\{s^n\}_{n=2}^{\infty}$ with $n : \mathbb{N}\backslash\{1\} \longrightarrow \mathbb{N}$ strictly increasing, such that $|s^{n(\tau)}(\theta')| \geq \delta$ for every $\tau \in \mathbb{N}$. Note $n(\tau)$ is the size of the society in the $\tau - th$ element of the subsequence. By monotonicity of $s^{n(\tau)}(\theta)$ with respect to $\theta \in [-1,1]$ for each $\tau \in \mathbb{N}$, it follows that if $\theta' \in (-1,0)$, then $s^{n(\tau)}(\theta) \leq -\delta$ for any $\theta \in [-1,\theta']$ and for any $\tau \in \mathbb{N}$, and if $\theta' \in (0,1)$, then $s^{n(\tau)}(\theta) \geq \delta$ for any $\theta \in [\theta', 1]$.

For each $n \in \mathbb{N}\backslash\{1\}$, and for each $k \in \{1, ..., n\}$, let $E[s^n(\bar{\theta})]$ denote the expectation of the random variable $s^n(\bar{\theta}_k)$, where we drop the subindex k because the expectation does not depend on k. For each $n \in N\backslash\{1\}$ and for each $k \in \{1,..,n\}$, define as well the independent, identically distributed random variables

$$q^n(\bar{\theta}_k) \equiv s^n(\bar{\theta}_k) - E[s^n(\bar{\theta})] \text{ and } q^n(\bar{\theta}) \equiv s^n(\bar{\theta}) - E[s^n(\bar{\theta})];$$

let $E[q^n(\bar{\theta})]$ and $Var[q^n(\bar{\theta})]$ denote their expectation and variance, which do not depend on k. Note that for each $n \in \mathbb{N}\backslash\{1\}$, and for each $k \in \{1,..,n\}$, $E[q^n(\bar{\theta})] = 0$. Since

$|s^{n(\tau)}(\theta)| \geq \delta$ for every $\tau \in \mathbb{N}$ either for any $\theta \in [\theta', 1]$ or for any for any $\theta \in [-1, \theta']$, there exists $\hat{\delta} \in \mathbb{R}_{++}$ such that $Var[q^{n(\tau)}(\bar{\theta})] > \hat{\delta}$ for any $\tau \in \mathbb{N}\backslash\{1\}$. Note $Var[q^{n(\tau)}(\bar{\theta})] \equiv E\left[\left(q^{n(\tau)}(\bar{\theta})\right)^2\right] - \left(E[q^{n(\tau)}(\bar{\theta})]\right)^2 = E\left[\left(q^{n(\tau)}(\bar{\theta})\right)^2\right]$, so $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|^2\right] > \hat{\delta}$, which implies $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|\right] > 0$ and $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|^3\right] > 0$. Since $E\left[\left|q^{n(\tau)}(\bar{\theta}_k)\right|\right] = E\left[\left|q^{n(\tau)}(\bar{\theta})\right|\right]$ for any $k \in \{1, ..., n(\tau)\}$, for any $\tau \in \mathbb{N}$, let $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|^2\right]$ and $E\left[\left|q^{n(\tau)}(\bar{\theta})\right|^3\right]$ respectively denote $Var[q^{n(\tau)}(\bar{\theta}_k)]$ and $E\left[\left|q^{n(\tau)}(\bar{\theta}_k)\right|^3\right]$ for any $k \in \{1, ..., n(\tau)\}$, for any $\tau \in \mathbb{N}$.

For each $\tau \in \mathbb{N}$, define $V^{\tau}(\bar{\theta}_{N^{n(\tau)}\backslash\{i\}})$ as the cumulative distribution of the random variable $\frac{\sum_{k \in N^{n(\tau)}\backslash\{i\}} q^{n(\tau)}(\bar{\theta}_k)}{\sqrt{n(\tau)-1}\sqrt{E[\left(q^{n(\tau)}(\bar{\theta})\right)^2]}}$. By the Berry-Esseen theorem (Berry 1941, Esseen 1942), there exists a $\kappa \in \mathbb{R}_{++}$ such that for any $\tau \in \mathbb{N}$ and any $x \in \mathbb{R}$,

$$|V^{\tau}(x) - N[0,1](x)| \leq \frac{\kappa E[\left|q^{n(\tau)}(\bar{\theta})\right|^3]}{\left(\sqrt{n(\tau)-1}\right)\left(E[\left(q^{n(\tau)}(\bar{\theta})\right)^2]\right)^{\frac{3}{2}}}.$$

For each $\tau \in \mathbb{N}$, define $\hat{H}^{\tau}(\bar{\theta}_{N^{n(\tau)}\backslash\{i\}})$ as the cumulative distribution of the random variable $\sum_{k \in N^{n(\tau)}\backslash\{i\}} q^{n(\tau)}(\bar{\theta}_k)$, and let $\hat{h}^{\tau}(\bar{\theta}_{N^{n(\tau)}\backslash\{i\}})$ be its density function. For any $z \in \mathbb{R}_{++}$ and any $x \in \mathbb{R}$, let $N[0,z](x)$ denote value at x of the cumulative distribution of a normal distribution with mean zero and variance z. Then,

$$\left|\hat{H}^{\tau}(x) - N[0, E[\left(q^{n(\tau)}(\bar{\theta})\right)^2]\,(n(\tau)-1)](x)\right| < \frac{\kappa E[\left|q^{n(\tau)}(\bar{\theta})\right|^3]}{\left(\sqrt{n(\tau)-1}\right)\hat{\delta}^{\frac{3}{2}}}, \tag{4}$$

Since $\{s^n(\bar{\theta})\}_{n=1}^{\infty}$ is bounded for any $n \in \mathbb{N}\backslash\{1\}$, both $\{E[s^{n(\tau)}(\bar{\theta})]\}_{n=1}^{\infty}$ and $\{q^{n(\tau)}(\bar{\theta})\}_{n=1}^{\infty}$ are bounded as well for any $\tau \in \mathbb{N}$, and hence $\{E[\left|q^{n(\tau)}(\bar{\theta})\right|^3]\}_{\tau=1}^{\infty}$ is bounded, and the right hand side of Inequality (4) converges to zero as τ diverges to infinity. Thus, the random variable $\sum_{k \in N^{n(\tau)}\backslash\{i\}} q_k^{n(\tau)}(\bar{\theta}) = \sum_{k \in N^{n(\tau)}\backslash\{i\}} \left(s_k^{n(\tau)}(\bar{\theta}) - E[s^{n(\tau)}(\bar{\theta})]\right)$ with cumulative distribution $\hat{H}^{\tau}(x)$ converges as $\tau \longrightarrow \infty$ to a mean zero Normal distribution with variance $E[\left(q^{n(\tau)}(\bar{\theta})\right)^2]\,(n(\tau)-1)$. Since $E[\left(q^{n(\tau)}(\bar{\theta})\right)^2] \geq \hat{\delta}$ for any $\tau \in \mathbb{N}$, it follows that $E[\left(q^{n(\tau)}(\bar{\theta})\right)^2]\,(n(\tau)-1)$ diverges to infinity as $\tau \longrightarrow \infty$. Therefore,

$$\lim_{\tau \longrightarrow \infty}\left(\hat{H}^{\tau}(x) - \hat{H}^{\tau}(-x)\right) = 0 \text{ for any } x \in \mathbb{R}_{++}. \tag{5}$$

Since G is strictly increasing and neutral ($G(x) = 1 - G(-x)$), and $\lim_{x \longrightarrow -\infty} G(x) = 0$, then for any $\varepsilon \in \left(0, \frac{1}{2}c(\delta)\right)$, there exist $\tilde{x} \in \mathbb{R}_{++}$ such that for any $x \in (-\infty, -\tilde{x}] \cup [\tilde{x}, \infty)$,

$$[G(x + c^{-1}(2p\gamma)) - G(x)]2p\gamma\theta' < \frac{1}{2}c(\delta) - \varepsilon$$

Since $|s^{n(\tau)}(\theta')| \geq \delta$ for every $\tau \in \mathbb{N}$ (first paragraph of this proof), it then follows that

$$[G(x + c^{-1}(2p\gamma)) - G(x)]2p\gamma\theta' < \frac{1}{2}c(s^{n(\tau)}(\theta')) - \varepsilon$$

for any $x \in (-\infty, -\tilde{x}] \cup [x, \infty)$. Further, since $\left|s^{n(\tau)}(\theta')\right| \leq c^{-1}(2p\gamma)$ (because $\left|s^{n(\tau)}(\theta')\right| > c^{-1}(2p\gamma)$ implies that $s_i = s^{n(\tau)}$ is a strictly dominated strategy), it follows that for any $x \in (-\infty, -\tilde{x}] \cup [\tilde{x}, \infty)$,

$$[G(x + s^{n(\tau)}(\theta')) - G(x)]2p\gamma\theta' < \frac{1}{2}c(s^{n(\tau)}(\theta')) - \varepsilon. \tag{6}$$

For each $\tau \in \mathbb{N}$, and for any arbitrary agent $i \in N^{n(\tau)}$ with $\theta_i = \theta'$, the expected utility of playing $a_i = s^{n(\tau)}(\theta')$, minus the expected utility of playing $a_i = 0$, is:

$$\begin{aligned}
&2\gamma\theta' \int_{-(n-1)c^{-1}(2p\gamma)}^{-\tilde{x}} (G(x + s^{n(\tau)}(\theta')) - G(x))h^{\tau}(x)dx \\
&+2\gamma\theta' \int_{-\tilde{x}}^{\tilde{x}} (G(x + s^{n(\tau)}(\theta')) - G(x))h^{\tau}(x)dx \\
&+2\gamma\theta' \int_{\tilde{x}}^{(n-1)c^{-1}(2p\gamma)} (G(x + s^{n(\tau)}(\theta')) - G(x))h^{\tau}(x)dx - \frac{1}{p}c(s^{n(\tau)}(\theta')),
\end{aligned}$$

which is equal to

$$\begin{aligned}
&2\gamma\theta' \int_{-(n-1)\left(c^{-1}(2p\gamma)+E[s^n(\bar{\theta})]\right)}^{-\tilde{x}} (G(x + s^{n(\tau)}(\theta')) - G(x))\hat{h}^{\tau}(x)dx \\
&+2\gamma\theta' \int_{-\tilde{x}}^{\tilde{x}} (G(x + s^{n(\tau)}(\theta')) - G(x))\hat{h}^{\tau}(x)dx \\
&+2\gamma\theta' \int_{\tilde{x}}^{(n-1)\left(c^{-1}(2p\gamma)-E[s^n(\bar{\theta})]\right)} (G(x + s^{n(\tau)}(\theta')) - G(x))\hat{h}^{\tau}(x)dx - \frac{1}{p}c(s^{n(\tau)}(\theta')).
\end{aligned} \tag{7}$$

By Expression (5), $\lim_{\tau \longrightarrow \infty} \left(\hat{H}^{\tau}(-\tilde{x}) - \hat{H}^{\tau}(\tilde{x})\right) = 0$, and thus $\lim_{\tau \longrightarrow \infty} \hat{h}^{\tau}(x) = 0$ for any $x \in (-\tilde{x}, \tilde{x})$, and hence

$$\lim_{\tau \longrightarrow \infty} 2\gamma\theta' \int_{-\tilde{x}}^{\tilde{x}} (G(x + s^{n(\tau)}(\theta')) - G(x))\hat{h}^{\tau}(x)dx = 0.$$

Therefore, the limit of Expression (7) as $\tau \longrightarrow \infty$ is equal to the limit of

$$\begin{aligned}
&2\gamma\theta' \int_{-(n-1)\left(c^{-1}(2p\gamma)+E[s^n(\bar{\theta})]\right)}^{-\tilde{x}} (G(x + s^{n(\tau)}(\theta')) - G(x))\hat{h}^{\tau}(x)dx \\
&+2\gamma\theta' \int_{\tilde{x}}^{(n-1)\left(c^{-1}(2p\gamma)-E[s^n(\bar{\theta})]\right)} (G(x + s^{n(\tau)}(\theta')) - G(x))\hat{h}^{\tau}(x)dx - \frac{1}{p}c(s^{n(\tau)}(\theta')),
\end{aligned}$$

which by Expression (6), is strictly smaller than

$$
\begin{aligned}
& \int_{-(n-1)\left(c^{-1}(2p\gamma)+E[s^n(\bar{\theta})]\right)}^{-\tilde{x}} \left(\frac{1}{2p}c(s^{n(\tau)}(\theta'))-\varepsilon\right)\hat{h}^{\tau}(x)dx \\
+ \quad & \int_{\tilde{x}}^{(n-1)\left(c^{-1}(2p\gamma)-E[s^n(\bar{\theta})]\right)} \left(\frac{1}{2p}c(s^{n(\tau)}(\theta'))-\varepsilon\right)\hat{h}^{\tau}(x)dx - \frac{1}{p}c(s^{n(\tau)}(\theta')) \\
< \quad & \frac{1}{p}c(s^{n(\tau)}(\theta'))-\varepsilon-\frac{1}{p}c(s^{n(\tau)}(\theta')) < -\varepsilon,
\end{aligned}
$$

so playing $a_i = 0$ is strictly better, and hence $s_i = s^{n(\tau)}(\theta')$ is not a best response, so $s^{n(\tau)}$ is not an equilibrium. Thus, we reach a contradiction. Thus, there does not exist $\theta' \in (-1,1)$ such that $\lim_{n\to+\infty} s^n(\theta') \neq 0$, and it must be that $\lim\limits_{n\to+\infty} s^n(\theta) = 0$ for each $\theta \in (-1,1)$. ■

The next lemma reformulates the First Order Condition (3) into a form that proves more convenient for subsequent results. Recall we use the notation $X \equiv [-c^{-1}(2p\gamma), c^{-1}(2p\gamma)]$, so $(n-1)X = [-(n-1)c^{-1}(2p\gamma), (n-1)c^{-1}(2p\gamma)]$.

Lemma 8 *For any tuple $(p, \{w_i^I\}_{i=1}^{\infty}, \gamma, F, c, G) \in \mathbb{R}_{++} \times \mathbb{R}_{+}^{\infty} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, for any sequence $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,p,\gamma,F,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$, for any $n \in \mathbb{N}\backslash\{1\}$, and for each $\theta \in [-1,1]$, there exists $z^{\theta} : (n-1)X \longrightarrow [s^n(\theta), 0) \cup (0, s^n(\theta)]$ such that $sgn(z^{\theta}(x)) = sgn(\theta)$ for any $x \in [-(n-1)X, (n-1)X]$, and*

$$
c'(s^n(\theta)) = 2p\gamma\theta\left(\int\limits_{x\in(n-1)X} g(x)h^n(x)dx + s^n(\theta)\int\limits_{x\in(n-1)X} g'(x+z^{\theta})h^n(x)dx\right). \qquad (8)
$$

Proof. For any given $n \in \mathbb{N}\backslash\{1\}$, only a compact subset of the domain of G, namely $[-nX, nX]$ is relevant, since $ns^n(\theta) \in nX$ for any θ. And G is twice continuously differentiable. Note that by the First Order Condition (3), for each $\theta \in [-1,1]$,

$$
c'(s^n(\theta)) = 2p\gamma\theta\int\limits_{x\in(n-1)X} g(x+s^n(\theta))h^n(x)dx.
$$

We want to show that for any $x \in (n-1)X$, and any $\theta \in [0,1]$, there exists a $z^{\theta}(x) \in (0, s^n(\theta))$ such that

$$
g(x+s^n(\theta)) = g(x) + s^n(\theta)g'(x+z^{\theta}(x)). \qquad (9)
$$

For each $x \in (n-1)X$, define $y_{\min} \equiv \arg\min\limits_{y\in[x,x+s^n(\theta)]} g'(y)$ and $y_{\max} \equiv \arg\max\limits_{y\in[x,x+s^n(\theta)]} g'(y)$. Then note

$$
(s^n(\theta))g'(y_{\min}) \leq g(x+s^n(\theta)) - g(x) \leq (s^n(\theta))g'(y_{\max})
$$

Since g is continuous, by the Intermediate Value Theorem, there exists some value $y(x) \in [x, x + s^n(\theta)]$ such that

$$(s^n(\theta))g'(y(x)) = g(x + s^n(\theta)) - g(x).$$

Then, define $z^\theta(x) \equiv y(x) - x$ and we obtain Equality (9).

An analogous argument, in this instance with $y(x) \in [x + s^n(\theta), x]$, establishes that for any $\theta \in [-1, 0]$, there exists a $z^\theta(x) \in [s^n(\theta), 0]$ such that Equality (9) holds. ■

The next lemma uses Lemma 8 to establish that the ratio of marginal costs of two agents converges to their ratio of attitudes.

Lemma 9 *For any tuple $(p, \{w_i^I\}_{i=1}^\infty, \gamma, F, c, G) \in \mathbb{R}_{++} \times \mathbb{R}_+^\infty \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, for any sequence of equilibria $\{s^n\}_{n=2}^\infty$, and for any $(\theta, \hat{\theta}) \in (-1, 1)^2$,*

$$\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\theta}{\hat{\theta}}.$$

Proof. For any tuple $(p, \{w_i^I\}_{i=1}^\infty, \gamma, F, c, G) \in \mathbb{R}_{++} \times \mathbb{R}_+^\infty \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, let $\{s^n\}_{n=2}^\infty$ be a sequence of equilibria, that is, $s^n \in E^{(n,p,\gamma,F,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$.

From Lemma 8, for each $\theta \in [-1, 1]$,

$$c'(s^n(\theta)) = 2p\gamma\theta \left(\int\limits_{x\in(n-1)X} g(x)h^n(x)dx + s^n(\theta) \int\limits_{x\in(n-1)X} g'(x + z^\theta(x))h^n(x)dx \right).$$

Notice that since g is strictly positive and continuous, and g' is continuous, for any $x, y \in \mathbb{R}$, $\frac{g'(y)}{g(x)}$ is continuous, and over any closed interval of $\mathbb{R}$, it is bounded. Further, by Condition (iii) of the definition of $\mathcal{G}$, $\exists \hat{\varepsilon} \in \mathbb{R}_{++}$ such that for any $\varepsilon \in (0, \hat{\varepsilon})$,

$$\lim_{x\to-\infty} \frac{g'(x+\varepsilon)}{g(x)} \in \mathbb{R} \text{ and } \lim_{x\to\infty} \frac{g'(x+\varepsilon)}{g(x)} \in \mathbb{R}. \tag{10}$$

Therefore, there exists $\kappa \in \mathbb{R}_{++}$ such that $\frac{g'(x+\varepsilon)}{g(x)} \in [-\kappa, \kappa]$, for any $\varepsilon \in (0, \bar{\varepsilon})$ and for any $x \in \mathbb{R}$. Equivalently,

$$-\kappa g(x) \leq g'(x + \varepsilon) \leq \kappa g(x) \; \forall \varepsilon \in (0, \bar{\varepsilon}), \; \forall x \in \mathbb{R}. \tag{11}$$

Since for any sequence $\{s^n\}_{n=1}^\infty$ of equilibria $\lim\limits_{n\to\infty} s^n(\theta) = 0$ for each $\theta \in (-1, 1)$ (Lemma 7), and since $z^\theta(x)$ defined in Lemma 8 satisfies $z^\theta(x) \in (0, s^n(\theta))$, it follows $\lim\limits_{n\to\infty} z^\theta(x) = 0$ for each $\theta \in [-1, 1]$ and for each $x \in (n-1)X$. Then, it follows from Expression (11), that that that there exists $\hat{n} \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ such that

$n > \hat{n}$, for each $x \in (n-1)X$, for any $\theta \in (-1,0) \cup (0,1)$, and for any equilibrium strategy s^n, we have:

$$-\kappa g(x) < g'(x + z^\theta(x)) < \kappa g(x).$$

Therefore,

$$\begin{aligned} g(x) - s^n(\theta)\kappa g(x) &< g(x) + s^n(\theta) g'(x + z^\theta(x)) < g(x) + s^n(\theta)\kappa g(x); \\ [1 - s^n(\theta)\kappa] g(x)\theta h^n(x) &< (g(x) + s^n(\theta) g'(x + z^\theta(x)))\theta h^n(x) < (1 + s^n(\theta)\kappa) g(x)\theta h^n(x). \end{aligned}$$

Once again since $\lim_{n\to\infty} s^n(\theta) = 0$ for each $\theta \in (-1,1)$ (Lemma 7), there exists $\tilde{n}$ such that $1 - s^n(\theta)\kappa > 0$ for every $n > \tilde{n}$.

Then we can integrate x over $(n-1)X$ on all sides and multiply by $2p\gamma$ to obtain:

$$\begin{aligned} & 2p\gamma[1 - s^n(\theta)\kappa]\theta \int\limits_{x\in(n-1)X} g(x)h^n(x)dx \\ < \; & 2p\gamma\theta \int\limits_{x\in(n-1)X} (g(x) + s^n(\theta)g'(x + z^\theta(x)))h^n(x)dx \\ < \; & 2p\gamma(1 + s^n(\theta)\kappa)\theta \int\limits_{x\in(n-1)X} g(x)h^n(x)dx, \end{aligned}$$

and hence, substituting Equality (8), for any $\theta \in (-1,0) \cup (0,1)$,

$$c'(s^n(\theta)) \in \left(2p\gamma(1 - s^n(\theta)\kappa)\theta \int\limits_{x\in(n-1)X} g(x)h^n(x)dx,\; 2p\gamma(1 + s^n(\theta)\kappa)\theta \int\limits_{x\in(n-1)X} g(x)h^n(x)dx \right). \tag{12}$$

Then, for any $\theta, \hat{\theta} \in (-1,0) \cup (0,1)$,

$$\begin{aligned} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} &\in \left(\frac{(1 - s^n(\theta)\kappa)\theta \int_{x\in(n-1)X} g(x)h^n(x)dx}{(1 + s^n(\hat{\theta})\kappa)\hat{\theta} \int_{x\in(n-1)X} g(x)h^n(x)dx}, \frac{(1 + s^n(\theta)\kappa)\theta \int_{x\in(n-1)X} g(x)h^n(x)dx}{(1 - s^n(\theta)\kappa)\hat{\theta} \int_{x\in(n-1)X} g(x)h^n(x)dx} \right) \\ &= \left(\frac{(1 - s^n(\theta)\kappa)\theta}{(1 + s^n(\hat{\theta})\kappa)\hat{\theta}}, \frac{(1 + s^n(\theta)\kappa)\theta}{(1 - s^n(\theta)\kappa)\hat{\theta}} \right). \end{aligned}$$

Note that because $\lim_{n\longrightarrow\infty} s^n(\tilde{\theta}) = 0$ for any $\tilde{\theta} \in (-1,0) \cup (0,1)$ (Lemma 7) and $s^n(0) = 0$ for any $n \in \mathbb{N}$, both limit points of the interval converge to $\frac{\theta}{\hat{\theta}}$ as n increases to infinity. Hence, for any $(\theta, \hat{\theta}) \in (-1,1)^2$,

$$\lim_{n\longrightarrow\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\theta}{\hat{\theta}}.$$

■

The next lemma proves the following observation: a cost elasticity greater than one near zero implies that the cost function is convex near zero.

Lemma 10 *For any $c \in C$, there exists $\lambda_c \in \mathbb{R}_{++}$ such that $c''(a) \in \mathbb{R}_{++}$ for any $a \in (0, \lambda_c]$.*

Proof. By definition of C, $c \in C$ implies that $\lim\limits_{a \longrightarrow 0} \frac{ac'(a)}{c(a)} \in (1, \mathbb{R})$, $c(0) = 0$ and $\lim\limits_{a \longrightarrow 0} ac'(a) = 0$. Let $z \equiv \lim\limits_{a \longrightarrow 0} \frac{ac'(a)}{c(a)}$. Then $\lim\limits_{a \longrightarrow 0} \frac{ac'(a)}{c(a)} = \frac{0}{0}$; applying L'Hopital rule,

$$z = \lim_{a \longrightarrow 0} \frac{ac'(a)}{c(a)} = \lim_{a \longrightarrow 0} \left(1 + \frac{ac''(a)}{c'(a)}\right)$$

so

$$\lim_{a \longrightarrow 0} \frac{ac''(a)}{c'(a)} = z - 1,$$

Hence, for any $\varepsilon \in \mathbb{R}_{++}$, there exists $\lambda_\varepsilon \in \mathbb{R}_{++}$ such that for any $a \in (0, \lambda_\varepsilon]$,

$$\frac{ac''(a)}{c'(a)} \in (z - 1 - \varepsilon, z - 1 + \varepsilon). \tag{13}$$

Select $\varepsilon = \frac{z-1}{2}$, and since $z > 1$, note that $z - 1 - \varepsilon > 0$. Further, for any $a \in \left(0, \lambda_{\frac{z-1}{2}}\right]$, by assumption $c'(a) > 0$. Thus, from Expression (13), it follows $c''(a) > \frac{c'(a)}{a}(\frac{z-1}{2}) > 0$ for any $a \in \left(0, \lambda_{\frac{z-1}{2}}\right]$. ■

Next we establish that the marginal effect of acquiring votes over the outcome converges to zero (Lemma 11).

Lemma 11 *For any tuple $(p, \{w_i^I\}_{i=1}^{\infty}, \gamma, F, c, G) \in \mathbb{R}_{++} \times \mathbb{R}_+^{\infty} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, and for any sequence of equilibria $\{s^n\}_{n=2}^{\infty}$,*

$$\lim_{n \longrightarrow \infty} \int\limits_{x \in (n-1)X} g(x) h^n(x) dx = 0.$$

Proof. By Lemma 10, there exists a $\lambda \in \mathbb{R}_{++}$ such that c' is strictly increasing in $(0, \lambda]$. Therefore, c' is invertible over $(0, \lambda]$. Let $(c')^{-1}$ denote the inverse of c' over $(0, \lambda]$. Then, for any $\theta \in (-1, 1)$, from Expression (12) in the proof of Lemma 9,

$$s^n(\theta) \in \left(\begin{array}{c} (c')^{-1}\left(2p\gamma(1 - s^n(\theta)\kappa)\theta \int\limits_{x \in (n-1)X} g(x)h^n(x)dx\right), \\ (c')^{-1}\left(2p\gamma(1 + s^n(\theta)\kappa)\theta \int\limits_{x \in (n-1)X} g(x)h^n(x)dx\right) \end{array} \right)$$

and, since $\lim_{n\longrightarrow\infty} s^n(\theta) = 0$ for any $\theta \in (-1,1)$ (Lemma 7), it follows that

$$\lim_{n\longrightarrow\infty} (c')^{-1}\left(2p\gamma(1-s^n(\theta)\kappa)\theta \int\limits_{x\in(n-1)X} g(x)h^n(x)dx\right) = 0,$$

which, since $c'(0) = 0$ and thus $(c')^{-1}(0) = 0$, implies

$$\lim_{n\longrightarrow\infty} \left(2p\gamma(1-s^n(\theta)\kappa)\theta \int\limits_{x\in(n-1)X} g(x)h^n(x)dx\right) = 0,$$

which, for any $\theta \in (-1,1)\backslash\{0\}$, implies

$$\lim_{n\longrightarrow\infty} \int\limits_{x\in(n-1)X} g(x)h^n(x)dx = 0.$$

■

Lemma 11 allows us to more easily strengthen Lemma 7 by showing that vote acquisitions converge to zero for every realization of attitudes, including $\theta \in \{-1,1\}$.

Lemma 12 *For any $(p, \{w_i^I\}_{i=1}^{\infty}, \gamma, F, c, G) \in \mathbb{R}_{++} \times \mathbb{R}_{+}^{\infty} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, and any sequence $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,p,\gamma,F,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$, and for any $\theta \in [-1,1]$, $\lim_{n\to\infty} s^n(\theta) = 0$.*

Proof. $\lim_{n\to\infty} s^n(\theta) = 0$ for any $\theta \in (-1,1)$ by Lemma 7. For $\theta_i \in \{-1,1\}$, note that the First Order Condition (3) for agent i is

$$2p\gamma\theta_i \int\limits_{x\in(n-1)X} h^n(x)g(x+a_i)dx = c'(a_i),$$

By definition of $\mathcal{G}$, and since $G \in \mathcal{G}$, G is strictly increasing and continuously differentiable, thus g is continuous and strictly positive, and hence g and $\frac{g(x+a_i)}{g(x)}$ are bounded over any closed interval of $\mathbb{R}$. Further, also by definition of $\mathcal{G}$, $\exists\hat{\varepsilon} \in \mathbb{R}_{++}$ such that $\lim_{x\to-\infty} \frac{g'(x+\varepsilon)}{g(x)} \in \mathbb{R}$ and $\lim_{x\to\infty} \frac{g'(x+\varepsilon)}{g(x)} \in \mathbb{R}$ for any $\varepsilon \in [0,\hat{\varepsilon})$. In particular, for $\varepsilon = 0$, $\frac{g'(x)}{g(x)}$ is bounded over $\mathbb{R}$, and $\frac{g(x)+\int_x^{x+a_i} g'(t)dt}{g(x)} = \frac{g(x+a_i)}{g(x)}$ is bounded over $\mathbb{R}$ as well, so there exists some $K \in \mathbb{R}_{++}$ such that $g(x+a_i) \le Kg(x)$ and

$$\int\limits_{x\in(n-1)X} h^n(x)g(x+a_i)dx \le K \int\limits_{x\in(n-1)X} h^n(x)g(x+a_i)dx$$

and hence, by Lemma 11,

$$\lim_{n\longrightarrow\infty}\int\limits_{x\in(n-1)X} h^n(x)g(x+a_i)dx=0$$

so

$$\lim_{n\longrightarrow\infty}2p\gamma\theta_i\int\limits_{x\in(n-1)X} h^n(x)g(x+a_i)dx=\lim_{n\longrightarrow\infty}c'(a_i)=0,$$

so $\lim\limits_{n\longrightarrow\infty}a_i=0.$ ■

As a corollary of Lemma 12, we can more strengthen Lemma 9 so that it holds for any $(\theta,\hat{\theta})\in[-1,1]^2$.

Corollary 13 *For any tuple $(p,\{w_i^I\}_{i=1}^{\infty},\gamma,F,c,G)\in\mathbb{R}_{++}\times\mathbb{R}_+^{\infty}\times\mathbb{R}_{++}\times\mathcal{F}\times C\times\mathcal{G}$, for any sequence of equilibria $\{s^n\}_{n=2}^{\infty}$, and for any $(\theta,\hat{\theta})\in[-1,1]^2$,*

$$\lim_{n\rightarrow\infty}\frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))}=\frac{\theta}{\hat{\theta}}.$$

The proof follows step-by-step the proof of Lemma 9, noting, where needed, that $\lim\limits_{n\rightarrow\infty}s^n(\theta)=0$ for $\theta\in\{-1,1\}$ by Lemma 12. We next define an auxiliary function and prove a lemma related to it. Define $J:\mathbb{R}_{++}^2\longrightarrow\mathbb{R}_+$ by

$$J(x,y)=\begin{cases}\frac{yc''(y)}{c'(y)} & \text{if } x=y\\ \frac{\ln c'(x)-\ln c'(y)}{\ln x-\ln y} & \text{otherwise}\end{cases}.$$

Lemma 14 *Let $\{x_n\}_{n=1}^{\infty}\in\mathbb{R}_{++}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}\in\mathbb{R}_{++}^{\infty}$ be two converging sequences with $\lim\limits_{n\longrightarrow\infty}x_n=\lim\limits_{n\longrightarrow\infty}y_n=0$ and define $z\equiv\lim\limits_{x\longrightarrow0}\frac{xc'(x)}{c(x)}$. Then $\lim\limits_{n\longrightarrow\infty}J(x_n,y_n)=z-1$.*

Proof. Note that for any $y\in\mathbb{R}_{++}$,

$$\lim_{x\longrightarrow0}J(x,y)=\frac{\ln c'(0)-\ln c'(y)}{\ln 0-\ln y}=\frac{-\infty}{-\infty},$$

applying L'Hopital rule,

$$\lim_{x\longrightarrow0}J(x,y)=\lim_{x\longrightarrow0}\frac{\frac{c''(x)}{c'(x)}}{\frac{1}{x}}=\lim_{x\longrightarrow0}\frac{xc''(x)}{c'(x)}.$$

Notice that $z \equiv \lim\limits_{x \longrightarrow 0} \frac{xc'(x)}{c(x)} = \frac{0}{0}$, so applying L'Hopital rule,

$$\begin{aligned} z &= \lim_{x \longrightarrow 0} \frac{c'(x) + xc''(x)}{c'(x)} = 1 + \lim_{x \longrightarrow 0} \frac{xc''(x)}{c'(x)} \\ z - 1 &= \lim_{x \longrightarrow 0} \frac{xc''(x)}{c'(x)}, \end{aligned} \tag{14}$$

so $\lim\limits_{x \longrightarrow 0} J(x, y) = z - 1$. Note as well that, using L'Hopital rule

$$\lim_{\varepsilon \longrightarrow 0} J(x, x + \varepsilon) = \frac{-\frac{c''(x)}{c'(x)}}{-\frac{1}{x}} = \frac{xc''(x)}{c'(x)}$$

so J is continuous.

Define the function $v : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$v(x) = \begin{cases} z - 1 & \text{if } x = 0 \\ \frac{xc''(x)}{c'(x)} & \text{if } x \in \mathbb{R}_{++} \end{cases}.$$

By Equality (14), $\lim\limits_{x \longrightarrow 0} \frac{xc''(x)}{c'(x)} = z - 1$ and hence $\lim\limits_{x \longrightarrow 0} v(x) = z - 1$ and v is continuous.

Define the correspondence $x^+ : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ by $x^+(w) = \arg \max\limits_{x \in [0,w]} v(x)$ for each $w \in \mathbb{R}_+$, and the correspondence $x^- : \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ by $x^-(w) = \arg \min\limits_{x \in [0,w]} v(x)$ for each $w \in \mathbb{R}_+$, and define the function $v^+ : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $v^+(w) = \max\limits_{x \in [0,w]} v(x)$ for each $w \in \mathbb{R}_+$ and the function $v^- : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $v^-(w) \equiv \min\limits_{x \in [0,w]} v(x)$ for each $w \in \mathbb{R}_+$. Since v is continuous, $x^+(w)$ and $x^-(w)$ are non-empty for each $w \in \mathbb{R}_+$, x^+ and x^- are upper hemi continuous, and v^+ and v^- are continuous (Berge's maximum theorem). Further, note that v^+ is non-decreasing and v^- is non-increasing.

Construct two sequences $\{x_t\}_{t=1}^{\infty} \in \mathbb{R}_+^{\infty}$ and $\{y_t\}_{t=1}^{\infty} \in \mathbb{R}_+^{\infty}$ such that $\lim\limits_{t \longrightarrow \infty} x_t = \lim\limits_{t \longrightarrow \infty} y_t = 0$. Then

$$\lim_{t \longrightarrow 0} \frac{x_t c''(x_t)}{c'(x_t)} = \lim_{t \longrightarrow 0} \frac{y_t c''(y_t)}{c'(y_t)} = z - 1.$$

Note that for any $y \in \mathbb{R}_{++}$, and for any $x \in (0, y)$, J is differentiable and

$$\begin{aligned} \frac{\partial J}{\partial x}(x, y) &= \frac{\frac{c''(x)}{c'(x)}(\ln x - \ln y) - (\ln c'(x) - \ln(c'(y))\frac{1}{x}}{(\ln x - \ln y)^2} \\ &= \frac{xc''(x)(\ln x - \ln y) - c'(x)(\ln c'(x) - \ln(c'(y))}{xc'(x)\,(\ln x - \ln y)^2}. \end{aligned}$$

Hence $\frac{\partial J}{\partial x}(x,y)=0$ if and only if

$$\begin{aligned} xc''(x)(\ln x-\ln y) &= c'(x)(\ln c'(x)-\ln(c'(y)) \\ \frac{xc''(x)}{c'(x)} &= \frac{\ln c'(x)-\ln c'(y)}{\ln x-\ln y}, \end{aligned}$$

that is, $\frac{\partial J}{\partial x}(x,y)=0$ if and only if $J(x,y)=\frac{xc''(x)}{c'(x)}$.

Since $x\in\arg\max_{x\in(0,y)} J(x,y)$ implies $\frac{\partial J}{\partial x}(x,y)=0$, it follows that for any $y\in\mathbb{R}_{++}$ and any $x\in\arg\max_{x\in(0,y)} J(x,y)$, $J(x,y)=v(x)$, so $J(x,y)\leq v^{+}(x)$. Since v^{+} is non-decreasing, it follows $\max_{x\in(0,y)} J(x,y)\leq v^{+}(y)$. If $\arg\max_{x\in(0,y)} J(x,y)=\varnothing$, then $\sup_{x\in(0,y)} J(x,y)\in\left\{\lim_{x\longrightarrow 0} J(x,y),J(y,y)\right\}=\{z-1,v(y)\}\leq v^{+}(y)$. So $\sup_{x\in(0,y)} J(x,y)\leq v^{+}(y)$ for any $y\in\mathbb{R}_{++}$. Similarly, it can be shown that $\sup_{y\in(0,x)} J(x,y)\leq v^{+}(x)$ for any $x\in\mathbb{R}_{++}$.

Moreover, since $x\in\arg\min_{x\in(0,y)} J(x,y)$ implies $\frac{\partial J}{\partial x}(x,y)=0$, it follows that for any $y\in\mathbb{R}_{++}$ and any $x\in\arg\min_{x\in(0,y)} J(x,y)$, $J(x,y)=v(x)$, so $J(x,y)\geq v^{-}(x)$. Since v^{-} is non-decreasing, it follows $\max_{x\in(0,y)} J(x,y)\geq v^{-}(y)$. If $\arg\min_{x\in(0,y)} J(x,y)=\varnothing$, then $\inf_{x\in(0,y)} J(x,y)\in\{\lim_{x\longrightarrow 0} J(x,y),J(y,y)\}=\{z-1,v(y)\}\geq v^{-}(y)$. So $\inf_{x\in(0,y)} J(x,y)\geq v^{-}(y)$ for any $y\in\mathbb{R}_{++}$. Similarly, it can be shown that $\sup_{y\in(0,x)} J(x,y)\geq v^{-}(y)$ for any $x\in\mathbb{R}_{++}$.

From all the above it follows that for any $t\in\mathbb{N}$, $J(x_t,y_t)\in[v^{-}(w_t),v^{+}(w_t)]$, where $w_t=\max\{x_t,y_t\}$. Notice that $\lim_{t\longrightarrow\infty} w_t=0$, and thus $\lim_{t\longrightarrow 0} v^{-}(w_t)=z-1$ and $\lim_{t\longrightarrow 0} v^{+}(w_t)=z-1$, and hence $\lim_{n\longrightarrow\infty} J(x_n,y_n)=z-1$. ■

We next establish a key intermediary result: equilibrium actions are asymptotically piecewise linear in $(\theta)^{\rho}$.

Lemma 15 *For any tuple $(p,\{w_i^I\}_{i=1}^{\infty},\gamma,F,c,G)\in\mathbb{R}_{++}\times\mathbb{R}_{+}^{\infty}\times\mathbb{R}_{++}\times\mathcal{F}\times C\times\mathcal{G}$, define $z\equiv\lim_{x\longrightarrow 0}\frac{xc'(x)}{c(x)}$. Then, for any $\{s^n\}_{n=1}^{\infty}$ such that $s^n\in E^{(n,p,\gamma,F,c,G)}$ for each $n\in\mathbb{N}\backslash\{1\}$, and for any $(\theta,\hat{\theta})^2\in[-1,0)^2\cup(0,1]^2$,*

$$\lim_{n\to\infty}\frac{s^n(\theta)}{s^n(\hat{\theta})}=\left(\frac{\theta}{\hat{\theta}}\right)^{\frac{1}{z-1}}.$$

Proof. For any $(\theta,\hat{\theta})\in[-1,0)^2\cup(0,1]^2$, by Lemma 9 and Corollary 13, $\lim_{n\to\infty}\frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))}=\frac{\theta}{\hat{\theta}}$, and taking logarithms on both sides,

$$\lim_{n\to\infty}(\ln c'(s^n(\theta))-\ln c'(s^n(\hat{\theta}))=\ln\left(\frac{\theta}{\hat{\theta}}\right). \tag{15}$$

By Lemma 14, for any $\{x_n\}_{n=1}^{\infty} \in \mathbb{R}_{++}^{\infty}$ with $\lim_{n\to\infty} x_n = 0$ and $\{y_n\}_{n=1}^{\infty} \in \mathbb{R}_{++}^{\infty}$ with $\lim_{n\to\infty} y_n = 0$,

$$\lim_{n\to\infty} \frac{\ln c'(x_n) - \ln c'(y_n)}{\ln \frac{x_n}{y_n}} = z - 1,$$

thus, in particular,

$$\begin{aligned} \lim_{n\to\infty} \frac{\ln c'(s^n(\theta)) - \ln c'(s^n(\hat{\theta}))}{\ln \frac{s^n(\theta)}{s^n(\hat{\theta})}} &= z - 1, \\ \lim_{n\to\infty} \left(\ln c'(s^n(\theta)) - \ln c'(s^n(\hat{\theta}))\right) &= \lim_{n\to\infty} \ln \left(\frac{s^n(\theta)}{s^n(\hat{\theta})}\right)^{z-1} \end{aligned}$$

and thus substituting the left hand side according to Equality 15, we obtain

$$\begin{aligned} \ln \frac{\theta}{\hat{\theta}} &= \lim_{n\to\infty} \ln \left(\frac{s^n(\theta)}{s^n(\hat{\theta})}\right)^{z-1}, \\ \lim_{n\to\infty} \frac{s^n(\theta)}{s^n(\hat{\theta})} &= \left(\frac{\theta}{\hat{\theta}}\right)^{\frac{1}{z-1}}. \end{aligned} \tag{16}$$

■

Further, we can strengthen this result, to obtain linearity in $(\theta)^{\rho}$.

Lemma 16 *For any tuple $(p, \{w_i^I\}_{i=1}^{\infty}, \gamma, F, c, G) \in \mathbb{R}_{++} \times \mathbb{R}_{+}^{\infty} \times \mathbb{R}_{++} \times \mathcal{F} \times C \times \mathcal{G}$, define $z \equiv \lim_{x\longrightarrow 0} \frac{xc'(x)}{c(x)}$. Then, for any $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,p,\gamma,F,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$, and for any $(\theta, \hat{\theta})^2 \in [-1, 0)^2 \cup (0, 1]^2$,*

$$\lim_{n\to\infty} \frac{s^n(\theta)}{s^n(\hat{\theta})} = sgn\left(\frac{\theta}{\hat{\theta}}\right)\left|\frac{\theta}{\hat{\theta}}\right|^{\frac{1}{z-1}}. \tag{17}$$

Proof. For any $(\theta, \hat{\theta}) \in [-1, 0]^2 \cup [0, 1]^2$, Equality (17) reduces to Equality (16), which holds by Lemma 15. We want to show that Equality (17) holds as well for any $(\theta, \hat{\theta}) \in ([-1, 0] \times [0, 1]) \cup ([0, 1] \times [-1, 0])$ (that is, if θ and $\hat{\theta}$ have different sign). For any $\theta \in [-1, 0) \cup (0, 1]$, by Lemma 9 and Corollary 13,

$$\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(-\theta))} = -1.$$

Hence, for any $(\theta, \hat{\theta}) \in ([-1, 0] \times [0, 1]) \cup ([0, 1] \times [-1, 0])$,

$$\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \lim_{n\to\infty} \frac{-c'(s^n(|\theta|))}{c'(s^n(|\hat{\theta}|))},$$

which, by Lemma 9 and Corollary 13, is equal to $-\frac{|\theta|}{|\hat{\theta}|}$. Thus,

$$-\lim_{n\to\infty}\frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))}=\frac{|\theta|}{|\hat{\theta}|}. \tag{18}$$

Note that the left hand side of Expression (18) is equal to $\lim_{n\to\infty}\frac{c'(|s^n(\theta)|)}{c'(|s^n(\hat{\theta})|)}\in\mathbb{R}_+$, so we can take logarithms on both side, and obtain

$$\lim_{n\to\infty}\left(\ln c'(|s^n(\theta)|)-\ln c'(|s^n(\hat{\theta})|)\right)=\ln\left(\frac{|\theta|}{|\hat{\theta}|}\right). \tag{19}$$

By Lemma 14, for any $\{x_n\}_{n=1}^{\infty}\in\mathbb{R}_{++}^{\infty}$ with $\lim_{n\to\infty}x_n=0$ and $\{y_n\}_{n=1}^{\infty}\in\mathbb{R}_{++}^{\infty}$ with $\lim_{n\to\infty}y_n=0$,

$$\lim_{n\to\infty}\frac{\ln c'(x_n)-\ln c'(y_n)}{\ln\frac{x_n}{y_n}}=z-1,$$

thus, in particular,

$$\begin{aligned}\lim_{n\to\infty}\frac{\ln c'(|s^n(\theta)|)-\ln c'(|s^n(\hat{\theta})|)}{\ln\frac{|s^n(\theta)|}{|s^n(\hat{\theta})|}} &= z-1,\\ \lim_{n\to\infty}\left(\ln c'(|s^n(\theta)|)-\ln c'(|s^n(\hat{\theta})|)\right) &= \lim_{n\to\infty}\ln\left|\frac{s^n(\theta)}{s^n(\hat{\theta})}\right|^{z-1}\end{aligned}$$

and thus substituting the left hand side according to Equality 19, we obtain

$$\begin{aligned}\ln\left(\frac{|\theta|}{|\hat{\theta}|}\right) &= \lim_{n\to\infty}\ln\left|\frac{s^n(\theta)}{s^n(\hat{\theta})}\right|^{z-1},\\ \lim_{n\to\infty}\left|\frac{s^n(\theta)}{s^n(\hat{\theta})}\right| &= \left|\frac{\theta}{\hat{\theta}}\right|^{\frac{1}{z-1}},\\ \lim_{n\to\infty}\frac{s^n(\theta)}{s^n(\hat{\theta})} &= sgn\left(\frac{\theta}{\hat{\theta}}\right)\left|\frac{\theta}{\hat{\theta}}\right|^{\frac{1}{z-1}}.\end{aligned}$$

■

So acquisitions of votes converge to linear in a power of valuations.

For any $F\in\mathcal{F}$, and for any function $\varphi:[-1,1]\longrightarrow\mathbb{R}$, let $E_F\left[\varphi(\bar{\theta})\right]$ denote the expectation of the random variable $\varphi(\bar{\theta})$, given that $\bar{\theta}$ is distributed according to F. If F is fixed and unambiguous, we drop the subindex. For any $\rho\in\mathbb{R}_{++}$, define $\mathcal{F}^{\rho}\subset\mathcal{F}$ by $\mathcal{F}^{\rho}\equiv\{F\in\mathcal{F}:E_F[sgn(\bar{\theta})|\bar{\theta}|^{\rho}]\neq 0\}$.

Lemma 17 *For any $\rho \in \mathbb{R}_{++}$, $\mathcal{F}^\rho$ is open and dense in $\mathcal{F}$.*

Proof. Consider an arbitrary $F \in \mathcal{F}^\rho$. By definition of $\mathcal{F}^\rho$, it follows from $F \in \mathcal{F}^\rho$ that $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] \neq 0$. Without loss of generality, assume $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] > 0$, that is,

$$\int_0^1 f(\theta)\theta^\rho d\theta - \int_{-1}^0 f(\theta)|\theta|^\rho d\theta = \kappa$$

for some $\kappa \in \mathbb{R}_{++}$. For any $\varepsilon \in \mathbb{R}_{++}$, let $N_\varepsilon(F)$ be the open $\varepsilon-$neighborhood around F, in the metric space $(\mathcal{F}, d_{\infty,\infty})$. For any $\varepsilon \in \mathbb{R}_{++}$, and for any $\hat{F} \in N_\varepsilon(F)$,

$$d_\infty(F, \hat{F}) + d_\infty(f, \hat{f}) < \varepsilon,$$

that is

$$\sup_{\theta\in[-1,1]} \left\{ \left| F(\theta) - \hat{F}(\theta) \right| \right\} + \sup_{\theta\in[-1,1]} \left\{ \left| f(\theta) - \hat{f}(\theta) \right| \right\} < \varepsilon,$$

which implies

$$\sup_{\theta\in[-1,1]} \left\{ \left| f(\theta) - \hat{f}(\theta) \right| \right\} < \varepsilon$$

and thus

$$\begin{aligned} \int_0^1 f(\theta)\theta^\rho d\theta - \int_{-1}^0 f(\theta)|\theta|^\rho d\theta - \left(\int_0^1 \hat{f}(\theta)\theta^\rho d\theta - \int_{-1}^0 \hat{f}(\theta)|\theta|^\rho d\theta \right) &< \varepsilon \int_{-1}^1 |\theta|^\rho d\theta \\ &= 2\varepsilon \frac{1}{\rho + 1} \end{aligned}$$

so for $\varepsilon < \frac{\rho+1}{2}\kappa$, it follows that

$$\int_0^1 f(\theta)\theta^\rho d\theta - \int_{-1}^0 f(\theta)|\theta|^\rho d\theta - \left(\int_0^1 \hat{f}(\theta)\theta^\rho d\theta - \int_{-1}^0 \hat{f}(\theta)|\theta|^\rho d\theta \right) < 2\varepsilon \frac{1}{\rho + 1},$$

$$0 < \kappa - 2\varepsilon \frac{1}{\rho + 1} < \left(\int_0^1 \hat{f}(\theta)\theta^\rho d\theta - \int_{-1}^0 \hat{f}(\theta)|\theta|^\rho d\theta \right),$$

so for any $\hat{F} \in N_\varepsilon(F)$, $E_{\hat{F}}[sgn(\theta)|\theta|^\rho] \neq 0$, that is, $N_\varepsilon(F) \subset \mathcal{F}^\rho$ so $\mathcal{F}^\rho$ is open in $(\mathcal{F}, d_{\infty,\infty})$.

To show that $\mathcal{F}^\rho$ is dense in $(\mathcal{F}, d_{\infty,\infty})$, let $F \in \mathcal{F}$ be such that $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] = 0$, and, for each $\delta \in \mathbb{R}_{++}$, take a cumulative distribution $F_\delta \in N_\delta(F)$ such that $F_\delta(\theta) < F(\theta)$ for any $\theta \in (-1, 1)$. Note that for each $\delta \in \mathbb{R}_{++}$, $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] > 0$ and thus $F_\delta \in \mathcal{F}^\rho$, and the sequence $\{F_\delta\}$ with $\delta \longrightarrow 0$ converges to F. Hence, $\mathcal{F}^\rho$ is dense in $\mathcal{F}$. ■

We also use the following lemma by Pólya, presented as Exercise 127 in Part II, Chapter 3 of Pólya and Szegő (1978).

Lemma 18 (Pólya) *If a sequence of monotone (continuous or discontinuous) functions converges on a closed interval to a continuous function it converges uniformly.*

We can now prove a main proposition.

Proposition 19 *For any $\rho \in \mathbb{R}_{++}$, the sequence of social choice correspondences SC_ρ is implementable over $\mathcal{F}^\rho$ by any vote-buying mechanism $c \in C$ such that $\lim\limits_{x \longrightarrow 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}$.*

Proof. Let c be any mechanism in C such that $\lim\limits_{x \longrightarrow 0} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}$. For any $(p, \{w_i^I\}_{i=1}^\infty, \gamma, F, G) \in \mathbb{R}_{++} \times \mathbb{R}_+^\infty \times \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{G}$, let $\{s^n\}_{n=1}^\infty$ be a sequence such that $s^n \in E^{(n,p,\gamma,F,c,G)}$ for each $n \in \mathbb{N} \backslash \{1\}$. Then, by Lemma 16, for any $\theta \in [-1, 1]$,

$$\lim_{n \to \infty} \frac{s^n(\theta)}{s^n(1)} = sgn\,(\theta)\,|\theta|^\rho\,. \tag{20}$$

For each $n \in \mathbb{N}\backslash\{1\}$, define the function $\psi^n : [-1,1] \longrightarrow [-1,1]$ by $\psi^n(\theta) = \frac{s^n(\theta)}{s^n(1)}$. For each $n \in \mathbb{N}\backslash\{1\}$, ψ^n is a monotone function defined on a closed interval, and by Expression (20), the sequence $\{\psi^n\}_{n=1}^\infty$ converges pointwise to the continuous function $sgn\,(\theta)\,|\theta|^\rho$. It follows from Polya's lemma (Lemma 18) that $\{\psi^n\}_{n=2}^\infty$ converges uniformly to function $sgn\,(\theta)\,|\theta|^\rho$. That is, for any $\varepsilon \in \mathbb{R}_{++}$, there exists $\hat{n}(\varepsilon)$ such that for any $\theta \in [-1,1]$, and for any $n > \hat{n}(\varepsilon)$,

$$\left| \frac{s^n(\theta)}{s^n(1)} - sgn\,(\theta)\,|\theta|^\rho \right| < \varepsilon. \tag{21}$$

Take any $F \in \mathcal{F}^\rho$ such that $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] > 0$, and any $\hat{\varepsilon} \in \left(0, E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho]\right)$. By the weak law of large numbers, the random variable $\frac{1}{n}\sum\limits_{k=1}^{n} sgn\left(\bar{\theta}_k\right)\left|\bar{\theta}_k\right|^\rho - \hat{\varepsilon}$, where $\bar{\theta}_k$ is distributed according to F for each $k \in \{1, ..., n\}$, converges to its expectation

$$E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] - \hat{\varepsilon} > 0;$$

and therefore,

$$\lim_{n \longrightarrow \infty} \Pr\left[\sum_{k=1}^{n} sgn\left(\bar{\theta}_k\right)\left|\bar{\theta}_k\right|^\rho - \hat{\varepsilon} > 0\right] = 1. \tag{22}$$

Since, by Inequality (21), for any $n > \hat{n}(\hat{\varepsilon})$, $\frac{s^n(\theta)}{s^n(1)} > sgn\,(\theta)\,|\theta|^\rho - \hat{\varepsilon}$, it follows that

$$\lim_{n \longrightarrow \infty} \Pr\left[\frac{s^n(\bar{\theta})}{s^n(1)} > sgn\left(\bar{\theta}\right)\left|\bar{\theta}\right|^\rho - \hat{\varepsilon}\right] = 1$$

and then from Equality (22),

$$\lim_{n \longrightarrow \infty} \Pr \left[\sum_{k=1}^{n} \frac{s^n(\bar{\theta}_k)}{s^n(1)} - \hat{\varepsilon} > 0 \right] = 1,$$

and thus

$$\lim_{n \longrightarrow \infty} \Pr \left[\sum_{k=1}^{n} s^n(\bar{\theta}_k) > 0 \right] = \lim_{n \longrightarrow \infty} (1 - H^n(0)) = 1, \tag{23}$$

so $\lim_{n \longrightarrow \infty} H^n(0) = 0$. Note that for any $F \in \mathcal{F}^\rho$ such that $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] > 0$, since $SC^n_\rho(\gamma, \theta_{N^n}) = A$ if and only if $\sum\limits_{k=1}^{n} sgn\left(\theta_k\right) |\theta_k|^\rho > 0$, and since $\lim\limits_{n \longrightarrow \infty} \Pr \left[\sum\limits_{k=1}^{n} sgn\left(\bar{\theta}_k\right) \left|\bar{\theta}_k\right|^\rho > 0 \right] =$ 1 (by the weak law of large numbers), it follows that

$$\lim_{n \longrightarrow \infty} \Pr \left[SC^n_\rho(\gamma, \bar{\theta}_{N^n}) = A \right] = 1. \tag{24}$$

From Lemma 11,

$$\lim_{n \longrightarrow \infty} \int\limits_{x \in (n-1)X} g(x) h^n(x) dx = 0, \tag{25}$$

and since $g(x) > 0$ for any $x \in \mathbb{R}$, from Equality (25) we obtain that for any $\hat{x} \in \mathbb{R}_{++}$,

$$\lim_{n \longrightarrow \infty} \int\limits_{-\hat{x}}^{\hat{x}} g(x) h^n(x) dx = 0.$$

Since g is continuous, it attains a minimum in $[-\hat{x}, \hat{x}]$, and this minimum is strictly positive. Since $h^n(x) \in \mathbb{R}_+$ for any $x \in \mathbb{R}$ and for any $n \in \mathbb{N} \backslash \{1\}$, it then follows that

$$\lim_{n \longrightarrow \infty} \int\limits_{-\hat{x}}^{\hat{x}} h^n(x) dx = 0,$$

which implies

$$\lim_{n \longrightarrow \infty} (H^n(\hat{x}) - H^n(-\hat{x})) = 0. \tag{26}$$

Note that $\lim\limits_{n \longrightarrow \infty} H^n(0) = 0$ and (26) together imply that

$$\lim_{n \longrightarrow \infty} \Pr \left[\sum_{k=1}^{n} s^n(\bar{\theta}_k) > \hat{x} \right] = \lim_{n \longrightarrow \infty} (1 - H^n(\hat{x})) = 1. \tag{27}$$

For any $\varepsilon_t \in \mathbb{R}_{++}$, and for any $\hat{x}_t \in \mathbb{R}_{++}$ such that $G(\hat{x}_t) > 1 - \varepsilon_t$, Equality (27) implies that $\lim_{n \longrightarrow \infty} \Pr[d_F^n(s, \bar{\theta}) = A] > 1 - \varepsilon_t$, and thus, choosing a sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ that converges to zero, $\lim_{n \longrightarrow \infty} \Pr[d_F^n(s, \bar{\theta}) = A] = 1$, and then, by Equation (24),

$$\lim_{n \longrightarrow \infty} \Pr[d_F^n(s, \bar{\theta}) = SC_\rho^n(\gamma, \bar{\theta}_{N^n})] = 1, \tag{28}$$

From equalities (23) and (24), it follows that the probability that the equilibrium outcome coincides with the alternative chosen by the social choice correspondence SC_ρ^n converges to one, so c asymptotically implements the sequence of social choice correspondences SC_ρ over the set $\left\{F \in \mathcal{F}^\rho \text{ such that } E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] > 0\right\}$.

Similarly, for any $F \in \mathcal{F}^\rho$ such that $E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] < 0$, $\lim_{n \longrightarrow \infty} \Pr\left[\sum_{k=1}^{n} s^n(\bar{\theta}_k) < 0\right] = 1$ and $\lim_{n \longrightarrow \infty} \Pr\left[SC_\rho^n(\gamma, \bar{\theta}_{N^n}) = B\right] = 1$, so c asymptotically implements SC_ρ over the set $\left\{F \in \mathcal{F}^\rho \text{ such that } E_F[sgn(\bar{\theta})|\bar{\theta}|^\rho] < 0\right\}$.

Hence, c asymptotically implements the sequence of social choice correspondences SC_ρ over the set of cumulative distributions $\mathcal{F}^\rho$. $\blacksquare$

After detailing sufficient conditions for generic implementability in Proposition 19, we next prove that these conditions are (almost) also necessary.

Proposition 20 *For any $SC \in \mathcal{SC}$ such that, for any $\rho \in \mathbb{R}_{++}$, SC and SC_ρ do not converge to each other generically, SC is not implementable generically over $\mathcal{F}$.*

Proof. We prove the contrapositive. For any $(p, \{w_i^I\}_{i=1}^{\infty}, \gamma, F, G) \in \mathbb{R}_{++} \times \mathbb{R}_+^{\infty} \times \mathbb{R}_{++} \times \mathcal{F} \times \mathcal{G}$, for any $n \in \mathbb{N}\backslash\{1\}$, for any $\theta_{N^n} \in [-1, 1]^n$, and for any equilibrium $s^n \in E^{n,p,\gamma,F,c,G}$, let $\bar{d}_F^n(\bar{\theta}_{N^n})$ denote the random variable that takes value $\bar{d}_F^n(\bar{\theta}_{N^n}) = A$ with probability

$$\int_{\theta_{N^n}} \prod_{k=1}^{n} f(\theta_k) \left(G\left(\sum_{k=1}^{n} s^n(\theta_k) \right) \right) d\theta_{N^n}$$

and $\bar{d}_F^n(\bar{\theta}_{N^n}) = B$ with probability

$$\int_{\theta_{N^n}} \prod_{k=1}^{n} f(\theta_k) \left(1 - G\left(\sum_{k=1}^{n} s^n(\theta_k) \right) \right) d\theta_{N^n}.$$

Assume c implements SC generically. We wish to show that there exists $\rho \in \mathbb{R}_{++}$ such that SC and SC_ρ converge to each other generically.

Recall that for any vote-buying mechanism $c \in C$, and for any $a \in \mathbb{R}$, $\eta_c(a) \equiv \frac{ac'(a)}{c(A)}$ denotes the elasticity of the cost function c evaluated at $a \in \mathbb{R}$, and recall as well that by definition of the class of mechanisms C, $\lim\limits_{a\longrightarrow 0}\eta_c(a)\in(1,\infty)$. Then note that from Proposition 19, for any $\rho\in\mathbb{R}_{++}$, any vote-buying mechanism $c\in C$ with $\lim\limits_{a\longrightarrow 0}\eta_c(a)=\frac{1+\rho}{\rho}$ implements SC_ρ, so defining $z\equiv\frac{1+\rho}{\rho}$, and hence $\rho=\frac{1}{z-1}$, for any $z\in(1,\infty)$, any vote-buying mechanism $c\in C$ with $\lim\limits_{a\longrightarrow 0}\eta_c(a)=z$ implements $SC_{\frac{1}{z-1}}=SC_\rho$. Since $\bigcup\limits_{z\in(1,\infty)}\{c\in C:\lim\limits_{a\longrightarrow 0}\eta_c(a)=z\}=C$, it follows that for any $c\in C$, $\exists\rho\in\mathbb{R}_{++}$ such that c implements SC_ρ generically (in particular, $\rho=\frac{1}{\lim_{a\longrightarrow 0}\eta_c(a)-1}$).

Therefore, for any $c\in C$, there exists $\rho\in\mathbb{R}_{++}$, and there exists an open $\mathcal{F}^D$ dense in $\mathcal{F}$ such that c implements SC_ρ over $\mathcal{F}^D$, so for any $F\in\mathcal{F}^D$, $\lim\limits_{n\longrightarrow\infty}\Pr\left[\bar{d}^n_F(\bar{\theta}_{N^n})\neq SC^n_\rho(\gamma,\bar{\theta}_{N^n})\right]=0$.

But since c is posited to also implement SC, there exists an open $\mathcal{F}^{D\prime}$ dense in $\mathcal{F}$ such that c implements SC_ρ over $\mathcal{F}^{D'}$, so for any $F\in\mathcal{F}^{D'}$ $\lim\limits_{n\longrightarrow\infty}\Pr\left[\bar{d}^n_F(\bar{\theta}_{N^n})\neq SC^n(\gamma,\bar{\theta}_{N^n})\right]=0$.

It follows that for any $F\in\mathcal{F}^{D'}\cap\mathcal{F}^D$, $\lim\limits_{n\longrightarrow\infty}\Pr\left[SC^n(\gamma,\bar{\theta}_{N^n})\neq SC^n_\rho(\gamma,\bar{\theta}_{N^n})\right]=0$.

Since the intersection of two open dense sets is dense (an implication of Baire's [3] Category Theorem), it follows that $\mathcal{F}^{D'}\cap\mathcal{F}^D$ it itself an open dense set in $\mathcal{F}$, so SC and SC_ρ converge to each other generically. ■

Proposition 19 and Proposition 20 together lead to our main result, the characterization of generically implementable sequences of social choice correspondences in Theorem 2.

Theorem 2 *A sequence SC of social choice correspondences is generically implementable by a vote-buying mechanism in C if and only if there exists $\rho\in\mathbb{R}_{++}$ such that SC and SC_ρ converge to each other generically, in which case, any vote-buying mechanism $c\in C$ such that $\lim\limits_{x\longrightarrow 0^+}\frac{xc'(x)}{c(x)}=\frac{1+\rho}{\rho}$ generically implements SC.*

Proof. By Proposition 19, for any $\rho\in\mathbb{R}_{++}$, any vote-buying mechanism $c\in C$ such that $\lim\limits_{x\longrightarrow 0^+}\frac{xc'(x)}{c(x)}=\frac{1+\rho}{\rho}$ implements SC_ρ over $\mathcal{F}^\rho$, and $\mathcal{F}^\rho$ is an open dense subset of $\mathcal{F}$ (Lemma 17). Hence, c implements SC_ρ generically.

For any $SC\in\mathcal{SC}$ such that SC and SC_ρ converge to each other generically, there exists an open dense set $\mathcal{F}^D\subseteq\mathcal{F}$ such that for any $F\in\mathcal{F}^D$, $\lim\limits_{n\longrightarrow\infty}\Pr\left[SC(\bar{\theta}_{N^n})\neq SC_\rho(\bar{\theta}_{N^n})\right]=0$.

Since SC and SC_ρ converge to each other over $\mathcal{F}^\rho\cap\mathcal{F}^D$, from

$$\lim_{n\longrightarrow\infty}\Pr\left[SC^n(\gamma,\bar{\theta}_{N^n})\neq SC^n_\rho(\gamma,\bar{\theta}_{N^n})\right]=0\text{ for any }F\in\mathcal{F}^\rho\cap\mathcal{F}^D,\text{ and}$$

$$\lim_{n\longrightarrow\infty}\Pr\left[\bar{d}^n_F(\bar{\theta}_{N^n})\neq SC^n_\rho(\gamma,\bar{\theta}_{N^n})\right]=0\text{ for any }F\in\mathcal{F}^\rho\cap\mathcal{F}^D,$$

it follows that

$$\lim_{n\longrightarrow\infty} \Pr\left[\bar{d}_F^n(\gamma,\bar{\theta}_{N^n}) \neq SC^n(\gamma,\bar{\theta}_{N^n})\right] = 0 \text{ for any } F \in \mathcal{F}^\rho \cap \mathcal{F}^D.$$

Since $\mathcal{F}^\rho$ is open and dense in $\mathcal{F}$ (Lemma 17), and since the intersection of two open dense sets is open dense (an implication of the Category Theorem by Baire (1899)), it follows that $\mathcal{F}^\rho \cap \mathcal{F}^D$ is itself an open dense set in $\mathcal{F}$, and thus c implements SC generically.

For any $SC \in \mathcal{SC}$ such that for any $\rho \in \mathbb{R}_{++}$, SC and SC_ρ do not converge to each other generically, SC is not implementable generically over $\mathcal{F}$, by Proposition 20. ■

We conclude by an implementation result restricted to the class of neutral distribution functions.

Proposition 21 *For any $\rho \in \mathbb{R}_{++}$, a sequence SC of social choice correspondences such that SC and SC_ρ converge to each other given any $F \in \mathcal{F}^*$ is implementable over $\mathcal{F}^*$ in symmetric, monotone, pure and neutral equilibria by any vote-buying mechanism $c \in C$ such that $\lim\limits_{x\longrightarrow 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}$.*

Proof. We first show that SC_ρ is asymptotically implemented by c such that $\lim\limits_{x\longrightarrow 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}$. We want to show that

$$\lim_{n\longrightarrow\infty} \begin{pmatrix} \int\limits_{\theta_{N^n}\in\Theta_A^\gamma(SC_\rho^n)} \left(\prod\limits_{i=1}^n f(\theta_i)\right) G\left(\sum_{i\in N^n} s^n(\theta_i)\right) d\theta_{N^n} \\ + \int\limits_{\theta_{N^n}\in\Theta_B^\gamma(SC_\rho^n)} \left(\prod\limits_{i=1}^n f(\theta_i)\right) \left(1 - G\left(\sum_{i\in N^n} s^n(\theta_i)\right)\right) d\theta_{N^n} \end{pmatrix} > 1-\varepsilon.$$

Note $\lim\limits_{x\longrightarrow 0} \frac{xc'(x)}{c(x)} = \frac{\rho}{1+\rho}$ so $\rho = \frac{1}{\lim\limits_{x\longrightarrow 0} \frac{xc'(x)}{c(x)} - 1}$. For each $n \in \mathbb{N}\backslash\{1\}$, for each $\theta \in (-1,1)$, by Lemma 15,

$$\lim_{n\longrightarrow\infty} \frac{s^n(\theta)}{s^n(1)} = sgn(\theta)|\theta|^\rho \text{ for each } \theta \in [-1,1]. \tag{29}$$

For each $n \in \mathbb{N}\backslash\{1\}$, define the random variable $\rho^n(\bar{\theta}) \equiv \frac{s^n(\bar{\theta})}{s^n(1)} - sgn(\bar{\theta})|\bar{\theta}|^\rho$. By Equality (29), for any $\delta \in \mathbb{R}_{++}$, there exists $\hat{n}_\delta \in \mathbb{N}$ such that for any $n > \hat{n}_\delta$, $\rho^n(\theta) \in (-\delta,\delta)$ for any $\theta \in [0,1]$; further, by neutrality of s^n, $\rho^n(\theta) = -\rho^n(-\theta)$ for any $\theta \in [-1,0]$. So, for any $n > \hat{n}_\delta$, $Var(\rho^n(\bar{\theta})) \leq \delta^2$. We can then construct a decreasing sequence $\{\delta_t\}_{t=1}^\infty$ such that $\delta_t \underset{t\to\infty}{\longrightarrow} 0$, and obtain

$$\lim_{n\longrightarrow\infty} Var(\rho^n(\bar{\theta})) = 0. \tag{30}$$

For each $n \in \mathbb{N}\backslash\{1\}$, and for each $k \in \{1,...,n\}$, define the random variable $\rho_k^n(\bar{\theta}) \equiv \frac{s^n(\bar{\theta})}{s^n(1)} - sgn(\bar{\theta})|\bar{\theta}|^\rho$. These are n independent, identically distributed random variables. Then note that

$$Var\left(\frac{1}{\sqrt{n}}\sum_{k=1}^n \rho_k^n(\bar{\theta})\right) = Var(\rho^n(\bar{\theta})), \tag{31}$$

so by equalities (30) and (31),

$$\lim_{n \longrightarrow \infty} Var\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \rho_k^n(\bar{\theta})\right) = 0;$$

that is, as $n \to \infty$ the realization of $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \rho_k^n(\bar{\theta})$ becomes arbitrarily close to zero with probability converging to one, so the cumulative distribution of $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \rho_k^n(\bar{\theta})$ converges to a step function that is zero below zero, and one above zero. Similarly, $Var\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} sgn(\bar{\theta}_k)|\bar{\theta}_k|^\rho\right) = Var(sgn(\bar{\theta})|\bar{\theta}|^\rho) > 0$, so the distribution of $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} sgn(\bar{\theta}_k)|\bar{\theta}_k|^\rho$ converges to a normal distribution with mean zero and strictly positive variance equal to $Var\left(sgn(\bar{\theta})|\bar{\theta}|^\rho\right)$. Hence,

$$\lim_{n \longrightarrow \infty} \Pr\left[sgn\left(\sum_{i \in N^n} \frac{s^n(\bar{\theta}_i)}{s^n(1)}\right) \neq sgn\left(\frac{1}{\sqrt{n}} \sum_{i \in N^n} sgn(\bar{\theta}_i)|\bar{\theta}_i|^\rho\right)\right] = 0,$$

or equivalently, since $s^n(1) > 0$ for each $n \in \mathbb{N}\backslash\{1\}$,

$$\lim_{n \longrightarrow \infty} \Pr\left[sgn\left(\sum_{i \in N^n} s^n(\bar{\theta}_i)\right) \neq sgn\left(\sum_{i \in N^n} sgn(\bar{\theta}_i)|\bar{\theta}_i|^\rho\right)\right] = 0. \tag{32}$$

From Lemma 11,

$$\lim_{n \longrightarrow \infty} \int_{x \in (n-1)X} g(x)h^n(x)dx = 0, \tag{33}$$

and since $g(x) > 0$ for any $x \in \mathbb{R}$, from Equality (33) we obtain that for any $\hat{x} \in \mathbb{R}_{++}$,

$$\lim_{n \longrightarrow \infty} \int_{-\hat{x}}^{\hat{x}} g(x)h^n(x)dx = 0. \tag{34}$$

Since g is continuous, it attains a minimum in $[-\hat{x}, \hat{x}]$, and this minimum is strictly positive. Since $h^n(x) \in \mathbb{R}_+$ for any $x \in \mathbb{R}$ and for any $n \in \mathbb{N}\backslash\{1\}$, it then follows that

$$\lim_{n \longrightarrow \infty} \int_{-\hat{x}}^{\hat{x}} h^n(x)dx = 0,$$

which implies

$$\lim_{n \longrightarrow \infty} \left(H^n(\hat{x}) - H^n(-\hat{x})\right) = 0. \tag{35}$$

For any $\varepsilon \in \mathbb{R}_{++}$, and for any $\hat{x}$ such that $G(\hat{x}) > 1 - \frac{\varepsilon}{2}$, it follows from Equation (35) that,

$$\lim_{n \longrightarrow \infty} \Pr\left[G\left(\sum_{i \in N^n} s^n(\bar{\theta}_i)\right) \in \left(\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right)\right] = 0. \tag{36}$$

It follows from $G(0) = \frac{1}{2}$ and from expressions (32) and (36) that

$$\begin{aligned} \lim_{n \longrightarrow \infty} \Pr\left[G\left(\sum_{i \in N^n} s^n(\bar{\theta}_i)\right) > 1 - \frac{\varepsilon}{2} \middle| sgn\left(\sum_{i \in N^n} sgn(\theta_i)|\theta_i|^\rho\right) > 0\right] &= 1, \text{ and} \\ \lim_{n \longrightarrow \infty} \Pr\left[G\left(\sum_{i \in N^n} s^n(\bar{\theta}_i)\right) < \frac{\varepsilon}{2} \middle| sgn\left(\sum_{i \in N^n} sgn(\theta_i)|\theta_i|^\rho\right) < 0\right] &= 1. \end{aligned} \tag{37}$$

Thus, subject to $\theta_{N^n} \in \Theta_A^\gamma(SC_\rho^n)$, with probability converging to one in n, $\sum_{i \in N^n} s^n(\theta_i)$ is strictly positive (Expression (32)), and subject to $\sum_{i \in N^n} s^n(\theta_i)$ being strictly positive, its magnitude is sufficiently large so that $G\left(\sum_{i \in N^n} s^n(\theta_i)\right) > 1 - \frac{\varepsilon}{2}$ (Expression (37)). Overall, subject to $(\gamma, \theta_{N^n}) \in \Theta_A^\gamma(SC_\rho^n)$, if n is sufficiently large, $G\left(\sum_{i \in N^n} s^n(\theta_i)\right) > 1 - \varepsilon$ as desired. Similarly, subject to $\theta_{N^n} \in \Theta_B^\gamma(SC_\rho^n)$, with probability converging to one in n, $\sum_{i \in N^n} s^n(\theta_i)$ is strictly negative (Expression (32)), and subject to $\sum_{i \in N^n} s^n(\theta_i)$ being strictly negative, its absolute value is sufficiently large so that $G\left(\sum_{i \in N^n} s^n(\theta_i)\right) < \frac{\varepsilon}{2}$ (Expression (37)). Overall, subject to $(\gamma, \theta_{N^n}) \in \Theta_B^\gamma(SC_\rho^n)$, if n is sufficiently large, $G\left(\sum_{i \in N^n} s^n(\theta_i)\right) < \varepsilon$ as desired.

Hence, any vote-buying mechanism $c \in C$ such that $\lim_{x \longrightarrow 0^+} \frac{xc'(x)}{c(x)} = \frac{1+\rho}{\rho}$ implements SC_ρ.

Further, for any $SC \in \mathcal{SC}$ such that SC and SC_ρ converge to each other,

$$\lim_{n \longrightarrow \infty} \Pr\left[SC_\rho^n(\gamma, \bar{\theta}_{N^n}) \neq SC^n(\gamma, \bar{\theta}_{N^n})\right] = 0,$$

so c also implements the sequence of social choice correspondences SC over the set of cumulative distributions $\mathcal{F}^*$. ■

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