

Working Paper 03-2026

***Conventional Inference under Nonstationary
Heavy Tailed Fractional Processes & Arrays***

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October 2026

Abstract

Existing regression inference methods for nonstationary heavy tailed data are largely limited to unit roots tests. Functional central limit theorems (FCLTs), which are commonly used for the asymptotic analysis of nonstationary models, are inapplicable to nonstationary data characterized by a memory parameter $d < 1$ and a stability parameter $\alpha < 2$. Furthermore, it is well known that inference under nonstationarity is often non-conventional, and this issue is exacerbated in the presence of heavy-tailed data. The interaction of long memory and heavy tails significantly enlarges the nonstationary region—i.e., the covariate model space—where conventional inference methods typically fail.

To address these challenges, we propose a new limit theory and methods that facilitate both workable asymptotics and conventional inference in regressions involving general nonstationary heavy-tailed covariate processes. Specifically, our approach develops a limit theory for heavy-tailed weakly nonstationary processes (hereafter, HT-WNPs), which lie on the threshold of nonstationarity (cf. Duffy and Kasparis, 2021, *Annals of Statistics*). We demonstrate that the proposed methods—including IVX-based and related instrumental-variable procedures—facilitate conventional inference for a broad class of stationary and nonstationary heavy-tailed covariates. This class includes nearly and mildly integrated arrays, as well as nonstationary fractional processes with memory parameter $1 - 1/\alpha < d < 2 - 1/\alpha$, where $\alpha \in (1, 2]$. Our analysis is applicable to both parametric and kernel regression models.

The finite-sample performance of the proposed methods is evaluated through a comprehensive simulation study. Additionally, we provide an empirical application to the predictability of stock returns using long-memory heavy-tailed predictors.

Keywords. Domains of attraction; fractional process; heavy-tailed process; kernel regression; IVX; mildly integrated process; nearly integrated process; stable distribution; weakly nonstationary process

1. Introduction

Heavy-tailed (HT) processes are prevalent across various fields, including economics, finance, urban studies, actuarial science, telecommunications, meteorology, and epidemiology. For instance, Cohen, Davis, and Samorodnitsky (2022) provide evidence that COVID-19 death counts in the U.S. exhibit infinite variance. Additional examples can be found in the works of Embrechts, Klüppelberg, and Mikosch (1997), Alder, Feldman, and Taqqu (1998), and Ibragimov, Ibragimov, and Walden (2015). While significant early research has focused on estimating stationary and unit root HT

autoregressions, considerably less attention has been paid to regressions that incorporate general HT covariates, which may display either stationary or nonstationary characteristics with varying persistence (e.g., memory).

In this study, we primarily investigate inference in parametric regressions of the form

$$y_t = \mu + \beta f(x_{t-1}) + u_t, \quad t = 1, \dots, n, \quad (1)$$

where x_t is some times series process that is potentially nonstationary and heavy tailed, u_t a finite variance m.d. sequence, and $[\mu, \beta]$ unknown regressions parameters. We also provide results for the case where y_t is given by a nonparametric regression of the form

$$y_t = m(x_{t-1}) + u_t, \quad t = 1, \dots, n, \quad (2)$$

with $m(\cdot)$ being some unknown regression function estimated by kernel methods are also provided. Our goal is to develop methods that enable conventional inference regardless of the covariate's stationarity properties and the existence of moments. To address these challenges, we focus on the asymptotic analysis of Heavy-Tailed Weakly Nonstationary Processes (HT-WNPs), a generalization of processes recently examined by Duffy and Kasparis (2021), encompassing processes at the boundary of nonstationarity.

In this study, WNPs may arise either from the model itself or from the estimation method. In the former case, the covariate (x_t) is assumed to belong to a particular class of WNPs. A prominent example of a method-specific WNP is provided by the so-called IVX methodology; see, inter alia, Phillips and Magdalinos (2009) and Kostakis, Magdalinos, and Stamatogiannis (2015) - KMS2015, hereafter. IVX instruments are constructed by applying an appropriate linear filter to the covariate (x_t), thereby generating a WNP that falls within the asymptotic framework developed in this study.

We consider three estimators: (i) a kernel regression estimator for the unknown function (m) in (1); (ii) a weighted least-squares (WIV) estimator, comparable to that studied by Samorodnitsky, Kurz-Kim, and Rachev (2007); and (iii) a weighted version of the IVX estimator, referred to as the W-IVX estimator. The kernel regression estimator is used for both estimation of and inference on the unknown regression function (m). We show that it supports conventional inference for a broad class of stationary and nonstationary covariates. The latter two estimators are used for estimation of and inference on the slope parameter in (1). The WIV estimator is primarily intended for point estimation, whereas the W-IVX estimator is designed principally for inference.

More specifically, the WIV estimator attains parametric convergence rates but does not always support conventional inference when the regressor is nonstationary. By contrast, under nonstationarity, the W-IVX estimator generally converges at semi-parametric rates but yields conventional inference for a broad class of stationary and nonstationary covariates. Consequently, practitioners may conduct W-IVX-based inference without precise knowledge of the persistence properties of x_t . Finally, when x_t is stationary, we show that the WIV and W-IVX estimators are asymptotically equivalent.

The interplay between heavy tails and persistence introduces inferential complications that go beyond those arising in the conventional near-unit-root framework. Existing analyses of highly persistent predictive regressors, such as Campbell and Yogo (2006), Mikusheva (2007), and Phillips (2014), maintain finite-variance, short-

memory innovations. Within that framework, nonstandard inference arises from local-to-unity persistence, with the localizing parameter c constituting the relevant nuisance parameter. Relaxing the finite-variance and short-memory assumptions introduces two additional complications. First, heavy tails enlarge the nonstationary region. For fractional processes, in particular, the conventional finite-variance stationarity boundary $d = 1/2$ is shifted to

$$d = 1 - \frac{1}{\alpha}, \quad 0 < \alpha \leq 2,$$

so that processes that would be classified as stationary under finite variance may become nonstationary under sufficiently heavy tails. Second, once fractional dependence and heavy-tailed innovations are allowed jointly, the limiting distributions can depend not only on the persistence parameter but also on additional nuisance parameters governing memory and the tail behavior of the innovations, including the stability index and skewness. The appearance of these additional nuisance parameters is specific to the more general framework considered here and constitutes an important departure from the existing near-unit-root literature.

Therefore, it is likely that data sets exhibiting long memory and heavy tail characteristics are more accurately described by nonstationary processes, where conventional, nuisance parameter-free inference generally does not hold. In economics, which is the focus of our empirical application, key examples include growth rates of macroeconomic and financial variables such as inflation (Charemza, Hristova, and Burridge, 2006), exchange rate returns (Falk and Wang, 2003; Nicolau and Rodrigues, 2019), and financial return volatility (Kim and Meddahi, 2020; Kim, Meddahi, and Yamashita, 2021). Additionally, economic activity metrics connected to extreme events or risk distress are relevant, as noted by Barigozzi, Cavaliere, and Trapani (2021).

The limit distributions of the estimators under consideration are also derived for cases where regression error (u_t) is a heavy-tailed (HT) term. When the regression error is heavy-tailed, both the instrumental variable (IV) and kernel methods may produce inconsistent estimators due to a weak signal-to-noise ratio. However, when the signal-to-noise ratio is substantial, the estimators are consistent; nonetheless, convergence rates are slower, and the limit distributions include nuisance parameters (e.g., tail and skewness parameters) related to the regression error distribution, which complicates the proposed tests. One potential solution to this issue is to combine the proposed methods with least absolute deviations (LAD) or quantile regression. Alternatively, bootstrap methods may be employed to address the nuisance parameter problem (see Hall and Yao, 2003; Cavaliere, Georgiev, and Taylor, 2016, 2018). However, the latter approach is less appealing for the models considered here, as those studies focus on bootstrap methods in the context of autoregressive time series models, where there is no deterioration in the signal-to-noise ratio in cases of heavy-tailed regression errors. To keep this work a manageable length, we will leave the exploration of quantile methods for future research.

The literature on nonstationary HT regression models has predominantly concentrated on random walks i.e. fractional processes with $d = 1$ -see e.g. Chan and Tran (1989), Knight (1989), Phillips (1991), Caner (1997, 1998), Hallin, van den Akker and Werker (2011), Cavaliere, Georgiev and Taylor (2018). The more recent work of Barigozzi, Cavaliere and Trapani (2021) develops cointegration rank test for systems with HT unit root processes. The asymptotic theory in the aforementioned studies relies on FCLTs for HT processes (e.g. Astrauskas, 1983; Avram and Taqqu, 1992).

For fractional parameter $d < 1$ and tail parameter $\alpha < 2$ however, FCLTs do not apply.¹ This technical difficulty possibly explains why there has been little attention to HT fractional processes that exhibit weaker signal than unit roots. Moreover, just like the stationary case limit distribution dependent on tail parameters. In particular, for simple unit root tests, limit distributions are determined by stochastic integrals depending on the tail (nuisance) parameter. Hallin et al. (2011) propose a distribution-free rank test for the unit root hypothesis while Georgiev and Taylor (2018) develop pivotal unit root tests based on bootstrap methods. The more recent work of Barigozzi, Cavaliere and Trapani (2021) develops a randomized test for common stochastic trends that is free of the tail parameter and at the same time circumvents the utilization of FCLTs. A comprehensive literature review on methods for heavy tailed time series models can be found in the aforementioned work.

It is important to note that developing pivotal tests within a nonstationary framework is a challenging endeavor, even when second moments exist. In nonstationary contexts, the limit distributions of various estimators are non-standard—typically determined by stochastic integrals—and depend on nuisance parameters, such as those related to near-unit roots and memory. When infinite variance is present, tail parameters also emerge in the limit distributions, adding an additional layer of complexity to an already intricate problem. Conventional estimators, such as ordinary least squares (OLS), do not follow the standard distributions under typical assumptions of finite second moments (see Chan and Wei, 1987; Phillips, 1987a). Furthermore, methods that provide valid and conventional inference under nonstationarity (e.g., Phillips and Hansen, 1990; Johansen, 1991; Phillips, 1995; Robinson and Hualde, 2003; Johansen and Nielsen, 2012a) lack robustness to local deviations from the (fractional) unit root paradigm (e.g., Phillips, 1987b; Chan and Wei, 1988; Buchmann and Chan, 2007). In general, inferential methods that are valid for nonstationary models may not apply to stationary models, and vice versa.

A number of relatively recent studies aim to bridge the gap between stationary and nonstationary environments under the assumption of finite second moments. One strand of this literature focuses on near-unit-root forms of nonstationarity. Notably, Campbell and Yogo (2006) and Mikusheva (2007), among others (see Phillips, 2014, for a review), develop conservative methods of inference. A drawback of these methods is that the associated test statistics can be complicated to construct and their critical values are not readily available from standard statistical tables. Moreover, although these procedures robustify inference to local deviations from a unit root, they do not accommodate fractional dependence or heavy-tailed innovations. Under fractional or heavy-tailed data, the corresponding limiting distributions generally acquire additional nuisance parameters associated with memory and the underlying stable law, rendering the resulting tests non-pivotal.

An alternative approach is provided by Phillips and Magdalinos (2009) and KMS2015, who develop IV-type estimators based on specific prefiltering of the covariates, commonly referred to as IVX, that facilitate conventional inference, including t-statistics with $N(0,1)$ limiting distributions. Existing IVX theory, however, is likewise developed under finite-variance assumptions and for covariates belonging to conventional

¹In this case partial sum processes converge in terms of finite dimensional distributions to fractional stable motions that exhibit discontinuities of the second kind (see e.g. Astrauskas, 1983). In fact, it can be demonstrated that a fractional stable motion has paths that *a.s.* unbounded on any interval of positive length, when $d < 1$ and $\alpha < 2$ -see Samorodnisky and Taqqu (1994) Corollary 10.2.4 and Example 10.2.5.

persistence classes, such as stationary short-memory linear processes, mildly integrated processes, and nearly integrated processes. Thus, neither fractional dependence nor heavy-tailed covariates are accommodated by the existing IVX results. Extending IVX inference to such environments is therefore nontrivial and constitutes part of the contribution of the present paper.

IV methods such as IVX are nevertheless particularly attractive in the present framework. Because inference is based on studentized estimators, nuisance parameters associated with tail thickness, skewness, and memory can potentially scale out of the limiting distributions, thereby restoring pivotal inference. In addition, the resulting test statistics are straightforward to construct and their critical values are available from standard statistical tables. For additional work on IVX methods, see, e.g., Yang, Long, Peng and Cai (2019), Demetrescu, Georgiev, Rodrigues and Taylor (2022), Magdalinos and Petrova (2022), and the references therein.

Another strand of the literature develops methods for inference in fractional systems under finite-variance conditions; see, for example, Robinson and Hualde (2003), Hualde and Robinson (2010), and Andersen and Varneskov (2021). These frameworks are more general than the one considered in the present paper in several important respects. In particular, they allow for systems of equations and can accommodate regression disturbances that themselves exhibit long memory. Their validity, however, relies on assumptions concerning fractional integration and the existence of finite second moments. Some of these methods bear a close resemblance to fully modified least squares (FMLS; Phillips, 1995), in that they exploit transformations of the covariates to recover short-memory innovations and correct for endogeneity. Whereas FMLS relies on ordinary differencing, Robinson and Hualde (2003), for example, employ fractional differencing together with GLS-type orthogonalization to accommodate fractional cointegrating relationships. However, when the covariate is a nearly integrated array, fractional differencing no longer recovers the underlying innovations exactly, leaving an additional component that depends on the localizing parameter. This component can affect the first-order asymptotic behavior of the estimators and compromise the validity of conventional inference. Furthermore, the existing results do not accommodate heavy-tailed innovations with infinite variance. Thus, although the fractional-cointegration literature permits substantially richer multivariate structures and more general dependence in the regression disturbances than considered here, it does not provide inference that is simultaneously robust to fractional dependence, near integration, and heavy tails.

Finally, a different approach is developed by Wang and Phillips (2012) and Duffy and Kasparis (2021), who investigate inferential procedures based on kernel methods. An important feature of this approach is that suitably normalized kernel functionals exhibit remarkably similar limiting behavior across very different persistence regimes. This facilitates conventional inference for regression models with covariates ranging from stationary processes to nonstationary fractional processes and nearly integrated arrays, without requiring the practitioner to know the precise persistence class of the covariate. In this sense, kernel methods provide an effective means of conducting inference while remaining agnostic about the degree and form of persistence in the data. The results developed in the present paper show that this robustness extends further to heavy-tailed environments. In particular, kernel-based procedures can remain pivotal despite the additional nuisance parameters associated with tail thickness, skewness, and memory. Hence, kernel methods provide a route to inference that is robust across stationary and nonstationary regimes, fractional and near-integrated

forms of persistence, and, as established here, heavy-tailed environments.

The remainder of the paper is organised as follows. Section 2 presents the models considered in this study. Section 3 develops basic limit theory for functionals of HT-WNPs relevant to both parametric and nonparametric estimation, while Section 4 establishes corresponding results for functionals of HT-SNPs. Section 5 considers estimation and inference in nonparametric regression models and shows that the resulting limit distributions are conventional for a broad class of heavy-tailed regressors, including stationary processes, WNPs, and SNPs. Section 6 considers estimation in parametric regression models, whereas Section 7 develops robust IVX-based methods for parametric inference. Section 8 presents simulation results examining the finite-sample performance of the proposed methods. Finally, Section 9 provides an empirical application to the predictability of stock returns using persistent and potentially heavy-tailed predictors.

Notation For two deterministic sequences $\{a_n\}$ and $\{b_n\}$, $a_n \sim b_n$, $a_n \asymp b_n$, and $a_n \preceq b_n$ denote $\lim_{n \rightarrow \infty} a_n/b_n = 1$, $\lim_{n \rightarrow \infty} a_n/b_n = C$, and $\lim_{n \rightarrow \infty} a_n/b_n \leq C \in (0, \infty)$ respectively. For two random variables X and Y , $X \sim Y$ and $X \stackrel{d}{=} Y$ denote distributional equality between X and Y . For a real number x , $\lfloor x \rfloor$ denotes its integer part. For $a, b \in \mathbb{R}$, $\sum_{j=a}^b (\cdot)_j$ will denote the sum $\sum_{j=\lfloor a \rfloor}^{\lfloor b \rfloor} (\cdot)_j$. $1\{A\}$ denotes the indicator function for the set A . As usual, $\overline{\mathbb{R}}$ is the extended real line. $\sigma(X)$ denotes the sigma algebra generated by random variable X . SV and RV stands for slowly varying and regularly varying function respectively. In addition, RV_p denotes a regularly varying function of index $p \in \mathbb{R}$ (see Bingham et al. 1987; p. 6 and p. 18 respectively). For $x, a \in \mathbb{R}$, $x^{(a)}$ stands for $\text{sgn}(x) |x|^a$ (signed polynomial) with $\text{sgn}(\cdot)$ being the sign function. As usual $v_t \sim I(\delta)$ stands for a process that is integrated of order δ (or equivalently has memory δ). All limits are taken as $n \rightarrow \infty$ unless otherwise indicated, and $:=$ denotes definitional equality.

2. Models and Main Assumptions

Next, we clearly state the set of assumptions that define the specifications of (1) and (2). As previously mentioned, the regression error u_t is assumed to form a martingale difference sequence with respect to an appropriate filtration and to possess finite conditional variance—this is the main assumption. Additionally, the possibility that u_t is an adapted heavy tailed sequence is also considered. For the regressor process x_t three classes are distinguished:

- **HT-SP:** HT Stationary Process.
- **HT-WNP:** HT Weakly Nonstationary Process.
- **HT-SNP:** HT Strongly Nonstationary Process.

Among these, the HT-SP class plays a particularly central role. It not only defines a class of covariate processes but also serves as a foundational building block for the other two. In fact, certain HT-WNP and HT-SNP processes can be constructed as linear filters of HT-SP processes. The first and the last class essentially comprise stationary and nonstationary fractional processes. These processes have been extensively studied under finite variance assumptions see e.g. Robinson and Hualde (2003), Phillips and Shimotsu (2004), Shimotsu and Phillips (2005), Christensen and Nielsen (2006), Hualde and Robinson (2010), Johansen and Nielsen (2012a) among others.

The second class comprises of fractional processes on nonstationarity boundary (i.e. $d = 1 - 1/\alpha$) and mildly integrated (MI, hereafter) arrays like those introduced by Giraitis and Phillips (2006), Phillips and Magdalinos (2007, 2009). A unified asymptotic framework for WNP was developed recently by Duffy and Kasparis (2021) (DK2021, hereafter) under finite variance assumptions. It is important to emphasize that in this work, MI processes are not only considered as potential models for the covariate process but also as instrumental variables. The so-called IVX method employs MI processes to construct t- and F-tests that are robust to the degree of dependence in the covariates. IVX instruments are obtained by applying a specific linear filter—referred to as the IVX filter—to the covariates. Within the current framework, applying the IVX filter to the HT-SNP class yields an HT-MI process, which can then be used as an instrument.

2.1. Covariates

The first assumption below specifies the innovations that drive the covariate term x_t .

Assumption INN (innovations).

(i) $\{\xi_j\}_{j \in \mathbb{Z}}$ is i.i.d. in the domain of attraction of a strictly stable law with stability parameter $\alpha \in (0, 2]$ such that as $\lambda \rightarrow 0$

$$\ln \mathbf{E}(e^{i\lambda\xi_1}) = -\sigma^\alpha |\lambda|^\alpha H(|\lambda|^{-1}) \left[1 + i\beta^* \operatorname{sgn}(\lambda) \tan\left(\frac{\pi\alpha}{2}\right) \right] [1 + o(1)], \quad (3)$$

where H is slowly varying at infinity (SV hereafter), σ^α ($\sigma > 0$) is the scale parameter, $|\beta| \leq 1$ the skewness parameter, and $\beta^* = \beta I_{\{\alpha \neq 1\}}$.

(ii) For $\alpha = 2$, and $x > 0$ we have $1 - P(\xi_1 \leq x) + P(\xi_1 \leq -x) = x^{-2}h(x)[1 + o(1)]$, with $h(x)$ SV.

(iii) ξ_1 has an absolutely continuous distribution and its characteristic function $\psi_\xi(\lambda)$ satisfies $\int_{\mathbb{R}} |\lambda| |\psi_\xi(\lambda)|^\theta d\lambda < \infty$ for some $\theta \geq 1$.

Notation. If the characteristic function of some random variable ξ admits the representation of (3), this will be denoted $\xi \sim \tilde{S}_\alpha\{\sigma^\alpha, \beta, H\}$.

Remark 1.

(a) Following Astrauskas (1983) and Jeganathan (2004, 2008), we assume that for $\alpha \neq 1$, ξ_1 has location parameter zero (i.e. strictly stable), and that for $\alpha = 1$, ξ_1 is symmetric (i.e. location and skewness are both zero).

(b) In general, ξ_1 is in the domain of attraction of an α -stable law with $\alpha \in (0, 2)$, if and only if² there are constants $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$, and h slowly varying such that as $0 < x \rightarrow \infty$

$$\begin{aligned} P(\xi_1 > x) &= c_1 x^{-\alpha} h(x) [1 + o(1)] \\ P(\xi_1 \leq -x) &= c_2 x^{-\alpha} h(x) [1 + o(1)]. \end{aligned}$$

(c) If $\operatorname{Var}(\xi_1) < \infty$ or **Assumption INN(ii)** holds, then ξ_1 is in the domain of attraction of normal law (i.e. (3) holds with $\alpha = 2$ —see Ibragimov and Linnik, 1971; p. 79, 84). Related results under finite variance are provided by DK2021.³ In this work for $\alpha = 2$ we assume that ξ_1 has RV_2 tails which allows for infinite variance.⁴

²See e.g. Ibragimov and Linnik (1971) or Aaronson and Denker (1998).

³DK2021 consider limit theory for mildly integrated process driven by short memory errors. In this work also allow for fractional errors that are either of positive memory or negative memory.

⁴Under **Assumption INN(ii)** we have $\operatorname{Var}(\xi_1 I_{\{|\xi_1| < x\}}) \sim 2 \int_0^x s^{-1} h(s) ds \sim H(x)$. Hence, ξ_1 has a finite variance if and only if H converges.

(d) The connection between h and the SV of (3) is the following (see Ibragimov and Linnik, 1971; proof of Thm 2.6.5)

$$H(x) = \begin{cases} h(x), & \alpha \in (0, 2) \\ -\int_0^x s^2 d\left(\frac{h(s)}{s^2}\right), & \alpha = 2 \end{cases}.$$

Further, when the skewness parameter is zero we have $c_1 = c_2$.

(e) In view of Theorem 1.8.2. in BGT 1987, $h(x)$ can be chosen to be smooth SV i.e. smooth regularly varying of order zero (see e.g. BGT 1987, p. 44).

2.1.1. Stationary and Strongly Nonstationary HT Processes

We begin our analysis with the first and last classes, which have received the most attention in the literature. First, we introduce a set of conditions that define the stationary process. As noted earlier, the stationary process serves as a foundational building block for both HT-WNP and HT-SNP classes. For this reason, we adopt a distinct notation—denoting the stationary process by v_t rather than x_t . However, when the covariate belongs to the HT-SP class, we may write $x_t = v_t$. Let v_t be a linear process of the form

$$v_t = \sum_{i=0}^{\infty} c_i \xi_{t-i}.$$

The following assumption, fully specifies v_t .

Assumption SP (stationary process). v_t is generated by (4), and one of the following conditions holds:

- SM** (short memory) $\sum_{s=0}^{\infty} c_s \neq 0$. Moreover, either: (i) $\sum_{s=0}^{\infty} |c_s|^\nu < \infty$ for some $\nu := \tau \wedge 1$, $\tau \in (0, \alpha)$, $\alpha \in (0, 2]$, and for $\alpha \in (1, 2]$ $\int_1^x h(u)/u du$ is bounded; or (ii) $\sum_{s=0}^{\infty} |c_s|^\nu < \infty$ for some $\nu \in (0, \alpha \wedge 1)$, $\alpha \in (0, 2]$.
- LM** (long memory) $c_j = m_0 \cdot j^{-m} (1 + O(j^{-\nu}))$, as $j \rightarrow \infty$, for some $m_0 \neq 0$, $0 < \nu < 1 - m$, $m \in (1/\alpha, 1)$, $\alpha \in (1, 2]$;
- NM** (negative memory) $c_j = m_0 \cdot j^{-m} (1 + O(j^{-\nu}))$, as $j \rightarrow \infty$, for some $m_0 \neq 0$, $0 < \nu$, $m \in (1, 1 + 1/\alpha)$, $\alpha \in (1, 2]$. In addition, $\sum_{i=0}^{\infty} c_i = 0$, $v_t = \sum_{i=0}^t c_i \xi_{t-i}$, and $H(x) \rightarrow 1$ as $x \rightarrow \infty$;

Remark 2. Under **Assumption SP-SM**, v_t is a short memory errors linear process. The summability requirement under condition **SM(i)** ensures that linear process v_t is well defined -see e.g. Avram and Taqqu (1992), Jeganathan (2004). **SM(i)** entails an additional restriction on the SV function h , when $\alpha > 1$. Condition **SM(ii)**, provides an alternative requirement with a slightly stronger condition on the linear process coefficients, but without any restrictions on h . Most of the technical results provided can be proved under **SM(i)** without any restrictions on h . Nevertheless, for the purposes of **Lemma 10** some additional restrictions are required. The aforementioned lemma shows that for any g bounded and Lipschitz an appropriate normalized WNP (MI) process (x_t) driven by **SM** errors, satisfies

$$\frac{1}{n} \sum_{t=1}^n g(\gamma_n^{-1} x_t) = \frac{1}{n} \sum_{t=1}^n \mathbf{E} [g(\gamma_n^{-1} x_t) | \mathcal{F}_0] + o_p(1).$$

Remark 3. Under **Assumption SP-LM**, v_t is a stationary long memory error v_t . In particular, our formulation allows for $v_t \sim I(\delta)$ with $0 < \delta < 1 - 1/\alpha$, $\alpha \in (1, 2]$. The explicit relation between the memory parameter δ and the exponent m is $\delta = 1 - m$, see e.g., Johansen and Nielsen (2012b; Definition 2). We remark that the restriction on the stability parameter ensures that v_t is well defined -see e.g. Samorodnitsky and Taqqu (1994), Theorem 7.13.2.

Most of the technical results presented here can be derived under the weaker assumption $c_j \sim m_0 \cdot j^{-m}$. However, for the purposes of **Lemma 1**, we impose the slightly stronger second-order condition $c_j = m_0 \cdot j^{-m} (1 + O(j^{-\nu}))$, $0 < \nu < 1 - m$. This condition remains weaker than the coefficient approximation typically satisfied by parametric ARFIMA models. In particular, for such models one has $c_j = m_0 \cdot j^{-m} (1 + O(j^{-1}))$; see, e.g., Giraitis and Surgailis (2012; Theorem 7.2.1).

Remark 4. Under **Assumption SP-NM**, v_t exhibits negative memory. In particular, v_t has a memory parameter $-1/\alpha < \delta < 0$, $\alpha \in (1, 2]$. In addition, we assume that v_t is a truncated linear of the form $\sum_{i=0}^{\infty} c_i \xi_{t-i} I_{\{t-i \geq 0\}}$, which corresponds to a type II fractional process -see e.g. Robinson and Hualde (2003), Phillips and Shimotsu (2004). The latter restriction is imposed to simplify derivations relating a technical result that pertains to the WNPs case (i.e. **Lemma FS $_{\alpha}$ M**), and can be possibly relaxed. Similarly to the **SP-LM** case, we impose a second-order condition on the coefficients c_j that is satisfied by parametric ARFIMA specifications. Finally, we note that the restriction on the stability parameter (i.e. $\alpha > 1$) postulates that the innovations have a mean. It is possible to formulate a negative memory linear process of order $-1/\alpha < \delta < 1 - 1/\alpha$ with $\alpha \in (0, 1]$ by restricting the exponent $m \in (1/\alpha, 1 + 1/\alpha)$. We leave this case for future research.

Next, we specify the two nonstationary classes. In general, these processes can be represented as arrays of the form $\{x_t(n)\}_{t=1}^n$, $n \in \mathbb{N}$, where each element is defined as

$$x_t := x_t(n) = \sum_{j=0}^{t-1} \phi_j(n) v_{t-j}, \quad \text{with} \quad v_t = \sum_{i=0}^{\infty} c_i \xi_{t-i}. \quad (4)$$

Moreover, for convenience, we assume that $x_0 = 0$. This assumption can be relaxed through an alternative formulation, as in KMS2015. In most cases, the process v_t is assumed to satisfy **Assumption SP**, except in the boundary fractional case with $d = 1 - 1/\alpha$ which is treated separately under **Assumption BFR** (see below). **Assumption SNP** next provides an explicit definition of the HT-SNP class that essentially comprises nonstationary heavy tailed fractional arrays.

Assumption SNP (strongly nonstationary process). x_t is generated by (4) with $\phi_j(n) = 1 - \frac{c}{n}$, $c \in \mathbb{R}$ and v_t satisfies **Assumption SP**.

Remark 5. Notice that under **Assumption SNP** x_t is a nearly integrated (**NI** hereafter) driven by stationary fractional errors (v_t) of memory δ . In the specific case in which the near-to-unity parameter satisfies $c = 0$, x_t reduces to a pure fractional process with memory parameter $d = 1 + \delta$. It is worth noting that even if $c \neq 0$, the effective memory parameter of x_t remains d . By effective memory, we mean the order of magnitude of the normalising sequence required for the appropriately rescaled x_n

to converge weakly to a nondegenerate limit. Under our assumptions the range of the memory (d) and stability (α) parameters is

$$d \in \begin{cases} \{1\}, \alpha \in (0, 2] \\ (1 - 1/\alpha, 1) \cup (1, 2 - 1/\alpha), \alpha \in (1, 2] \end{cases}.$$

Notice that for the specific case where $\alpha = 2$, this specializes to the usual nonstationary fractional class $1/2 < d < 3/2$ that has been widely considered in the literature—see e.g. Robinson and Hualde (2003), Phillips and Shimotsu (2004) among others.

Hereafter, we will refer to x_t that satisfies **Assumption SNP** as **HT-SNP**.

2.1.2. Processes on the Nonstationarity Boundary: HT-WNPs

WNPs play a central role in this work, with MI processes being particularly important for the development of robust inferential procedures. Accordingly, we provide a dedicated treatment of this class in the present subsection. Following DK2021, we consider two types of WNPs. The first class consists of boundary fractional processes (hereafter BFR), i.e., processes that lie exactly on the boundary of nonstationarity. In the current framework, this corresponds to a memory parameter of $d = 1 - 1/\alpha$, which represents the lower bound of the memory parameter range for nonstationarity. Under the assumption of finite variance (i.e. $\alpha = 2$), the nonstationarity threshold becomes $d = 1/2$ recovering the results of DK2021 as special case. The second class of WNPs comprises MI arrays, i.e., autoregressive processes with coefficients that drift slowly toward unity—at a rate slower than n . Both types can be incorporated into the specification of (4), with the latter class relying on **Assumption SP** to define v_t . For the former type, i.e. BFR, a different formulation of v_t is required. The following assumption characterizes the BFR class in detail.

Assumption BFR (boundary fractional process). $x_t(n)$ is generated by (4), as $x \rightarrow \infty$

$$L(x) := \int_1^x \frac{H(u)}{u} du \rightarrow \infty$$

and either:

BFR1 $\phi_j = 1 \ \forall j \geq 0$, $c_s \sim s^{-(1+\frac{1}{\alpha})}$, and $\sum_{s=0}^{\infty} c_s = 0$; or

BFR2 $\phi_j \sim j^{-1/\alpha}$, $\phi_0 \neq 0$, **Assumption SM** holds with, $c_\infty := \sum_{s=0}^{\infty} c_s \neq 0$. Further, one of the following holds:

(a) $\limsup_{j \rightarrow \infty} \bar{c}_j |c_j| < \infty$, where $\bar{c}_j := \varsigma(j)j^{\frac{1}{\alpha \wedge 1}}$ with $\varsigma(x)$ SV such that

$$\begin{aligned} \lim_{x \rightarrow \infty} \varsigma(x)^{-1} &= 0, \text{ if } \alpha < 1, \\ \lim_{x \rightarrow \infty} \varsigma(x)^{-1} \ln(x) &= 0, \text{ if } \alpha = 1, \\ \varsigma(x) &= 1, \text{ if } \alpha \in (1, 2]; \end{aligned}$$

or

(b) $\sum_{j=1}^{\infty} j^{1/\alpha} |c_j| < \infty$.

Remark 6.

(a) The condition $L(x) \rightarrow \infty$ effectively requires x_t being nonstationary with $x_{\lfloor nr \rfloor}$ converging weakly upon standardization. The function $L(x)$ defines a normalizing sequence $\gamma_n \rightarrow \infty$ required for the term $\gamma_n^{-1}x_n$ to converge weakly under **Assumption BFR** –see **Definition 1** and **Lemma 1** below.

(b) $L(x)$ is SV (see e.g. BGT 1987; Proposition 1.5.9a).

(c) Notice for some $x_0 > 0$ large enough, $\int_{x_0}^x \frac{H(u)}{u} du$, is well defined (see e.g. BGT 1987; Lemma 1.3.2). For simplicity in **Assumption BFR** we assume $x_0 = 1$.

Remark 7. Conditions (a) and (b) of **Assumption BFR2** are technical restrictions on the short memory component (v_t) that ensure

$$\sum_{j=0}^s \phi_j c_{s-j} \sim s^{-1/\alpha} \sum_{j=0}^{\infty} c_j.$$

Remark 8. Under **BFR1** and **BFR2** we have type I and type II fractional $I(1 - 1/\alpha)$, $\alpha \in (0, 2]$ processes respectively. The parametric counterpart of these processes is an ARFIMA of memory parameter $d = 1 - 1/\alpha$. Clearly, when ξ_t possesses second moment, $\alpha = 2$ and therefore we get $d = 1/2$ as a special case (cf. DK2021).

We now introduce the MI class. Similar to HT-SNPs, MI processes can be modeled using (4) under **Assumption SP**. The key distinction between these two classes lies in the specification of the parameter $\phi_j(n)$. For HT-SNPs, $\phi_j(n)$, rendering the process a near unit root model applied to the linear process errors described in **Assumption SP**. In contrast, for MI processes, $\phi_j(n)$ is a sequence that drifts slowly toward unity, as shown in the equation below. Let

$$\rho_n = 1 - \kappa_n^{-1}, \tag{5}$$

where $\kappa_n > 0$ such that

$$\kappa_n = n^{\alpha_\kappa} \ell_\kappa(n),$$

with ℓ_κ SV and $\alpha_\kappa \in [0, 1)$, $\kappa_n \rightarrow \infty$ and $\sup_{n \geq 1, 1 \leq t \leq n} \sup \kappa_t / \kappa_n < \infty$.

Assumption MI (mildly integrated process). $x_t(n)$ is generated by (4) with $\phi_j(n) = \rho_n^j$ and v_t satisfies **Assumption SP**.

Remark 9. **Assumption MI** encompasses three types of MI processes, extending the framework of DK2021, who consider only MI processes driven by short-memory errors. This generalization is crucial for the development of IVX methods applicable to HT-SNP covariates.

2.2. Regression Errors

Next, specify the regression error u_t of (1) and (2). We consider two possibilities for the regression error. First, we assume that u_t is (together with an appropriate filtration) forms a martingale difference error with finite conditional variance –see **Assumption MDE** below. We show that under this assumption the IVX method proposed here delivers conventional inference. We also consider the possibility of heavy-tailed regression errors (Assumption HTE below) in a number of illustrative

examples involving the kernel and WIV estimators. These examples highlight the importance of the noise-to-signal ratio for the asymptotic behavior of the estimators. In particular, when this ratio is sufficiently large, the estimators may become inconsistent. Under more favorable noise-to-signal conditions, consistency may be preserved, although convergence generally occurs at reduced rates. Moreover, the resulting limiting distributions typically depend on additional nuisance parameters associated with the heavy-tailed regression errors, rendering conventional regression tests non-pivotal. These findings illustrate the challenges of extending conventional inference to models with heavy-tailed regression disturbances. A potentially promising approach is to combine the methods developed here with LAD or quantile regression techniques. We leave this development for future research.

Assumption MDE (martingale difference error).

- (i) $\{x_t, u_t\}_{t \geq 1}$ are $\mathcal{F}_t$ -measurable with $\mathcal{F}_t = \mathcal{F}_0 \vee \sigma(u_1, \dots, u_t, \xi_1, \dots, \xi_t)$, $t \geq 1$, and $\mathcal{F}_0 = \sigma(\dots, \xi_{-1}, \xi_0)$;
- (ii) $\{u_t, \mathcal{F}_t\}$ is martingale difference such that $\mathbf{E}(u_t^2 \mid \mathcal{F}_{t-1}) = \sigma_u^2 < \infty$ a.s.;
- (iii) For any deterministic $0 < A_n \rightarrow \infty$, $\max_{1 \leq t \leq n} \mathbf{E}(u_t^2 I\{|u_t| \geq A_n\} \mid \mathcal{F}_{t-1}) = o_p(1)$.

The requirements of **Assumption MDE** are standard in the literature see e.g. Park and Phillips (2001), Wang and Phillips (2009), Wang (2014), DK2021. The following alternative assumption relaxes these requirements by removing the need for a finite conditional variance.

Assumption HTE (heavy-tailed error).

- (i) $\{x_t, u_t\}_{t \geq 1}$ are $\mathcal{F}_t$ -measurable with $\mathcal{F}_t = \mathcal{F}_0 \vee \sigma(u_1, \dots, u_t, \xi_1, \dots, \xi_t)$, $t \geq 1$, and $\mathcal{F}_0 = \sigma(\dots, \xi_{-1}, \xi_0)$;
- (ii) Let $\alpha_u \in (0, 2]$, $\sigma_u > 0$, $\beta_u \in [-1, 1]$, $\beta_u^* = \beta_u I_{\{\alpha_u \neq 1\}}$ and H_u SV. The conditional characteristic function satisfies

$$\ln \mathbf{E}(e^{i\lambda u_t} \mid \mathcal{F}_{t-1}) = -\sigma_u^{\alpha_u} |\lambda|^{\alpha_u} H_u(|\lambda|^{-1}) \left[1 + i\beta_u^* \operatorname{sgn}(\lambda) \tan\left(\frac{\pi\alpha_u}{2}\right) \right] [1 + R_t(\lambda)],$$

where for all $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P\left(\sup_{|\lambda| \leq \delta} \sup_{1 \leq t \leq n} |R_t(\lambda)| > \varepsilon\right) = 0.$$

- (iii) $\lim_{x \rightarrow \infty} H_u(x) = 1$.

Condition (ii) postulates a conditional characteristic-function expansion corresponding to a regression error in the domain of attraction of an α_u -stable law, i.e. $u_t \mid \mathcal{F}_{t-1} \sim S_{\alpha_u}\{\sigma_u^{\alpha_u}, \beta_u, H_u\}$ with probability approaching one. The remainder condition is uniform over the triangular array $1 \leq t \leq n$. In particular, the stronger condition $\sup_{t \geq 1} |R_t(\lambda)| = o_{a.s.}(1)$, as $\lambda \rightarrow 0$ is sufficient, since it implies $\sup_{|\lambda| \leq \delta} \sup_{t \geq 1} |R_t(\lambda)| = o_p(1)$, as $\delta \rightarrow 0$, and hence also the requirement of condition (ii) above.

3. Limit Theory for Functionals of HT-WNPs

This section develops a foundational limit theory for functionals of HT-WNPs. We begin by introducing the necessary notation and definitions. In view of (4), we can write

$$x_t(n) = \sum_{k=0}^{\infty} a_{k,t}(n) \xi_{t-k}, \quad (6)$$

where (suppressing the dependence of these quantities on n for the sake of a readability, as we shall do freely below)

$$a_{k,t} = \sum_{j=0}^{(t-1) \wedge k} \phi_j c_{k-j} = \begin{cases} \sum_{j=0}^k \phi_j c_{k-j} =: a_k & \text{if } 0 \leq k \leq t-1, \\ \sum_{j=0}^{t-1} \phi_j c_{k-j} =: a_{k,t}^- & \text{if } k \geq t. \end{cases} \quad (7)$$

Further, decompose

$$x_t(n) = \sum_{k=0}^{t-1} a_k(n) \xi_{t-k} + \sum_{k=t}^{\infty} a_{k,t}^-(n) \xi_{t-k} =: x_t^+(n) + x_t^-(n). \quad (8)$$

After appropriate normalization –denoted γ_n – the process $x_{\lfloor nr \rfloor}(n)$, $r > 0$ converges weakly and possesses finite sub- α moments. Both properties are essential for establishing the basic limit theory. The following definition formalizes the normalizing sequence γ_n .

Definition 1 (normalizing sequence γ_n)

(i) **Under Assumption BFR:** $\gamma(x) > 0$ is SV such that $\gamma_n := \gamma(n)$ satisfies

$$\gamma_n \sim L(n^{1/\alpha} \gamma_n)^{1/\alpha}.$$

(ii) **Under Assumption MI:** $\gamma(x) > 0$ and $\gamma_n := \gamma(n)$ satisfies

$$\gamma_n \sim \kappa_n^{b/\alpha} \ell_1(\kappa_n)^{1/\alpha},$$

where $\ell_1 > 0$ is SV such that

$$\ell_1(x) \sim H(x^{1/\alpha} \ell_1(x)^{1/\alpha}),$$

and

$$b := \begin{cases} 1, & \text{under SM,} \\ 1 + \alpha - \alpha m, & \text{under LM \& NM.} \end{cases}$$

Remark 10. The existence of the SV sequences in **Definition 1** follows from asymptotic inversion of regularly varying functions, see e.g., Arvanitis and Louka (2024). Moreover, notice that by definition the following relation holds.

$$b \in \begin{cases} (1, \alpha), & \text{under LM, } \alpha \in (1, 2] \\ \{1\}, & \text{under SM, } \alpha \in (0, 2] \\ (0, 1), & \text{under NM, } \alpha \in (1, 2] \end{cases}.$$

This implies that under **Assumption MI** γ_n is regularly varying in κ_n of order

$$b/\alpha \in \begin{cases} (1/\alpha, 1), & \text{under } \mathbf{LM}, \alpha \in (1, 2] \\ \{1/\alpha\}, & \text{under } \mathbf{SM}, \alpha \in (0, 2] \\ (0, 1/\alpha), & \text{under } \mathbf{NM}, \alpha \in (1, 2] \end{cases}.$$

It is worth noting that when the MI is obtained from IVX filtering of an $I(d)$, $1 - 1/\alpha < d < 2 - 1/\alpha$ process, b/α corresponds to the *Hurst* ($\mathcal{H}$) index of underlying process i.e.

$$b/\alpha = \mathcal{H} = d - 1 + 1/\alpha.$$

Hence, γ_n regularly varying in κ_n of index $\mathcal{H}$.

Lemma FS _{α} M (finite sub- α moments). *Suppose that **Assumption INN** holds and α' is an arbitrary number in $(0, \alpha)$. The following hold:*

(a) Under **Assumption BFR** and **MI**,

$$\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{-1} x_t \right|^{\alpha'} < \infty;$$

(b) Under **Assumption BFR1** for, $0 < s < r \leq 1$ as $n \rightarrow \infty$,

$$\mathbf{E} \left| \gamma_n^{-1} (x_{[nr]}^- - x_{[ns]}^-) \right|^{\alpha'} = o(1);$$

(c) Under **Assumption BFR2** and **MI**, for all $0 < r \leq 1$ as $n \rightarrow \infty$,

$$\mathbf{E} \left| \gamma_n^{-1} x_{[nr]}^- \right|^{\alpha'} = o(1).$$

Using the same notation as in **Assumption INN**, we have the following definition.

Definition 2 (stable distribution) *A random variable ξ follows an α -stable distribution $S_\alpha \{\sigma^\alpha, \beta\}$ if,*

$$\mathbf{E}(e^{i\lambda\xi}) = \exp \left\{ -\sigma^\alpha |\lambda|^\alpha \left[1 + i\beta^* \operatorname{sgn}(\lambda) \tan \left(\frac{\pi\alpha}{2} \right) \right] \right\}, \quad \beta^* = \beta I_{\{\alpha \neq 1\}}. \quad (9)$$

Next, we show that the “partial sum” process $\gamma_n^{-1} x_{[nr]}$, $0 < r \leq 1$ converges weakly. Recall that $c_\infty := \sum_{s=0}^\infty c_s$ when the sum exists. We have the following result.

Lemma 1. (weak convergence). *Suppose that **Assumption INN**, and either **Assumption BFR** or **MI** hold. Then for all $0 < r \leq 1$, we have*

$$\gamma_n^{-1} [x_{[nr]}^+, x_{[nr]}^-] \xrightarrow{d} [X^+, X^-],$$

where under **Assumption BFR1** $X^+, X^- \sim iid$, whilst under **Assumption BFR2** and **MI** $X^- = 0$. In addition,

(i) Under **Assumption BFR1**

$$X^+ \sim S_\alpha \left\{ \sigma^\alpha \alpha^{\alpha+1}, \beta^* \right\}.$$

(ii) Under **Assumption BFR2**

$$X^+ \sim S_\alpha \left\{ \sigma^\alpha \alpha |c_\infty|^\alpha, \beta^* \operatorname{sgn}(c_\infty) \right\}.$$

(iii) Under **Assumption MI-SM**

$$X^+ \sim S_\alpha \left\{ \sigma^\alpha \alpha^{-1} |c_\infty|^\alpha, \beta^* \operatorname{sgn}(c_\infty) \right\}.$$

(iv) Under **Assumption MI-LM**

$$X^+ \sim S_\alpha \left\{ \sigma^\alpha |m_0|^\alpha \mathbb{V}_1, \beta^* \operatorname{sgn}(m_0) \right\},$$

where

$$\mathbb{V}_1 := \int_{x=0}^{\infty} e^{-x} \left(\int_{y=0}^x e^y y^{-m} dy \right)^\alpha dx.$$

(v) Under **Assumption MI-NM**

$$X^+ \sim S_\alpha \left\{ \sigma^\alpha |m_0|^\alpha \mathbb{V}_2, -\beta^* \mathbb{V}_3 \operatorname{sgn}(m_0) \right\},$$

where

$$\mathbb{V}_2 := |1-m|^{-\alpha} \int_{x=0}^{\infty} \underbrace{\left| x^{1-m} - e^{-x} \int_{y=0}^x e^y y^{1-m} dy \right|^\alpha}_{=:p(x)} dx,$$

and

$$\mathbb{V}_3 = \mathbb{V}_2^{-1} \cdot |1-m|^{-\alpha} \int_0^\infty p(x)^{\langle \alpha \rangle} dx.$$

We remark that if $\alpha = 2$ (i.e. Gaussian case), by definition all skewness parameters equal zero. In addition, $\mathbb{V}_1$ reduces to

$$\mathbb{V}_1 = \Gamma[2(1-m)] \cdot B[1-m, 2m-1],$$

where Γ and B are Gamma and Beta functions respectively.

Next, we introduce some notation. Hereafter, Φ_{X^+} is the density of X^+ .

Theorem 1. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lebesgue integrable. Furthermore, suppose:

- (i) x_t satisfies **Assumption INN** and either **BFR** or **MI** hold;
- (ii) there is some $\lambda \in (0, \alpha)$ such that $|f(x)| = O(|x|^\lambda)$, as $|x| \rightarrow \infty$;
- (iii) (a) either for each t' fixed $\frac{1}{n} \sum_{t=1}^{t'} f(\gamma_n^{-1} x_t) = o_p(1)$, or (b) **Assumption INN** holds with $\theta = 1$.

Then, as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} x_t) \xrightarrow{d} \int_{\mathbb{R}} f(x + X^-) \Phi_{X^+}(x) dx, \quad (10)$$

where X^+ , X^- are as in **Lemma 1**.

Furthermore, suppose that $\tilde{f} : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is such that $\tilde{f}(x) = f(x)$ almost everywhere and (a) either for each t' fixed $\frac{1}{n} \sum_{t=1}^{t'} \tilde{f}(\gamma_n^{-1} x_t) = o_p(1)$, or (b) **Assumption INN** holds with $\theta = 1$. Then (10) holds with f replaced by $\tilde{f}$.

Assumption K (kernel). $K : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is such that K and K^2 are Lebesgue integrable, with $\int_{\mathbb{R}} K(x) dx \neq 0$.

Theorem 2. Suppose that

- (i) x_t satisfies **Assumption INN** and either **BFR** or **MI** hold;
- (ii) $\{h_n\}$ a positive sequence with $\gamma_n^{-1} h_n + \gamma_n (nh_n)^{-1} \rightarrow 0$;
- (iii) K satisfies **Assumption K**;
- (iv) for each t' fixed $\frac{\gamma_n}{h_n n} \sum_{t=1}^{t'} K\left(\frac{x_t - x}{h_n}\right) = o_p(1)$.

Then, as $n \rightarrow \infty$,

$$\frac{\gamma_n}{h_n n} \sum_{t=1}^n K\left(\frac{x_t - x}{h_n}\right) \xrightarrow{d} \Phi_{X^+}(-X^-) \int_{\mathbb{R}} K(x) dx, \quad (11)$$

where X^+ , X^- are as in **Lemma 1**.

We note that condition (iv) in the theorem above is very mild and satisfied for all bounded kernels.

To simplify notation set

$$\varrho(x) := \begin{cases} \Phi_{X^+}(x - X^-), & \text{under BFR1} \\ \Phi_{X^+}(x), & \text{under BFR2, MI} \end{cases}$$

Therefore, we can write the limit term of (10) as

$$\int_{\mathbb{R}} f(x + X^-) \Phi_{X^+}(x) dx = \int_{\mathbb{R}} f(x) \varrho(x) dx.$$

Remark 11.

- (a) In view of Condition (ii), Condition (iv) is satisfied when the kernel functional is bounded. Under some additional requirements on the coefficients $a_{k,t}(n)$, it can be shown that **Conditions HL7** and **HL9** (Appendix) hold with $t_0 = 1$. In this case condition (iv) can be dispensed with.
- (b) Note that the limit term above can be also written as $\varrho(0) \int_{\mathbb{R}} K(x) dx$.
- (c) It follows from the above that for $h_n = 1$,

$$\frac{\gamma_n}{n} \sum_{t=1}^n K(x_t) \xrightarrow{d} \Phi_{X^+}(-X^-) \int_{\mathbb{R}} K(x) dx.$$

Remark 12. (comparison with the stationary case).

(a) (finite moment functionals) Consider the fractional linear process

$$x_t = \sum_{j=0}^{\infty} c_j \xi_{t-j}, \quad c_j \sim \text{cons.} \cdot j^{d-1}, \quad d \in (-\infty, 1 - 1/\alpha), \quad \alpha \in (0, 2],$$

with $\xi_t \sim \text{id.} S_{\alpha} \{\sigma^{\alpha}, \beta, \mu\}$ i.e. α -stable with scale, skewness and location parameters $\sigma^{\alpha}, \beta, \mu$ respectively (see e.g. Samorodnitsky and Taqqu, 1994; p. 381). If in addition $\mathbf{E}|f(x_t)| < \infty$, by ergodicity we have (e.g. Shiryaev, 1996; p. 413)

$$\frac{1}{n} \sum_{t=1}^n f(x_t) \xrightarrow{L_1} \mathbf{E}f(x_t).$$

Further, if ξ_t has a bounded density the ergodic theorem yields

$$\frac{1}{h_n n} \sum_{t=1}^n K\left(\frac{x_t - x}{h_n}\right) \xrightarrow{p} \Phi_{x_1}(x) \int_{\mathbb{R}} K(x) dx,$$

with $\Phi_{x_1}(\cdot)$ being the density of x_1 (cf. Wu and Mielniczuk, 2002; p. 1451).

(b) (infinite moment functionals) Davis and Resnick (1985a,b) provide limit theory sample covariance functionals of stationary linear processes of the form $x_t = \sum_{j=0}^{\infty} c_j \xi_{t-j}$, with $\sum_{j=0}^{\infty} |c_j|^{\tau \wedge 1} < \infty$, $\tau \in (0, \alpha)$ and ξ_t are iid possessing regularly varying tails $RV_{-\alpha}$ $\alpha \in (0, 2]$ -see e.g. Davis and Resnick (1985a; p. 183). Under some additional regularity conditions on $\{c_j\}$, we have the following limit result that is relevant for regression analysis

$$\frac{1}{a_n^2} \sum_{t=1}^n x_t^2 \xrightarrow{d} \sum_{j=0}^{\infty} c_j^2 \cdot X_0,$$

with a_n being an appropriate diverging sequence and X_0 an $\alpha/2$ -stable variate that corresponds to the weak limit of $a_n^{-2} \sum_{t=1}^n \xi_t^2$ -see e.g. Davis and Resnick (1985b; p. 271).

Remark 13. (comparison with the SNP case). Theorem 2 is complementary to the limit theory of Jeganathan (2004, 2008) for kernel functionals of HT-SNPs. The aforementioned papers consider type I fractional x_t with $1 - 1/\alpha < d < 2 - 1/\alpha$. Note that the requirement on the memory range implies that $\alpha > 1$, when $d \geq 1$. In particular, Suppose that x_t satisfies **Assumption SNP** with $c = 0$ (near-to-unity parameter). It is known (e.g. Astrauskas, 1983; Kasahara and Maejima, 1988,) that for an appropriate $\gamma_n^* \rightarrow \infty$ sequence⁵, the following finite dimensional distribution result holds

$$\gamma_n^{*-1} x_{\lfloor nr \rfloor} \xrightarrow{f.d.d.} \Lambda_{\alpha,d}(r), \quad r \in [0, 1], \quad (12)$$

with $\Lambda_{\alpha,d}$ being the Linear Fractional Stable Motion (LFSM hereafter)

$$\Lambda_{\alpha,d}(r) := \begin{cases} \sigma_0 \int_{\mathbb{R}} \left[(r-u)_+^{d-1} - (-u)_+^{d-1} \right] dZ_{\alpha}(u), & d \neq 1 \\ \sigma_0 Z_{\alpha}(r), & d = 1 \end{cases},$$

⁵The sequence γ_n^* is regularly varying in n of order $\mathcal{H} = d - 1 + 1/\alpha$. Notice that $\mathcal{H}$ is the Hurst index of $\Lambda_{\alpha,d}(r)$ -see e.g. Jeganathan (2004).

where $(x)_+ := xI_{\{x>0\}}$, $\sigma_0 > 0$, and $\{Z_\alpha(r)\}_{r>0}$ is an α -stable Lévy process. The limit of (12) can be strengthened to weak convergence in the $D[0, 1]$ space, if $\alpha = 2$ or $\alpha < 2$ with $d \geq 1$ (Astrauskas, 1983; Avram and Taqqu, 1992). Note that for $\alpha = 2$ $\Lambda_{\alpha,d}(r)$ reduces to a fractional BM with $1/2 < d < 3/2$.

Jeganathan (2004, 2008) shows that, for integrable K and $\frac{\gamma_n^*}{h_n n} \rightarrow 0$,

$$\frac{\gamma_n^*}{h_n n} \sum_{t=1}^n K\left(\frac{x_t - x}{h_n}\right) \xrightarrow{d} L_{\Lambda_{\alpha,d}}(0, 1) \int_{\mathbb{R}} K(x) dx, \quad (13)$$

where $L_{\Lambda_{\alpha,d}}(0, 1)$ denotes the local time of $\Lambda_{\alpha,d}$ at the origin up to time 1. It should be emphasized that this limit result holds under (12) and does not require convergence of $\gamma_n^{*-1}x_{\lfloor nr \rfloor}$ in the $D[0, 1]$ space. In fact, under **Assumption SNP**, with $c = 0$, and for any locally integrable $f : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ that satisfies certain polynomial growth conditions, it is possible to show that (see **Theorem 6** in the following section)

$$\frac{1}{n} \sum_{t=1}^n f\left(\frac{x_t}{\gamma_n^*}\right) \xrightarrow{d} \int_0^1 f(\Lambda_{\alpha,d}(r)) dr, \quad (14)$$

where γ_n^* and $\Lambda_{\alpha,d}(r)$ are given in (12). It is worth noting that (12) does not require an FCLT and is valid even if $d < 1$ or $\alpha < 2$.

Preliminary results obtained by Kasparis and Wang (2026) suggest that (14) extends to the case $c \neq 0$ with $\Lambda_{\alpha,d}(r)$ replaced by the fractional α -stable Ornstein–Uhlenbeck process

$$J_{\alpha,c,d}(r) := \Lambda_{\alpha,d}(r) + c \int_0^r e^{c(r-u)} \Lambda_{\alpha,d}(u) du.$$

A rigorous treatment of this generalization is left for future work.

For the purposes of our regression analysis we need some additional limit results involving sample covariance terms of $f(\gamma_n^{-1}x_t)$ or $K\left(\frac{x_t - x}{h_n}\right)$ and some term u_t corresponding to regression error -cf. (1). We will primarily assume that the regression error is a finite variance martingale difference. Some limit results are also provided for the case where u_t is an iid process in the domain of attraction of an alpha-stable law. In the latter case the proposed estimators do not yield pivotal statistical tests due nuisance tail parameters related to the regression error. Usable inference in the later case can developed using analogous methods to those considered by Zhu and Ling (2012) or Cavaliere, Georgiev and Taylor (2016, 2018). We leave explorations in this direction for future work.

Hereafter, in order to simplify conditions we assume that **Assumption INN** holds with $\theta = 1$. As demonstrated in **Theorem 1** and **Theorem 2**, this requirement can be relaxed if finite sums of HT-WNPs functionals are asymptotically negligible.

The following two results generalize **Theorems 1** and **2**. In particular, **Theorems 3** and **4** below are relevant for parametric and nonparametric analysis respectively.

Theorem 3. Suppose that $f^2, g : \mathbb{R} \rightarrow \mathbb{R}$ are locally Lebesgue integrable. In addition, suppose that:

- (i) **Assumption INN** holds with $\theta = 1$, and x_t satisfies either **Assumption BFR** or **MI**;

- (ii) there is some $\lambda \in (0, \alpha)$ such that $f(x)^2, |g(x)| = O(|x|^\lambda)$, as $|x| \rightarrow \infty$.
- (iii) x_t, u_t satisfy **Assumption MDE**.

Then, as $n \rightarrow \infty$,

$$\left[\frac{1}{\sqrt{n}} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1}) u_t, \frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1})^2, \frac{1}{n} \sum_{t=1}^n g(\gamma_n^{-1} x_{t-1}) \right] \quad (15)$$

$$\xrightarrow{d} \left[MN \left[0, \sigma_u^2 \int_{\mathbb{R}} f(x)^2 \varrho(x) dx \right], \int_{\mathbb{R}} f(x)^2 \varrho(x) dx, \int_{\mathbb{R}} g(x) \varrho(x) dx \right]$$

where X^+, X^- are defined in **Lemma 1**.

Remark. Under conditions analogous to those imposed in **Theorem 1**, it can be shown that (10) continues to hold when f and g are extended-real-valued functions.

Theorem 4. Suppose that:

- (i) **Assumption INN** holds with $\theta = 1$, and x_t satisfies either **Assumption BFR** or **MI**;
- (ii) x_t, u_t satisfy **Assumption MDE**;
- (iii) $\{h_n\}$ is a positive sequence with $\gamma_n^{-1} h_n + \gamma_n (nh_n)^{-1} \rightarrow 0$;
- (iv) K, K^2 and L satisfy **Assumption K**.

Then, as $n \rightarrow \infty$,

$$\left[\sqrt{\frac{\gamma_n}{h_n n}} \sum_{t=1}^n K \left(\frac{x_{t-1} - x}{h_n} \right) u_t, \frac{\gamma_n}{h_n n} \sum_{t=1}^n K \left(\frac{x_{t-1} - x}{h_n} \right)^2, \frac{\gamma_n}{h_n n} \sum_{t=1}^n L \left(\frac{x_{t-1} - x}{h_n} \right) \right] \quad (16)$$

$$\xrightarrow{d} \left[MN \left[0, \sigma_u^2 \varrho(0) \int_{\mathbb{R}} K(x)^2 dx \right], \varrho(0) \int_{\mathbb{R}} K(x)^2 dx, \varrho(0) \int_{\mathbb{R}} L(x) dx \right].$$

It follows directly from **Theorem 3** and **Theorem 4** that, when the regression error u_t possesses a finite second moment, the “score” term admits a mixed Gaussian weak limit. This property underpins the validity of conventional inference.

The next result provides a limit theory for the score terms in the case where the regression error is an adapted process lying in the domain of attraction of an α_u -stable distribution, with $0 < \alpha_u \leq 2$. To keep the technical exposition simple, we exclude the case of a **BFR1** covariate x_t .

Theorem 5. Suppose that:

- (i) **Assumption INN** holds with $\theta = 1$, and x_t satisfies either **Assumption BFR** or **MI**;
- (ii) x_t, u_t satisfy **Assumption HTE**;
- (a) Suppose that in addition to conditions (i)-(ii) the following hold:
- (iii) $|f|^{\alpha_u} : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lebesgue integrable, and $\int_{\mathbb{R}} |f(x)|^{\alpha_u} \varrho(x) dx > 0$;

(iv) there is some $\lambda \in (0, \alpha)$ such that $|f(x)|^{\alpha_u} = O(|x|^\lambda)$, as $|x| \rightarrow \infty$.

Then, as $n \rightarrow \infty$,

$$n^{-1/\alpha_u} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1}) u_t \xrightarrow{d} S_{\alpha_u} \left\{ \sigma_u^{\alpha_u} \int_{\mathbb{R}} |f(x)|^{\alpha_u} \varrho(x) dx, \beta_u^* \frac{\int_{\mathbb{R}} f(x)^{\langle \alpha_u \rangle} \varrho(x) dx}{\int_{\mathbb{R}} |f(x)|^{\alpha_u} \varrho(x) dx} \right\}.$$

(b) Suppose that addition to (i)-(ii) the following hold:

(v) $\{h_n\}$ is a positive sequence with $\gamma_n^{-1} h_n + \gamma_n(nh_n)^{-1} \rightarrow 0$;

(vi) $|K|^{\alpha_u}$ satisfies **Assumption K**, and $\int_{\mathbb{R}} |K(x)|^{\alpha_u} dx > 0$.

Then, as $n \rightarrow \infty$,

$$\left(\frac{\gamma_n}{h_n n} \right)^{1/\alpha_u} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n} \right) u_t \xrightarrow{d} S_{\alpha_u} \left\{ \sigma_u^{\alpha_u} \varrho(0) \int_{\mathbb{R}} |K(x)|^{\alpha_u} dx, \beta_u^* \frac{\int_{\mathbb{R}} K(x)^{\langle \alpha_u \rangle} dx}{\int_{\mathbb{R}} |K(x)|^{\alpha_u} dx} \right\}.$$

Under the assumptions of **Theorem 5**, the score terms possess stable limit distributions. In the special case where $\alpha_u = 2$ (i.e. the error is in the domain of attraction of a normal law), regression estimators and t-statistics admit conventional limit distributions. By contrast, for $\alpha_u < 2$, limit distributions are non-standard, involving nuisance parameters related to the regression error—namely α_u , β_u^* , σ_u . We conjecture that **Theorem 5** can be extended to settings with a covariate satisfying **Assumption BFR1**. In such cases, it is natural to anticipate mixed-stable limit distributions, with X^- serving as the mixing variate. We leave these extensions to future work.

4. Limit Theory for Functionals of HT-SNPs

This section develops limit theory for sample functionals of heavy-tailed strongly nonstationary processes (**HT-SNPs**). The results provide the theoretical foundation for parametric estimation procedures, particularly the weighted instrumental-variable (WIV) estimator, and for the consistent estimation of nuisance parameters, including the regression-error variance (σ_u^2), under **Assumption MDE**. Unlike the W-IVX procedure, which facilitates conventional inference at the cost of slower, semiparametric convergence rates, WIV generally achieves faster convergence rates but does not support conventional inference in all persistence regimes. Nevertheless, its superior convergence rates make WIV particularly attractive for point estimation of regression parameters and associated nuisance quantities, even when the limiting distributions of the corresponding test statistics are non-pivotal. Thus, the asymptotic theory developed in this section is relevant not only for inference but also for establishing the consistency and convergence rates of parametric estimators.

The main objects of interest are normalized sample averages of the form $\frac{1}{n} \sum_{t=1}^n f\left(\frac{x_t}{\gamma_n^*}\right)$, where x_t is an HT-SNP and γ_n^* is the appropriate normalizing sequence. The goal is to show that, for a suitable class of functions f ,

$$\frac{1}{n} \sum_{t=1}^n f\left(\frac{x_t}{\gamma_n^*}\right) \xrightarrow{d} \int_0^1 f(\Lambda_{\alpha,d}(r)) dr, \quad (17)$$

where $\Lambda_{\alpha,d}$ denotes linear fractional stable motion (LFSM). This result is useful even in regions where a functional central limit theorem (FCLT) in the usual Skorokhod space is not available.

When x_t has finite variance, results such as Marinucci and Robinson (2000) establish weak convergence of $x_{\lfloor nr \rfloor} / \gamma_n^*$ in $D[0, 1]$ for $1/2 < d < 3/2$, under moment restrictions on the innovations. In particular, if the innovations ξ_t are i.i.d. with $\mathbf{E}|\xi_t|^q < \infty$, the relevant restriction is

$$q > 2 \vee \frac{2}{2d - 1}.$$

Thus, as $d \downarrow 1/2$, progressively stronger moment assumptions are required. At the boundary $d = 1/2$, the FCLT no longer applies; a different weakly nonstationary limit theory is required, as in Duffy and Kaspas (2021) -see also Johansen and Nielsen (2012b) for impossibility results.

For HT innovations in the domain of attraction of an α -stable law, with $1 < \alpha \leq 2$, Astrauskas (1983) obtains weak convergence of the continuous polygonal interpolation of $x_{\lfloor nr \rfloor} / \gamma_n^*$ in $C[0, 1]$ when

$$1 < d < 2 - \frac{1}{\alpha}.$$

The case $d = 1$ corresponds to stable Lévy motion. Avram and Taqqu (1992) establish weak convergence in $D[0, 1]$ for this integrated short-memory case under additional technical conditions on the short-memory component.

The remaining nonstationary region is fundamentally different. When

$$1 - \frac{1}{\alpha} < d < 1, \alpha < 2,$$

the finite-dimensional distributions of $x_{\lfloor nr \rfloor} / \gamma_n^*$ still converge to those of $\Lambda_{\alpha, d}$; see, for example, Astrauskas (1983) and Kasahara and Maejima (1988). However, $\Lambda_{\alpha, d}$ is not a $D[0, 1]$ -valued process in this range. Indeed, Corollary 10.2.4 and Example 10.2.5 of Samorodnitsky and Taqqu (1994) imply that, for $d < 1$ and $\alpha < 2$, LFSM has sample paths that are almost surely unbounded on every interval of positive length. Since every cadlag function on a compact interval is bounded, weak convergence in $D[0, 1]$ cannot hold. Consequently, the proof of (17) cannot rely on a continuous mapping argument applied to an FCLT. Instead, it combines finite-dimensional convergence with suitable moment and smoothness bounds for the transformed process.

As explained earlier, mildly integrated (MI) processes arise from applying the IVX filter to strongly nonstationary processes (SNPs). For the limit theory developed in the previous section, additional technical restrictions were imposed on the MI/SNP construction. We now introduce **Assumptions SP*** and **SNP***, which are variants of **Assumptions SP** and **SNP**, respectively. **Assumption SP*** is weaker than **Assumption SP** and allows for a more general specification of the stationary component v_t , whereas **Assumption SNP*** imposes stronger conditions than **Assumption SNP**. The more general specification of v_t accommodates a broader class of strongly nonstationary processes. To keep the paper to a manageable length, we restrict attention to the case in which the localizing parameter satisfies $c = 0$, so that $\rho_n = 1$. We begin by specifying v_t .

Assumption SP* (stationary process). *The process v_t is generated by the linear process representation in (4), and one of the following conditions holds:*

SM* (short memory) $\sum_{s=0}^{\infty} c_s \neq 0$, $\sum_{s=0}^{\infty} |c_s|^\nu < \infty$ for some $\nu := \tau \wedge 1$, $\tau \in (0, \alpha)$, $\alpha \in (0, 2]$.

LM* (long memory) $c_j \sim m_0 \cdot j^{-m}$, as $j \rightarrow \infty$, for some $m_0 \neq 0$, $m \in (1/\alpha, 1)$, $\alpha \in (1, 2]$;

NM* (negative memory) $c_j \sim m_0 \cdot j^{-m}$, as $j \rightarrow \infty$, for some $m_0 \neq 0$, $m \in (1, 1+1/\alpha)$, $\alpha \in (1, 2]$. In addition, $\sum_{i=0}^{\infty} c_i = 0$.

In view of **Assumption SP***, we now introduce a weaker version of **Assumption SNP**.

Assumption SNP* (strongly nonstationary process). *The process x_t is generated by (4) with $\varphi_j(n) = 1$, for all $j \geq 0$, and the stationary component v_t satisfies **Assumption SP***.*

The most notable extension concerns the negative-memory case. Under **Assumption SNP*-NM**, the process x_t is a Type I fractional process; in particular, it is not truncated. We next define the normalizing sequence required for the finite-dimensional convergence of $(\gamma_n^*)^{-1}x_{\lfloor nr \rfloor}$.

Definition 3 (normalizing sequence γ_n^*). *Under **Assumption SNP***, let $\gamma^*(x) > 0$ and set $\gamma_n^* := \gamma^*(n)$, where*

$$\gamma_n^* \sim n^{b/\alpha} \ell_1(n)^{1/\alpha}.$$

Here $\ell_1 > 0$ is slowly varying and satisfies

$$\ell_1(x) \sim H \left(x^{1/\alpha} \ell_1(x)^{1/\alpha} \right),$$

and

$$b := \begin{cases} 1, & \text{under } \mathbf{SM}^*, \\ 1 + \alpha - \alpha m, & \text{under } \mathbf{LM}^* \text{ \& } \mathbf{NM}^*. \end{cases}$$

It is useful to compare this normalization with the corresponding one for MI processes. Under **Assumption SNP***, γ_n^* is regularly varying in n with index $\mathcal{H} = \frac{b}{\alpha} = d - 1 + \frac{1}{\alpha}$, where $\mathcal{H}$ is the Hurst, or self-similarity, index of the limiting LFSM. Under **Assumption MI**, the analogous normalizing sequence is regularly varying in the localizing sequence κ_n with the same index $\mathcal{H}$. Thus, the SNP normalization uses calendar time n , whereas the MI normalization uses the effective persistence scale κ_n .

The finite-dimensional convergence

$$(\gamma_n^*)^{-1}x_{\lfloor nr \rfloor} \xrightarrow{f.d.d.} \Lambda_{\alpha,d}(r), \quad r \in [0, 1], \quad (18)$$

established, for example by Astrauskas, (1983), and Kashara and Maejima (1986), together with the following moment-continuity result of **Lemma SS $_{\alpha}$ M** below is sufficient for (17).

Before we present the limit theory for nonlinear functionals of HT-SNP processes, we introduce some class of functions that is relevant to our asymptotic analysis and provide a preliminary Lemma.

Definition LH (local Hölder). Let $f : \mathbb{R} \rightarrow \mathbb{R}$. We will say that f is **LH**, if for some $C_{\alpha'} \in (0, \infty)$,

$$|f(x) - f(y)| \leq C_{\alpha'} \cdot \begin{cases} |x|^{\alpha'-1}|y - x| + |y - x|^{\alpha'}, & 1 < \alpha' < \alpha, \alpha > 1, \\ |y - x|^{\alpha'}, & 0 < \alpha' < \alpha, \alpha \leq 1. \end{cases} \quad (19)$$

The smoothness restriction in (19) is a Hölder-type condition designed for functions with polynomial growth. The requirement is mild enough to include many functions used in regression applications, while being strong enough to facilitate a Riemann-sum approximation underlying (17).

Remark 14. Note that (19) implies the following growth condition for f : For some $C'_{\alpha'} \in (0, \infty)$,

$$|f(x)| \leq C'_{\alpha'} \cdot (1 + |x|^{\alpha'}) \quad 0 < \alpha' < \alpha.$$

This growth restriction ensures uniform integrability at sub- α orders.

Lemma SS $_{\alpha}$ M (smooth sub- α moments). Suppose that:

- (i) x_t satisfies **Assumption INN** and **Assumption SNP***;
- (ii) either:
 - (a) f is **LH**; or
 - (b) f is bounded and Lipschitz

Then

$$\sup_{n \geq 1} \sup_{r \in [0,1]} \mathbf{E} \left| f \left(\gamma_n^{*-1} x_{\lfloor nr \rfloor} \right) \right| < \infty, \quad (20)$$

and

$$\lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{r,s \in [0,1] \\ |r-s| \leq \delta}} \mathbf{E} \left| f \left(\gamma_n^{*-1} x_{\lfloor nr \rfloor} \right) - f \left(\gamma_n^{*-1} x_{\lfloor ns \rfloor} \right) \right| = 0. \quad (21)$$

It follows from Gikhman and Skorokhod (1969) that (18), (20) and (21) are sufficient for the limit result of (17). Consequently, (17) holds whenever f is **LH** or bounded and Lipschitz. It can further be shown that, if f locally integrable and satisfies the polynomial growth condition $|f(x)| = O(|x|^{\lambda})$ for some $0 < \lambda < \alpha$, then sample functionals involving f can be approximated by corresponding sample functionals of a bounded and Lipschitz function, for which (17) holds by the preceding argument. This approximation argument is implicit in the following theorem, which provides the main limit result for sample functionals of HT-SNP processes.

Theorem 6. Suppose that:

- (i) x_t satisfies **Assumption INN** and **Assumption SNP***;
- (ii) $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally integrable;
- (iii) As $|x| \rightarrow \infty$, $|f(x)| = O(|x|^{\lambda})$ for some $0 < \lambda < \alpha$;

- (iv) (a) either for each t' fixed $\frac{1}{n} \sum_{t=1}^{t'} f(\gamma_n^{*-1} x_t) = o_p(1)$, or (b) **Assumption INN** holds with $\theta = 1$.

Then, as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{t=1}^n f\left(\frac{x_t}{\gamma_n^*}\right) \xrightarrow{d} \int_0^1 f(\Lambda_{\alpha,d}(r)) dr. \quad (22)$$

Furthermore, suppose that $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ is such that $\tilde{f}(x) = f(x)$ almost everywhere and (a) either for each t' fixed $\frac{1}{n} \sum_{t=1}^{t'} \tilde{f}(\gamma_n^{*-1} x_t) = o_p(1)$, or (b) **Assumption INN** holds with $\theta = 1$. Then (22) holds with f replaced by $\tilde{f}$.

Remark 15. **Theorem 6** requires that the function f is locally integrable and satisfy a sub- α polynomial growth rate. This is a technical condition imposed to establish (22), and it accommodates many regression functions used in practice. Under these conditions, it can also be shown that the limiting random variable is well defined, in the sense that $\int_0^1 |f(\Lambda_{\alpha,d}(r))| dr < \infty$ a.s. See Proposition 5 in the Appendix and, for more general conditions, Samorodnisky and Taqqu (1994), Theorem 11.3.2. A more general version of (22) is under development by Kasparis and Wang (2026). In particular, preliminary arguments indicate that (22) continues to hold when f is locally integrable and satisfies $|f(x)| = O(|x|^\lambda)$, for some $\lambda > 0$ such that $\mathcal{H} > \frac{1}{\alpha} - \frac{1}{\lambda}$ where recall that the Hurst index is $\mathcal{H} = d - 1 + 1/\alpha$.

5. Kernel Regression

Next, we consider estimation and inference in regression models of the form

$$y_t = m(x_{t-1}) + u_t, \quad (23)$$

where m is an unknown regression function and x_t, u_t are $\mathcal{F}_t$ -measurable for some appropriate filtration $\mathcal{F}_t$, $t \geq 0$ -e.g. **Assumption MDE**. In particular, we consider the limit distribution of the kernel regression estimator (local level) for the function $m(\cdot)$ in (23) for the case where x_t is an HT-WNP. This result is complementary to e.g. Wu and Mielniczuk (2002), who consider kernel regressions with stationary linear process covariates. In addition, Jeganathan (2004, 2008) develops foundational limit theory for kernel functionals of HT-SNPs under conditions analogous to those imposed by **Assumption SNP*** in the present paper. These results can be adapted with relatively minor modifications to the asymptotic analysis of kernel regression estimators. In particular, the aforementioned studies consider nonstationary fractional covariates of memory $1 - 1/\alpha < d < 2 - 1/\alpha$, $\alpha \in (0, 2]$.⁶

Set $K_{n,t}(x) := K\left(\frac{x_t - x}{h_n}\right)$ and consider the Nadaraya-Watson (NW hereafter) kernel regression estimator

$$\hat{m}(x) = \frac{\sum_{t=1}^n K_{n,t-1}(x) y_t}{\sum_{t=1}^n K_{n,t-1}(x)}.$$

Theorem 7. Suppose that $\{y_t\}$ is generated by (23). In addition:

- (i) **Assumption INN** holds with $\theta = 1$ and x_t satisfies either **Assumption BFR** or **Assumption MI**;

⁶Notice that for $d = 1$ this restrictions are equivalent to $\alpha \in (1, 2]$.

- (ii) x_t, u_t satisfy **Assumption MDE**;
- (iii) K and K^2 satisfy **Assumption K**;
- (iv) $\{h_n\}$ a positive sequence such that $h_n + \gamma_n(nh_n)^{-1} \rightarrow 0$;
- (v) There is some function $\bar{m}(x, z)$ such that $|m(x + hz) - m(x)| \leq h^\nu \bar{m}(x, z)$, with $\nu \in (0, 1]$ for all h sufficiently small, and $\int_{\mathbb{R}} \bar{m}(x, z) |K(z)| dz < \infty$ for all $x \in \mathbb{R}$;

Then, as $n \rightarrow \infty$,

$$\hat{m}(x) \xrightarrow{p} m(x).$$

If in addition, $\left(\frac{nh_n}{\gamma_n}\right)^{1/2} h_n^\nu \rightarrow 0$, then

$$\left(\frac{nh_n}{\gamma_n}\right)^{1/2} [\hat{m}(x) - m(x)] \xrightarrow{d} MN \left[0, \frac{\sigma_u^2}{\varrho(0) \int_{\mathbb{R}} K(z)^2 dz}\right]. \quad (24)$$

Next, we consider a non parametric t-test for the hypothesis $H_0 : m(x) = m_0(x)$, where $m_0(x)$ is a prespecified function. To construct the test statistic, we employ the following local estimator of the regression-error variance, based on nonparametric residuals:

$$\hat{\sigma}_u^2(x) = \frac{\sum_{k=1}^n [y_t - \hat{m}(x)]^2 K_{n,t-1}(x)}{\sum_{k=1}^n K_{n,t-1}(x)}, \quad (25)$$

can be shown to be consistent for σ_u^2 under our assumptions. Set $\hat{t}(x, m_0)$

$$\hat{t}(x, m_0) := \frac{\sum_{k=1}^n K_{n,t-1}(x) [\hat{m}(x) - m_0(x)]}{\sqrt{\hat{\sigma}_u^2(x) \left[\sum_{k=1}^n K_{n,t-1}^2(x)\right]}}.$$

Under the null hypothesis the t-statistic has approximately standard normal limit distribution. This is demonstrated by the following result.

*Theorem 8. Suppose that in addition to the conditions of **Theorem 7**, there exists some $p > 1$ such that:*

- (i) $K > 0$ and K^p and K^{2p} satisfy **Assumption K**;
- (ii) $\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left[|u_t|^{2p} \mid \mathcal{F}_{t-1} \right] < \infty$, a.s.,

Then for each x , under H_0 ,

$$\hat{t}(x, m_0) \xrightarrow{d} N(0, 1).$$

- **Comparison with the stationary case:** For the case where x_t is a stationary linear process on i.i.d. innovations (possibly heavy tailed), under some additional technical conditions on the smoothness of m it can be shown (cf. Wu and Mielniczuk, 2002; p. 1451) that (24) holds with $\gamma_n = 1$ and $\varrho(0)$ replaced with $\Phi_{x_1}(0)$ i.e. the density of x_1 at the origin.

- **Comparison with the strongly nonstationary case:** For x_t fractional $1 - 1/\alpha < d < 2 - 1/\alpha$, $\alpha \in (0, 2]$, under smoothness requirements similar to those of condition (v) of the Theorem above, (24) holds (cf. Jeganathan, 2008; Theorem 4) with $\gamma_n = \gamma_n^*$ and $\varrho(0)$ replaced by $L_{\Lambda_{\alpha,d}}(0, 1)$ i.e. the local time of the fractional stable motion $\Lambda_{\alpha,d}$ –see (12).

These results show that the kernel regression estimator has conventional limit distribution irrespective of the stationarity properties of the data and moments properties of the covariate. Therefore, kernel based inference is conventional.

Finally, we consider the case of heavy-tailed regression errors satisfying **Assumption HTE**. We restrict attention to WNP covariates satisfying either **Assumption BFR2** or **Assumption MI**. Before stating the corresponding limit result, we introduce some notation. Set

$$\tau_1 := \frac{\sigma_u^{\alpha_u} \varrho(0)^{1-\alpha_u} \int_{\mathbb{R}} |K(x)|^{\alpha_u} dx}{\left(\int_{\mathbb{R}} K(x) dx\right)^{\alpha_u}}, \quad \tau_2 := \frac{\int_{\mathbb{R}} K(x)^{\langle \alpha_u \rangle} dx}{\int_{\mathbb{R}} |K(x)|^{\alpha_u} dx}.$$

These quantities are the scale and skewness parameters of the limit distribution of the rescaled score term associated with the NW estimator.

Theorem 9. Suppose that $\{y_t\}$ is generated by (23). In addition:

- (i) **Assumption INN** holds with $\theta = 1$ and x_t satisfies either **Assumption BFR2** or **Assumption MI**;
- (ii) x_t, u_t satisfy **Assumption HTE**;
- (iii) K and $|K|^{\alpha_u}$ satisfy **Assumption K**, and $\int K > 0$;
- (iv) $\{h_n\}$ a positive sequence such that $h_n + \gamma_n(nh_n)^{-1} \rightarrow 0$;
- (v) There is some function $\bar{m}(x, z)$ such that $|m(x + hz) - m(x)| \leq h^\nu \bar{m}(x, z)$, with $\nu \in (0, 1]$ for all h sufficiently small, and $\int_{\mathbb{R}} \bar{m}(x, z) |K(z)| dz < \infty$ for all $x \in \mathbb{R}$.

(a) Suppose that $\alpha_u > 1$. Then, as $n \rightarrow \infty$,

$$\hat{m}(x) \xrightarrow{p} m(x).$$

If in addition $\left(\frac{nh_n}{\gamma_n}\right)^{1-1/\alpha_u} h_n^\nu \rightarrow 0$, then

$$\left(\frac{nh_n}{\gamma_n}\right)^{1-1/\alpha_u} [\hat{m}(x) - m(x)] \xrightarrow{d} S_{\alpha_u} \{\tau_1, \beta_u^* \tau_2\}. \quad (26)$$

(b) Suppose that $\alpha_u = 1$. Then, as $n \rightarrow \infty$,

$$\hat{m}(x) \xrightarrow{d} m(x) + S_{\alpha_u} \{\tau_1, \beta_u^* \tau_2\}. \quad (27)$$

(c) Suppose that $\alpha_u < 1$. Then, as $n \rightarrow \infty$,

$$\left(\frac{nh_n}{\gamma_n}\right)^{1-1/\alpha_u} \hat{m}(x) \xrightarrow{d} S_{\alpha_u} \{\tau_1, \beta_u^* \tau_2\}. \quad (28)$$

Remark 16. **Theorem 9** highlights important limitations of the NW estimator in the presence of heavy tailed regression errors. In particular, the estimator is consistent only when $\alpha_u > 1$. For $\alpha_u = 1$, it converges in distribution to a nondegenerate random pseudo-true limit, whereas for $\alpha_u < 1$, it diverges in absolute value. Even when consistency holds, the convergence rate is slower than the conventional Gaussian rate for $\alpha_u \in (1, 2)$, and the limiting distribution is non-Gaussian and depends on unknown nuisance parameters, including the tail index, scale, and skewness of the regression errors. Consequently, conventional Gaussian-based inference is generally invalid, while inference based on the stable limiting distribution is non-pivotal, complicating the construction of valid statistical tests.

6. IV Methods for Parametric Estimation

The focus of this section is the parametric estimation of regressions of the form

$$y_t = \mu + \beta f(x_{t-1}) + u_t. \quad (29)$$

As in the previous section, x_t and u_t are assumed to be $\mathcal{F}_t$ -measurable with respect to an appropriate filtration $\mathcal{F}_t$, $t \geq 0$, and $\{u_t, \mathcal{F}_t\}$ forms a martingale difference sequence. The regression function $f(x)$ is assumed to be known, whereas the parameter vector $\mathbf{b} := [\mu, \beta]' \in \mathbb{R}^2$ is unknown.

We consider an IV estimator that uses reduced-growth transformations, or weights, of the covariates as instruments. We refer to this estimator as the weighted IV estimator (WIV hereafter). Comparable estimators have been considered by Blattberg and Sargent (1971), Samorodnitsky, Kurz-Kim, and Rachev (2007), and Kim and Meddahi (2020), among others. Samorodnitsky et al. (2007) employ signed-power functions as weights and show that, under assumptions more restrictive than those imposed in the present work, the resulting estimator is efficient. In our framework, when the covariate is stationary, reduced-growth transformations ensure the existence of sufficiently high moments for an LLN to apply; see, for example, Remark 12. Similarly, such transformations facilitate the development of our limit theory when the covariate is either an HT-WNP or an HT-SNP.

It should be emphasised, however, that the WIV estimator does not always support conventional inference under strong nonstationarity. To address this limitation, we consider a more sophisticated IV method in the subsequent section. The WIV estimator of the slope parameter in (1) takes the form

$$\tilde{\mathbf{b}}_{WIV} = \left[\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right]^{-1} \left[\sum_{t=1}^n \mathbf{w}_{t-1} y_t \right], \quad (30)$$

with $\mathbf{x}'_t = [1, f(x_t)]$ and $\mathbf{w}'_t = [w_0(x_t), w_1(x_t)]$. In particular, we assume that the weight functions $w_i(\cdot)$, $i = \{0, 1\}$ satisfy the following growth

$$|f(x)w_i(x)| = O(|x|^{\lambda_i}), \quad \lambda_i \in (0, \alpha). \quad (31)$$

For example, the *Cauchy* estimator (see e.g. So and Shin (1999), Phillips, Park and Chang (2004), Choi, Jacewitz and Park (2016), Kim and Meddahi (2020) among others) is a special case of the WIV estimator with $w_1(x) = \text{sgn}(x)$.

It follows from (30) that the WIV estimator involves nonlinear transformations through both the weight functions and the regression function f . For the case where

x_t is a nonstationary process, we consider a class of functions introduced by Park and Phillips (1999, 2001). In particular, we focus on locally integrable functions that are not globally integrable. Although **Theorem 2** (with $h_n = 1$) also enables the analysis to be extended to integrable transformations, we restrict attention to the nonintegrable case for brevity and because such transformations arise more frequently in applications. In particular, we consider functions that are asymptotically homogeneous, as defined below (see also DK2021).

Definition 4 (Asymptotically Homogeneous Function).

- (i) Let $\{x_t\}$ a random sequence and μ_n a strictly positive sequence. $g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is an **asymptotically homogeneous function** for $\{x_t, \mu_n\}$ (**AHF** $\{x_t, \mu_n\}$, hereafter), if for all $\lambda > 0$ it admits the decomposition

$$g(x) = k_g(\lambda)g_H(x/\lambda) + R_g(x, \lambda),$$

where $k_g : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$, $g_H : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is locally integrable, $R_g : \mathbb{R} \times \mathbb{R}_+^* \rightarrow \overline{\mathbb{R}}$ and,

$$R_{g,n} := \frac{1}{k_g(\mu_n)n} \sum_{t=1}^n \mathbf{E} |R_g(x_t, \mu_n)| = o(1),$$

as $n \rightarrow \infty$. g_H and k_g are the limit homogeneous function and asymptotic order of g respectively.

- (ii) If $R_g(x, \lambda) = 0$, g is a **homogeneous function** (**HF**, hereafter).

- (iii) A matrix $\mathbf{A}$ is **AHF** $\{x_t, \mu_n\}$, if its elements are **AHF** $\{x_t, \mu_n\}$.

Next, set

$$\mathbf{A}(x) := \begin{bmatrix} w_0(x) & (w_0 \cdot f)(x) \\ w_1(x) & (w_1 \cdot f)(x) \end{bmatrix}, \quad \mathbf{B}(x) := \begin{bmatrix} (w_0 \cdot f)(x) \\ (w_1 \cdot f)(x) \end{bmatrix},$$

$$\mathbf{W}(x) := \begin{bmatrix} w_0^2(x) & (w_0 \cdot w_1)(x) \\ (w_0 \cdot w_1)(x) & w_1^2(x) \end{bmatrix},$$

Under nonstationarity, we assume that the elements of the matrices above are **AHF**, for some appropriate random array and sequence. We denote the corresponding limit homogeneous matrix-valued functions by $\mathbf{A}_H(x)$, $\mathbf{B}_H(x)$, $\mathbf{W}_H(x)$.

Moreover, we consider following t-statistic for the null hypothesis $H_0 : \beta = \beta_0$,

$$\tilde{t}_{WIV} = \frac{\tilde{\beta}_{WIV} - \beta_0}{\sqrt{\tilde{\sigma}_u^2 \left[\left[\sum_{t=1}^n \mathbf{A}(x_{t-1}) \right]^{-1} \sum_{t=1}^n \mathbf{W}(x_{t-1}) \left[\sum_{t=1}^n \mathbf{A}'(x_{t-1}) \right]^{-1} \right]_{22}}},$$

where

$$\tilde{\sigma}_w^2 = \frac{\sum_{k=1}^n [y_t - \tilde{m}_{t-1}]^2 w_u(x_{t-1})}{\sum_{k=1}^n w_u(x_{t-1})}, \quad w_u(x) \geq 0 \quad (32)$$

and $\tilde{m}_{t-1} := \mathbf{x}'_{t-1} \tilde{\mathbf{b}}$. In addition, to simplify our exposition we assume that the weight w_u is an HF. i.e. for all $\lambda > 0$,

$$w_u(\lambda x) = k_{w_u}(\lambda)w_u(x).$$

The decomposition above is relevant only when x_t is nonstationary. The following two results characterize the limiting behavior of $\tilde{t}_{WIV}$ is an SP and a WNP, respectively. In both cases inference is asymptotically Gaussian.

Theorem 10. Suppose that y_t is generated by (29).

(a) In addition, suppose that the following conditions hold:

- (i) **Assumption INN** holds;*
- (ii) x_t satisfies **Assumption SP**;*
- (iii) $\mathbf{E} \|\mathbf{A}(x_t)\|, \mathbf{E} \|\mathbf{W}(x_t)\|^{1+\delta} < \infty$, for some $\delta > 0$;*
- (iv) **Assumption MDE** holds;*

Then as $n \rightarrow \infty$,

$$\sqrt{n} (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) \xrightarrow{d} [\mathbf{E} (\mathbf{A}(x_t))]^{-1} \cdot N \left[\mathbf{0}, \sigma_u^2 \mathbf{E} (\mathbf{W}(x_t)) \right].$$

(b) In addition, suppose that for some $1 < p < 2$ the following conditions hold:

- (v) $\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} [|u_t|^{2p} | \mathcal{F}_{t-1}] < \infty$ a.s.*
- (vi) $\mathbf{E} [f(x_t)^2 w_u(x_t)], \mathbf{E} [w_u(x_t)^p] < \infty$.*

Then under H_0 , as $n \rightarrow \infty$,

$$\hat{t}_{WIV} \xrightarrow{d} N(0, 1).$$

The following result gives the limit distribution of the WIV estimator when the covariate is an HT-WNPs process.

Theorem 11. Suppose that $\{y_t\}$ is generated by (29).

(a) In addition:

- (i) **Assumption INN** holds with $\theta = 1$ and x_t satisfies either **Assumption BFR** or **Assumption MI**;*
- (ii) x_t, u_t satisfy **Assumption MDE**;*
- (iii) $\mathbf{A}(x), \mathbf{W}(x)$ are **AHF** $\{x_t, \gamma_n\}$ and their elements are $O(|x|^\lambda)$, $\lambda \in (0, \alpha)$;*
- (iv) The inverse of $\int_{\mathbb{R}} \mathbf{A}_H(x) \varrho(x) dx$ exists a.s.*

Then as $n \rightarrow \infty$,

$$\begin{aligned} & \sqrt{n} \text{diag}\{1, k_f(\gamma_n)\} (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) \\ & \xrightarrow{d} \left[\int_{\mathbb{R}} \mathbf{A}_H(x) \varrho(x) dx \right]^{-1} \cdot MN \left[\mathbf{0}, \sigma_u^2 \int_{\mathbb{R}} \mathbf{W}_H(x) \varrho(x) dx \right]. \end{aligned} \quad (33)$$

(b) In addition to the conditions (i)-(iv), suppose there is some $1 < p < 2$ such that:

- (v) w_u is an **HF** and $f w_u, f^2 w_u$ are **AHF** $\{x_t, \gamma_n\}$;*
- (vi) $f^2 w_u, w_u^p$, are of order $O(|x|^\lambda)$, $\lambda \in (0, \alpha)$;*

(vii) $\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left[|u_t|^{2p} \mid \mathcal{F}_{t-1} \right] < \infty$ a.s.

Then as $n \rightarrow \infty$,

$$\tilde{t}_{WIV} \xrightarrow{d} N(0, 1).$$

Example 1 (Cauchy estimator). Suppose that $f(x) = x$ and $w_1(x) = \text{sgn}(x)$, $w_0(x) = 1$. Then (33) holds for all $1 < \alpha \leq 2$. In particular, the IV estimator for the slope parameter β satisfies

$$\sqrt{n}\gamma_n (\tilde{\beta}_{WIV} - \beta) \xrightarrow{d} MN \left[0, \sigma_u^2 \frac{1 - [\int_{\mathbb{R}} \text{sgn}(x) \varrho(x) dx]^2}{[\int_{\mathbb{R}} |x| \varrho(x) dx - X - \int_{\mathbb{R}} \text{sgn}(x) \varrho(x) dx]^2} \right]$$

Example 2. Suppose that $f(x) = x$ and $w_1(x) = \text{sgn}(x) |x|^{-b_1}$, $w_0(x) = \text{sgn}(x) |x|^{-b_0}$ with $0 < 2b_1, 2b_0 < 1$, $b_1 \neq b_0$. Then (33) holds for all $1 - (b_1 \wedge b_0) < \alpha \leq 2$.

The following result establishes the convergence rate of the WIV estimator when the covariate is an SNP process satisfying **Assumption SNP***. In particular, it shows that WIV attains a parametric rate of convergence and is therefore more suitable for estimation than IVX. Recall that, under nonstationarity, IVX generally attains only semiparametric convergence rates, see e.g. Phillips and Magdalinos (2009), KMS2015. Moreover, this convergence-rate result is used to establish the consistency of an estimator of σ_u^2 based on WIV residuals. This variance estimator is subsequently employed in the studentization of IVX-based t-statistics.

The following result characterizes the limiting behavior of the WIV estimator when the regression errors satisfy Assumption HTE and the covariate is a weakly nonstationary process (WNP). To simplify the exposition, we exclude BFR1 processes and restrict attention to homogeneous weight functions (**HF**).

Theorem 12. Suppose that $\{y_t\}$ is generated by (29). In addition:

- (i) **Assumption INN** holds with $\theta = 1$ and x_t satisfies either **Assumption BFR2** or **Assumption MI**;
- (ii) x_t, u_t satisfy **Assumption HTE**;
- (iii) $\mathbf{A}(x)$, is **AHF** $\{x_t, \gamma_n\}$ and its elements are $O(|x|^\lambda)$, $\lambda \in (0, \alpha)$;
- (iv) for $i = \{0, 1\}$, $w_i(x)$, $|w_i(x)|^{\alpha_u}$, is **HF** and $|w_i(x)|^{\alpha_u} = O(|x|^{\lambda_i})$, $\lambda_i \in (0, \alpha)$;
- (v) The inverse of $\int_{\mathbb{R}} \mathbf{A}_H(x) \varrho(x) dx$ exists.

Then as $n \rightarrow \infty$,

$$n^{1-1/\alpha_u} \text{diag}\{1, k_f(\gamma_n)\} (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) \xrightarrow{d} \left[\int_{\mathbb{R}} \mathbf{A}_H(x) \varrho(x) dx \right]^{-1} \cdot \mathbf{S}_u, \quad (34)$$

where for $\mathbf{v} \in \mathbb{R}^2$,

$$\mathbf{E} \left(e^{i\mathbf{v}'\mathbf{S}_u} \right) = \exp \left\{ -\sigma_u^{\alpha_u} \left[\int_{\mathbb{R}} |\mathbf{v}'\mathbf{w}(x)|^{\alpha_u} [1 + i\beta_u^* \text{sgn}(\mathbf{v}'\mathbf{w}(x))] \tan \left(\frac{\pi\alpha_u}{2} \right) \varrho(x) dx \right] \right\}.$$

Remark 17. The limit result of (34) shows that, in contrast to (33), the WIV estimator generally has a non-Gaussian and non-pivotal stable limiting distribution involving the nuisance parameters $\sigma_u, \alpha_u, \beta_u^*$. Moreover, for $\alpha_u < 2$, the convergence rates are slower than those obtained under **Assumption MDE**. An important implication of this result is that the consistency of the WIV estimator depends on the relative strength of the regression signal and the heavy-tailed noise. In particular, when $\alpha_u \leq 1$, the intercept estimator is inconsistent. For $\alpha_u = 1$, its estimation error converges to a nondegenerate stable distribution, whereas for $\alpha_u < 1$, the estimation error diverges in absolute value in probability. By contrast, the slope estimator may remain consistent even when $\alpha_u \leq 1$. Specifically, provided its normalized limiting distribution is nondegenerate, consistency holds if and only if

$$n^{1-1/\alpha_u} k_f(\gamma_n) \rightarrow \infty$$

Thus, sufficiently strong growth of the regression function can compensate for the heavy-tailed regression disturbances, preserving consistency of the slope estimator even when the intercept estimator is inconsistent.

Theorem 13. Suppose that $\{y_t\}$ is generated by (23).

(a) In addition suppose that the following hold:

- (i) x_t satisfies **Assumption INN** and **Assumption SNP***;
- (ii) $\mathbf{A}(x)$, $\mathbf{W}(x)$ are **AHF** $\{x_t, \gamma_n^*\}$ and their elements satisfy the assumptions imposed on f in **Theorem 6**;
- (iii) x_t and u_t satisfy **Assumption MDE**.

Then as $n \rightarrow \infty$,

$$\sqrt{n} \text{diag}\{1, k_f(\gamma_n^*)\} (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) = O_p(1).$$

(b) Suppose that in addition to the conditions (i)-(iii) there is some $1 < p < 2$ such that:

- (iv) w_u is an **HF** and $f w_u, f^2 w_u$ are **AHF** $\{x_t, \gamma_n^*\}$;
- (v) $w_u f^2, w_u f, w_u^p$ satisfy the assumptions imposed on f in **Theorem 6**;
- (vi) $\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} [|u_t|^{2p} \mid \mathcal{F}_{t-1}] < \infty$ a.s.

Then, as $n \rightarrow \infty$,

$$\tilde{\sigma}_w^2 \xrightarrow{p} \sigma_u^2.$$

Remark 18. (WIV with SNP Covariates). (a) It least for the case where FCLTs apply, it can demonstrated directly that the IV estimator has limit distribution determined by stochastic integrals. Therefore, in this case inference is non-conventional and test statistics have limit distributions that depend on nuisance parameters. To see this suppose that weak limit of (12) can be strengthened to weak convergence in the $D[0, 1]$ space i.e.

$$\gamma_n^{*-1} x_{[nr]} \Rightarrow \Lambda_{\alpha, d}(r), \quad r \in [0, 1].$$

This is true if $\alpha = 2$ or $d \geq 1$, see e.g. Astrauskas (1983), Avram and Taqu (1992). Arguments similar to those used in Park and Phillips (2001), Robsinon and Hualde (2003)

$$\sqrt{n} \text{diag}\{1, k_f(\gamma_n^*)\} (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) \xrightarrow{d} \left[\int_0^1 \mathbf{A}_H(\Lambda_{\alpha,d}(r)) dr \right]^{-1} \cdot \int_0^1 \mathbf{B}_H(\Lambda_{\alpha,d}(r)) dU(r), \quad (35)$$

where U is a Brownian Motion relating to the regression error u_t . Notice that stochastic integral above mixed normal only under very restrictive conditions i.e. $\Lambda_{\alpha,d} \perp U$. In practice, this requires strong exogeneity in the model.

(b) Some preliminary results (cf. Kasparis and Wang (2026)) indicate that for $0 < \alpha < 2$ the $\tilde{\mathbf{b}}_{WIV}$ estimator has a mixed normal distribution with the associate score term having asymptotic distribution

$$MN \left[\mathbf{0}, \sigma_u^2 \int_0^1 \mathbf{B}_H^2(\Lambda_{\alpha,d}(r)) dr \right].$$

An intuitive explanation is provided by the following result. As shown, for example, by Jeganathan (2008; eq. (133)), for $0 < \alpha < 2$, there is some SV $\varsigma(x)$ satisfying $\varsigma(x)^\alpha \sim H(x^{1/\alpha} \varsigma(x))$ such that, as $n \rightarrow \infty$,

$$\left[\frac{1}{n^{1/\alpha} \varsigma(n)} \sum_{t=1}^{\lfloor nr \rfloor} \xi_t, \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor nr \rfloor} u_t \right] \xrightarrow{f.d.d.} [Z_\alpha(r), U(r)],$$

where Z_α is an α -stable Lévy process, U a Brownian motion and $Z_\alpha \perp U$. This asymptotic independence suggests that, in this setting, the WIV estimator may support conventional inference. Nevertheless, two qualifications are important:

(i) The WIV estimator is not robust in general. In particular, if ξ_t belongs to the domain of attraction of a normal law, the resulting inference is nonconventional and is governed by stochastic-integral limits, as in (35).

(ii) The simulation results presented in this work show that WIV-based t-tests exhibit substantial finite-sample size distortions when the stability parameter satisfies $\alpha > 1$.

7. IVX for Parametric Inference

The results of the previous section demonstrate that the WIV estimator does not, in general, support conventional inference when the covariate is an HT-SNP. To address this limitation, we develop a weighted extension of IVX (W-IVX) that facilitates conventional inference for the slope parameter in (29). A central feature of IVX is the construction of instruments that are either exactly or approximately mildly integrated (MI), depending on the persistence properties of the original covariate. This construction reduces the degree of persistence of the instruments and facilitates asymptotically Gaussian inference. In the present heavy-tailed framework, however, the MI structure alone is not sufficient to ensure the desired limiting behavior of the relevant sample functionals. The weight functions are therefore introduced to control the growth of functionals of the IVX instruments, ensuring, in particular, that condition (ii) of **Theorem 1** is satisfied. This allows us to exploit the limit theory developed in Section 3. and establish conventional inference for the resulting W-IVX estimator. In this respect, W-IVX can be viewed as the semiparametric counterpart of WIV, in much the same way that the IVX estimator of Phillips and

Magdalinos (2009) provides a semiparametric counterpart to OLS in conventional linear predictive regressions. The distinction reflects a trade-off between estimation efficiency and robustness of inference: while WIV typically achieves faster, parametric convergence rates, W-IVX sacrifices some of this convergence speed in exchange for conventional, asymptotically pivotal inference across a broader range of persistence regimes.

IVX methods were introduced by Phillips and Magdalinos (2009) and further developed by KMS2015 and Magdalinos and Petrova (2022), among others. These studies show that IVX-based tests support conventional inference for a wide range of regressors. Specifically, the original IVX results apply to linear models in which the regressors are finite-variance nearly integrated or mildly integrated processes; see Phillips and Magdalinos (2009). KMS2015 extend these results to stationary short-memory regressors, while Magdalinos and Petrova (2022) further generalize the method to accommodate potentially explosive or mildly explosive covariates driven by short-memory innovations. Extensive simulation studies indicate that IVX-based tests exhibit very good finite-sample performance and outperform competing methods in most scenarios. For further details, see KMS2015 and Magdalinos and Petrova (2022).

Existing studies, however, do not consider fractionally integrated or heavy-tailed regressors or nonlinear regression functions. Thus, the validity of IVX inference with fractionally integrated regressors has not previously been established. Indeed, Andersen and Varneskov (2021, Remark 9) describe their LCM methodology as an analogue of IVX for general multivariate fractionally integrated processes and appear to suggest that IVX does not, in general, deliver Gaussian inference in fractional settings:

“...however, [for IVX] the dependent variable is restricted from exhibiting long-range dependence. Whereas the IVX inference is not Gaussian uniformly in the persistence regimes, the corresponding Wald significance test is robust to the specific form of persistence.”

In this study, we extend the IVX framework by allowing for nonlinearities in both the regression and instrument functions. We also accommodate heavy-tailed, SNP, MI and SP fractional regressors. Contrary to the suggestion that IVX does not support Gaussian inference in fractional settings, we show that the resulting IVX-based t-statistics have standard Gaussian limit distributions even when the regressors are both fractionally integrated and heavy-tailed.

7.1. IVX Instruments

We establish the validity of the IVX method when the covariates are HT-SNPs, HT-MI or HT-SPs. Specifically, given a series x_t , the IVX instruments $z_t(n)$ are obtained by applying the following linear filter:

$$z_t := z_t(n) := \sum_{j=1}^t \rho_{z,n}^{t-j} (x_j - x_{j-1}), \quad (36)$$

where $\rho_{z,n}$ is defined as in (5) i.e.

$$\rho_{z,n} := 1 - \kappa_{z,n}^{-1}, \quad \kappa_{z,n} := n^{\alpha_{z,\kappa}} \ell_{z,\kappa}(n), \quad (37)$$

with $\kappa_{z,n} > 0$ such that $\kappa_{z,n} \rightarrow \infty$ and $\sup_{n \geq 1, 1 \leq t \leq n} \sup \kappa_{z,t} / \kappa_{z,n} < \infty$, $\ell_{z,\kappa}$ SV and $\alpha_{z,\kappa} \in [0, 1)$.

Recall that ρ_n denotes the model-specific sequence associated with the autoregressive parameter of x_t when x_t is a MI process satisfying **Assumption MI**, whereas $\rho_{z,n}$ denotes the analogous method-specific sequence associated with the IVX method. In this context, the localizing sequence $\kappa_{z,n}$ functions analogously to a bandwidth parameter and is selected by the practitioner. The IVX estimator achieves conventional inference under minimal conditions on $\kappa_{z,n}$, namely $\kappa_{z,n}^{-1} + \kappa_{z,n} n^{-1} \rightarrow 0$.

If x_t is an HT-SNP or HT-MI process, then z_t behaves asymptotically as an HT-MI process whose underlying autoregressive sequence is either ρ_n or $\rho_{z,n}$. Finally, if x_t is a HT-SP, then z_t is asymptotically equivalent to x_t . These results parallel those established by KMS2015.

To keep the length of the paper manageable, we impose some restrictions on the parameter space of x_t . Specifically, we only consider array (i.e. NI and MI) covariates driven by short memory errors and stability parameter $1 < \alpha \leq 2$. The latter restriction is not binding when x_t is a pure $I(1)$ process, that is, when $\rho_n = 1$. The asymptotic behavior of the IVX instruments depends on the underlying properties of x_t and may be summarized as follows:

HT-SNP: If $x_t \sim \text{HT-SNP}$ with $\rho_n = 1$, then we have

$$d \in \left\{ \begin{array}{l} (1 - 1/\alpha, 1), \\ \{1\}, \\ (1, 2 - 1/\alpha) \end{array} \right\} \xrightarrow{\text{IVX filtering}} z_t \sim \left\{ \begin{array}{l} \text{MI-NM, on } \rho_{z,n} \\ \text{MI-SM, on } \rho_{z,n} \\ \text{MI-LM on } \rho_{z,n} \end{array} \right.$$

HT-SNP/MI: (arrays) If $x_t \sim \text{HT-SNP-SM}$ or HT-MI-SM , with $\kappa_n / \kappa_{z,n} \rightarrow \infty$, then we have

$$x_t \sim \text{SNP / MI-SM, on } \rho_n \xrightarrow{\text{IVX filtering}} z_t \overset{\text{approx.}}{\sim} \text{MI-SM, on } \rho_{z,n}.$$

In addition, if $\kappa_{z,n} / \kappa_n \rightarrow \infty$,

$$x_t \sim \text{MI-SM, on } \rho_n \xrightarrow{\text{IVX filtering}} z_t \overset{\text{approx.}}{\sim} \text{MI-SM, on } \rho_n.$$

HT-SP: If $x_t \sim \text{HT-SP}$,

$$x_t \sim \text{HT-SP} \xrightarrow{\text{IVX filtering}} z_t \overset{\text{approx.}}{\sim} x_t.$$

As indicated above, when $x_t \sim \text{HT-SNP}$ with $\rho_n = 1$, the IVX instruments z_t are exactly MI processes. When x_t is a nonstationary array, either NI or MI, the IVX instruments behave asymptotically as MI processes. Finally, when x_t is stationary, the IVX instruments approximate x_t . The latter two approximation results are formally stated in the following lemma. Before presenting the lemma, we introduce an additional normalising sequence that supplements **Definition 1**.

Definition 1*. Given a localizing sequence $\kappa_{z,n}$, under **Assumption HT-SNP** or **Assumption HT-MI**, define $\gamma_{z,n}$ by

$$\gamma_{z,n} \sim \kappa_{z,n}^{b/\alpha} \ell_1(\kappa_{z,n})^{1/\alpha},$$

where b and ℓ_1 are defined in **Definition 1**.

Lemma 2 below introduces random arrays that asymptotically approximate the IVX instruments z_t suitable suitable normalization. The corresponding normalizing sequences, which characterize the orders of magnitude of these instruments, are specified in the following definition.

Definition 5. Suppose that x_t satisfies either **Assumption SNP** or **Assumption MI**. Define the normalizing sequence μ_n by

$$\mu_n := \begin{cases} \gamma_{z,n}, & \text{under } \mathbf{Assumption\ SNP} \text{ or } \mathbf{Assumption\ MI} \text{ with } \kappa_n/\kappa_{z,n} \rightarrow \infty, \\ \gamma_n, & \text{under } \mathbf{Assumption\ MI} \text{ with } \kappa_{z,n}/\kappa_n \rightarrow \infty. \end{cases}$$

Lemma 2. Suppose that z_t is defined by (36) and $\rho_{z,n}$ by (37). The following hold:

- (a) Suppose that x_t satisfies **Assumption SNP-SM** or **Assumption MI-SM** with $\kappa_n/\kappa_{z,n} \rightarrow \infty$. Then, for all $1 < \alpha' < \alpha$ as $n \rightarrow \infty$,

$$\frac{1}{\mu_n} \sup_{1 \leq t \leq n} \mathbf{E} \left| z_t(n) - \sum_{j=1}^t \rho_{z,n}^{t-j} v_j \right|^{\alpha'} = o(1).$$

- (b) Suppose that x_t satisfies **Assumption MI-SM** with $\kappa_{z,n}/\kappa_n \rightarrow \infty$. Then, for all $1 < \alpha' < \alpha$ as $n \rightarrow \infty$,

$$\frac{1}{\mu_n} \sup_{1 \leq t \leq n} \mathbf{E} |z_t(n) - x_t|^{\alpha'} = o(1).$$

- (c) Suppose that $x_t = v_t$, where v_t satisfies **Assumption SP**. Then, for all $0 < \alpha' < \alpha$ as $n \rightarrow \infty$,

$$\sup_{1 \leq t \leq n} \mathbf{E} |z_t(n) - x_t|^{\alpha'} = o(1).$$

Remark 19. It can be easily seen from **Lemma 1** and **Lemma 2** that for nonstationary x_t ,

$$z_n = \begin{cases} O_p(\gamma_{z,n}), & \text{if } x_t \text{ HT-SNP or HT-MI with } \kappa_n/\kappa_{z,n} \rightarrow \infty \\ O_p(\gamma_n), & \text{if } x_t \text{ HT-MI with } \kappa_{z,n}/\kappa_n \rightarrow \infty \end{cases}.$$

It follows from the preceding discussion that the IVX instrument is obtained by applying a linear filter to the covariate x_t . As discussed in the previous section, when x_t is an SNP, parametric estimators such as W-IV generally do not have Gaussian limit distributions. By contrast, IVX filtering attenuates the signal in x_t sufficiently for an MCLT to apply, thereby validating conventional inference. This signal attenuation is reflected in the relative orders of magnitude of x_n and z_n when x_t is an HT-SNP. In particular, under our assumptions,

$$\gamma_{z,n}, \gamma_n = o(\gamma_n^*).$$

Thus, the normalizing sequence required for the weak convergence of x_n dominates those required for the weak convergence of z_n and of the corresponding mildly integrated process. To see this, note the following.

- γ_n^* is $RV_{\mathcal{H}}$ in n (i.e. regularly varying of order $\mathcal{H}$) with $\mathcal{H} = d - (1 - 1/\alpha)$ being the Hurst index -see e.g. Jeganathan (2004, 2008).
- γ_n is $RV_{\alpha_\kappa \cdot \mathcal{H}}$ in n with, $\alpha_\kappa \in [0, 1)$.⁷
- $\gamma_{z,n}$ is $RV_{\alpha_{\kappa_z} \cdot \mathcal{H}}$ in n with, $\alpha_{\kappa_z} \in [0, 1)$.

Before we present the IVX estimator, we introduce some notation. As usual, for any sequence $\{a_t\}_{t=1}^n$ set

$$\bar{\alpha} = \frac{\sum_{t=1}^n a_t w_0(z_{t-1})}{\sum_{t=1}^n w_0(z_{t-1})}$$

and $\bar{\alpha}_t = a_t - \bar{\alpha}$. Note that for $w_0 = 1$, $\bar{\alpha}$ reduces to the sample mean. Consider the weighted IVX estimator (W-IVX, hereafter) estimator for the slope parameter of (1)

$$\hat{\beta}_{IVX} = \left[\sum_{t=1}^n w_1(z_{t-1}) \bar{f}(x_{t-1}) \right]^{-1} \sum_{t=1}^n w_1(z_{t-1}) \bar{y}_t, \quad (38)$$

where z_t is given in (37), and w is some weight function. If we set $w(x) = x$ (and $f(x)$) we get the IVX estimator of Phillips and Magdalinos (2009) as special case. As mentioned before, the possibility of fractional nonstationary covariates has not been considered in the IVX literature. In addition, all existing results concern covariates with finite variance. Our limit theory is sufficient for the characterization of IVX based test-statistics for the case where x_t is an HT-SNP. Consider the IVX t-statistic for the $H_0 : \beta = \beta_0$

$$\hat{t}_{IVX} = \sum_{t=1}^n w_1(z_{t-1}) \bar{f}(x_{t-1}) \cdot \frac{\hat{\beta}_{IVX} - \beta_0}{\sqrt{\hat{\sigma}^2 \sum_{t=1}^n \bar{w}_1(z_{t-1})^2}}, \quad (39)$$

where $\hat{\sigma}^2$ is some consistent estimator for σ_u^2 e.g. eq. (32).

We start with the limiting behaviour of the IVX estimator when x_t is an HT-SP. KMS2015 show that, for finite-variance short-memory linear processes, the IVX estimator is asymptotically equivalent to OLS. In our framework, the parametric counterpart of W-IVX is WIV. This equivalence is important from a practical perspective, as it implies that W-IVX attains parametric convergence rates under stationarity. Consequently, W-IVX-based tests have very good asymptotic power under stationarity, comparable to that of tests based on parametric estimators. The following theorem establishes that WIV and W-IVX are asymptotically equivalent when the covariate is an HT-SP.

Theorem 14. Suppose that y_t is generated by (29). In addition, suppose that the following conditions hold:

- (i) **Assumption INN** holds with $\theta = 1$;
- (ii) z_t is defined by (36) and $\rho_{z,n}$ by (37);

⁷Recall that under **Assumption MI**, γ_n is RV_p in n with $p = \alpha_\kappa b/\alpha$, $b = 1 + \alpha - \alpha m$. Moreover, the relationship between the memory parameter δ of v_t and the exponent m is $\delta = 1 - m$, while $d = 1 + \delta$. Consequently, $p = \alpha_\kappa [d - (1 - 1/\alpha)]$.

- (iii) x_t satisfies **Assumption SP**;
- (iv) **Assumption MDE** holds;
- (v) $\mathbf{E} \|\mathbf{A}(x_t)\|, \mathbf{E} \|\mathbf{W}(x_t)\|^{1+\delta} < \infty$, for some $\delta > 0$;
- (vi) the elements of $\mathbf{A}(x), \mathbf{W}(x)$ are $O(|x|^\lambda)$, $\lambda \in (0, \alpha)$;
- (vii) $\hat{\sigma}^2$ is consistent for σ_u^2 .

The following hold:

- (a) As $n \rightarrow \infty$,

$$\sqrt{n}(\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) = \sqrt{n}(\hat{\mathbf{b}}_{IVX} - \mathbf{b}) + o_p(1).$$

- (b) Under H_0 , as $n \rightarrow \infty$,

$$\hat{t}_{IVX} \xrightarrow{d} N(0, 1).$$

The conditions of **Theorem 14** are comparable to those of **Theorem 10**. However, the latter imposes an additional requirement, namely condition (vi), on the growth rates of certain nonlinear transformations involving the regression and weight functions. This technical condition facilitates the approximation established in **Lemma 2** without requiring any smoothness restrictions on these functions.

Next, we consider the case in which the covariate is either a nearly integrated or an MI heavy-tailed array. Here, we restrict attention to covariates driven by short-memory errors and assume that $1 < \alpha \leq 2$. These restrictions are imposed to facilitate the approximation established in **Lemma 2**. Finally, we require the relevant nonlinear functionals to satisfy **Definition LH**. This latter restriction can be dispensed with if $\mu_t^{-1}z_t$ possesses a uniformly bounded density from some index onward. We conjecture that this property holds under our assumptions, but leave its formal verification, and hence the corresponding extension of the result, for future work.

Theorem 15. Suppose that y_t is generated by (29). In addition, suppose that the following hold:

- (i) **Assumption INN** holds with $1 < \alpha \leq 2$;
- (ii) z_t is defined by (36) and $\rho_{z,n}$ by (37);
- (iii) x_t satisfies either **Assumption SNP-SM**, or **Assumption MI-SM** with $\kappa_n/\kappa_{z,n} \rightarrow \infty$ or **Assumption MI-SM** with $\kappa_{z,n}/\kappa_n \rightarrow \infty$;
- (iv) **Assumption MDE** holds;
- (v) $\mathbf{W}(x)$ is **AHF** $\{z_t, \mu_n\}$ and **LH**;
- (vi) $\int_{\mathbb{R}} [w_{1,H}(x) - a_H w_{0,H}]^2 \varrho(x) dx > 0$, $a_H := \frac{\int_{\mathbb{R}} w_{1,H}(x) \varrho(x) dx}{\int_{\mathbb{R}} w_{0,H}(x) \varrho(x) dx}$;
- (vii) $\hat{\sigma}^2$ is consistent for σ_u^2 .

Then under H_0 , as $n \rightarrow \infty$,

$$\hat{t}_{IVX} \xrightarrow{d} N(0, 1).$$

Finally, the following limit result establishes that $\hat{t}_{IVX}$ is asymptotically standard normal when the covariate is HT-SNP process with $\rho_n = 1$.

Theorem 16. Suppose that y_t is generated by (29). In addition, suppose that the following conditions hold:

- (i) **Assumption INN** holds;
- (ii) z_t is defined by (36) and $\rho_{z,n}$ by (37);
- (iii) x_t satisfies **Assumption SNP** with $\rho_n = 1$;
- (iv) **Assumption MDE** holds;
- (v) $\mathbf{W}(x)$ is **AHF** $\{z_t, \mu_n\}$ and its elements are $O(|x|^\lambda)$, $\lambda \in (0, \alpha)$;
- (vi) $\int_{\mathbb{R}} [w_{1,H}(x) - a_H w_{0,H}]^2 \varrho(x) dx > 0$, $a_H := \frac{\int_{\mathbb{R}} w_{1,H}(x) \varrho(x) dx}{\int_{\mathbb{R}} w_{0,H}(x) \varrho(x) dx}$;
- (vii) $\hat{\sigma}^2$ is consistent for σ_u^2 .

Then under H_0 , as $n \rightarrow \infty$,

$$\hat{t}_{IVX} \xrightarrow{d} N(0, 1).$$

8. Simulations

The tables and figures discussed in this section are collected in Section 20 at the end of the appendix.

In this section we explore the finite sample performance of the weighted IVX t-statistic -cf. (39) for the hypothesis

$$H_0 : \beta = 0$$

$$H_1 : \beta = b \neq 0$$

in the context of the parametric regression of (1) i.e.

$$y_t = \mu + \beta f(x_{t-1}) + u_t, \quad t = 1, \dots, n.$$

Data are generated from 20,000 replication paths as follows. The covariate x_t is a type-II fractional process (e.g. Robinson and Hualde, 2003; Shimotsu and Phillips, 2005) generated from

$$(I - L)^d x_t = v_t I \{t > 0\},$$

where v_t is some innovation process that is specified in detail below. For of the memory parameter we consider a range of values in the interval $d \in [d_* - 0.6, d_* + 0.6]$ with $d_* := d_*(\alpha)$ being the nonstationarity boundary

$$d_* = 1 - 1/\alpha.$$

The regression errors are generated from

$$u_t = \alpha^{-1} \cdot \varepsilon_t^{(\alpha/\nu)}, \quad \nu = 2.1, \quad (40)$$

where ε_t is a secondary heavy tailed innovation specified below. Recall that $x^{(a)} = \text{sgn}(x)|x|^a$. Finally, we specify the innovation vector $[v_t, \varepsilon_t]$. For $t = 1, \dots, n$, $n = \{250, 500, 1000\}$, we consider three alternative DGPs:

- **DGP1:** $[v_t, \varepsilon_t]$ follows a bi-variate t-distribution with location parameter zero, α -degrees of freedom (tail parameter) and scale matrix

$$\Sigma_\delta = \begin{bmatrix} 1 & \delta \\ \delta & 1 \end{bmatrix}, \quad |\delta| < 1. \quad (41)$$

- **DGP2:** $[v_t, \varepsilon_t]$ is sub-Gaussian with stability parameter α , and scale matrix Σ_δ . This is equivalent to $[v_t, \varepsilon_t]$ having characteristic function $\phi(\mathbf{t}) = \exp\left(-\left|\frac{1}{2}\mathbf{t}'\Sigma_\delta\mathbf{t}\right|^{\alpha/2}\right)$. For more details, see Samorodnitsky and Taqqu (1994; p. 78).

The following values for the stability parameter are considered

$$\alpha = [2, 1.99, 1.8, 1.6, 1.4, 1.2, 1, 0.8].$$

Under **DGP1**, when $\alpha = 2$, $[v_t, \varepsilon_t]$ is in the domain of attraction of normal law but has infinite variance⁸. Under **DGP2**, when $\alpha = 2$, the vector $[v_t, \varepsilon_t]$ is Gaussian. In all cases, for $\alpha < 2$ ($\alpha < 1$), v_t, ε_t are infinite variance (mean) with tails $P(|v_t| > x), P(|\varepsilon_t| > x) \sim \text{cons.}x^{-\alpha}$. The formulation of (40) together with **DGP1-DGP2** postulates dependence between v_t (the innovations that drive the covariate) and u_t (regression error) are dependent with each other. In particular, for $\alpha < 2$, v_t is always infinite-variance whereas u_t is heavy tailed but has finite variance with tail exponent $\nu = 2.1$.

We consider two functional forms for the regression function $f(x)$ along with multiple choices of the weight function $w(x)$ for each regression function:

- **Specification 1:** $[f(x), w(x)] = [x, x^{(j)}], j = \{0, 0.5, 0.75, 1\}$;
- **Specification 2:** $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$.

The first specification corresponds to a linear-in-variables model with four alternative choices for the weight function. In particular, three of these weight functions grow at a slower rate than f . To assess the robustness of the proposed methods to more general specifications, the simulation set-up includes models that are more general than those permitted under our theoretical assumptions, specifically condition (v) of Theorem 16. The second specification involves a nonlinear regression function with an identical weight function; this specification is directly relevant for the empirical application presented in the following section. We note that for $[f(x), w(x)] = [x, x^{(0.5)}], [x^{(0.5)}, x^{(0.5)}]$ our theoretical assumptions on the growth rates of the transformation functions are satisfied whenever the stability parameter of the covariate x_t is $\alpha > 1$. Following KMS2015, we set the IVX bandwidth term

⁸In this case, as $x \rightarrow \infty$ $\mathbf{E}(v_t^2 I\{|v_t| \leq x\}) \sim 2 \ln(x)$, see e.g., Abadir and Magnus (2004)

to $\kappa_n^{-1} = n^{-0.95}$ and employ the studentization method recommended therein. This approach uses HAC estimates based on auto-regressive residuals. Whereas KMS2015 obtain regression residuals using OLS autoregressions, here we instead use WIV autoregressions (i.e., IV autoregressions that instrument with $w(x_{t-1})$).

We report size and power results for three methods:

- weighted IVX which uses the instrument $w(z_{t-1})$;
- WIV which uses the instrument $w(x_{t-1})$;
- OLS, interpreted as IV with instrument with instrument $f(x_{t-1})$.

It is worth noting that for Specification 1 with $w(x) = x$ and Specification 2, the IV and the OLS estimators coincide.

The simulation results under **DGP1** are nearly identical to those under **DGP2**. For this reason, we primarily report results for **DGP1**. Tables 6–13 present empirical size results for t-tests based on IVX, WIV, and OLS for Specification 1 with weight function $w(x) = x^{(0.5)}$, under **DGP1**. It is can easily seen from Tables 6-13 that IVX tests exhibit very good size for all configurations even if endogeneity is extreme (i.e. $\delta = \pm 0.99$) and sample sizes relatively small (e.g. $n = 250$). On the other hand the IV and OLS test exhibit severe size distortions when the covariate is in the nonstationary region with the former performing slightly better in general. The IVX tests in some cases are slightly oversized with empirical size around 8%. In contrast, the IV and OLS tests display severe size distortions when the covariate lies in the nonstationary region, with IV generally performing slightly better than OLS. In some cases, the IVX tests are mildly oversized, with empirical size around 8%. By comparison, for the same configurations the IV and OLS tests exhibit empirical sizes exceeding 35%, i.e., more than six times the nominal level. Interestingly, the IVX tests continue to exhibit very good size even when Condition iv of Theorem 16 is violated, suggesting that the IVX t-statistics still converge to a standard normal distribution, despite the fact that the basic limit theory (cf. Theorem 1), does not apply in this setting.

The simulation results under **DGP1** and **DGP2** are nearly identical. We therefore focus primarily on **DGP1**. Tables 6–13 report the empirical sizes of t-tests based on IVX, WIV, and OLS for Specification 1, with weight function $w(x) = x^{(0.5)}$. The IVX tests exhibit very good size control across the configurations considered, even under extreme endogeneity (i.e. $\delta = \pm 0.99$) and relatively small sample sizes ($n = 250$). Although mild over-rejection occurs in some cases, with empirical sizes reaching approximately 8%, the corresponding distortions are substantially smaller than those associated with WIV and OLS. In particular, when the covariate lies in the nonstationary region, the latter procedures exhibit severe size distortions, with empirical rejection frequencies exceeding 35% in some configurations, compared with a nominal level of 5%. Overall, WIV tends to perform slightly better than OLS. Interestingly, the IVX tests continue to exhibit very good size control even when condition (v) of **Theorem 16** is violated. This finding suggests that the validity of conventional IVX inference may extend beyond the sufficient conditions established by our limit theory, particularly since **Theorem 1** does not apply in these configurations.

Next, Tables 14–21 examine the effects of alternative weight functions on the empirical size of IVX tests under **DGP1**. Specifically, we consider the following choices for $w(x)$: $w(x) = x^{(j)}$, $j = \{0, 0.5, 0.75, 1\}$. Overall, size performance is very

good across specifications, although the choice $w(x) = x^{(0)}$ (the Cauchy estimator weight function) exhibits somewhat more pronounced over-sizing in certain cases, with empirical size approaching 9%.

Tables 22–29 report empirical size results for IVX and OLS tests under Specification 2 (i.e., a nonlinear regression function) for **DGP1**. We note that in this setting the WIV and OLS estimators coincide. The overall size performance is comparable to that reported in Tables 6–13: the IVX test continues to perform very well across all configurations, whereas the OLS test exhibits severe over-sizing when the covariate lies in the nonstationary region.

Finally, Tables 30–37 report empirical size results for IVX tests under **DGP2** for different choices of the weight function. The simulation design is comparable to that used in Tables 14–21. Once again, size performance is very good across all weight-function choices and across different memory and tail parameter configurations.

Finally, empirical power results for the three tests are presented in Figures 1–9. We focus on the case $n = 1000$ and $\delta = -0.95$, which aligns the Monte Carlo design with that of KMS2015. Regression errors are generated according to DGP1. The following values for the stability and memory parameters are considered:

$$\alpha = [2, 1.8, 1.6, 1.4], \quad d \in [d_* - 0.6, d_* + 0.6].$$

Figures 1–4 display empirical power results for IVX-, WIV-, and OLS-based tests under Specification 1. For nonstationary predictors, WIV and OLS yield higher empirical power, albeit at the cost of severe over-sizing under the null hypothesis. The asymptotic power advantage of these tests over IVX diminishes progressively as the process approaches the stationary region. For smaller values of the memory parameter, namely $d = d_* - 0.4, d_* - 0.6$, the power performance of all three tests is nearly identical. This finding is consistent with **Theorem 14**, which establishes the asymptotic equivalence of the IVX and WIV estimators under stationary short-memory covariates.

Figures 5–9 report size-adjusted power (SAP) results.⁹ For highly persistent regressors (e.g., $d = d_* + 0.4, d_* + 0.6$), IVX tests outperform WIV and OLS across a wide range of slope parameter values. Under Specification 1, IVX exhibits particularly strong performance for the weight functions $w(x) = x^{(j)}, j = \{0.75, 1\}$. Consistent with the results shown in Figures 1–4, under stationarity the IVX tests exhibit size performance comparable to that of the WIV and OLS tests, particularly for the weight functions $w(x) = x^{(0.75)}, x^{(1)}$.

9. Application to the Predictability of Stock Returns

The predictability of stock returns using publicly available information has been the subject of a vast literature in empirical finance and econometrics. For reviews and recent developments, see, among others, Campbell and Yogo (2006), Welch and Goyal (2008), Bollerslev et al. (2013), Kasparis et al. (2015), KMS2015, Phillips (2015), and Andersen and Varneskov (2021). The most common approach to testing the so-called

⁹Under the alternative hypothesis H_1 , SAP is computed as

$$\text{Empirical Power} = \max \{(\text{Empirical Size} - \text{Nominal Size}), 0\}.$$

predictability hypothesis relies on t-tests in predictive regressions of the form

$$r_t = \beta_0 + \beta_1 x_{t-1} + u_t,$$

where r_t denotes stock returns over a given horizon, x_{t-1} is a lagged predictive variable, and u_t is a martingale difference error term. The null hypothesis $H_0 : \beta_1 = 0$ corresponds to the absence of predictive power of x_t for future returns.

Testing this hypothesis is challenging due to uncertainty surrounding the stationarity—and more generally, the persistence properties—of the predictive variables. The predictors most commonly used in empirical work include financial variables such as dividend yields, return volatility, and earnings-price ratios, as well as macroeconomic variables such as inflation and short-term interest rates. These variables typically exhibit high persistence, often consistent with nonstationary behavior. It is well documented, for example, that measures of stock market risk such as realized variance display long-memory dynamics (see Christensen and Nielsen, 2006, 2007; Bollerslev et al., 2013; Andersen and Varneskov, 2021). By contrast, the possibility that predictive variables may also exhibit heavy-tailed behavior has received comparatively little attention. Recent work by Kim and Meddahi (2020) and Kim, Meddahi, and Yamashita (2021) shows that return volatility—a widely used predictor—exhibits thick tails.

In this study, we apply IVX tests to a set of predictive variables from Welch and Goyal (2024) that appear to exhibit long memory and, in some cases, heavy-tailed behavior. As demonstrated earlier, the IVX tests considered here are agnostic to the exact values of the memory and tail parameters. Nevertheless, to underscore the relevance of this dataset for the methods employed, we report preliminary estimates of memory parameters using the Local Whittle and Exact Local Whittle (ELW) estimators (see Shimotsu and Phillips, 2005), along with Hill estimates for the tail index.

Shimotsu and Phillips (2005) propose the ELW estimator, which remains valid under nonstationarity provided that second moments are finite. The validity of the Hill and Whittle estimators for stationary heavy-tailed processes has been established by Hill (2010) and Kokoszka and Taqqu (1996), respectively. More specifically, for stationary heavy-tailed fractional processes (i.e., $d < 1 - 1/\alpha$), Hill (2010) establishes consistency and asymptotic normality of the Hill estimator, while Kokoszka and Taqqu (1996) establish analogous results for the Whittle estimator.

Given the uncertainty surrounding the persistence and tail properties of the predictive variables, we conducted supplementary estimations using both first-differenced and fractionally differenced series, namely Δx_t and $\Delta^{\hat{d}_{ELW}} x_t$, where $\hat{d}_{ELW}$ denotes the ELW estimate of the memory parameter. These transformations were employed to assess the robustness of the estimated memory and tail indices to the persistence properties of the underlying series. After accounting for the effects of differencing on the memory parameter, the supplementary estimates were broadly consistent with those obtained from the original series in levels. Since these exercises yielded qualitatively similar conclusions, we report only the estimates based on the original series.

Rebalancing Memory and Tails in Predictive Regressions. A well-known stylized fact in the return predictability literature (see, e.g., Marmer, 2007; Kasparis et al., 2015; Phillips, 2015) is that stock returns exhibit markedly different memory properties from those of commonly used predictive variables such as return volatility, dividend yield, earnings–price ratios, and inflation. This discrepancy creates both

modeling and statistical challenges. First, differences in memory between the dependent and independent variables may signal model misspecification. Second, it is well known that regressing a stationary variable on a nonstationary one yields coefficient estimates that vanish asymptotically. Phillips (2015) refers to this phenomenon as misbalanced regression. As shown in Kasparis et al. (2015) and Hu, Kasparis, and Wang (2004), such misbalancing can substantially reduce the power of predictability tests. While memory misbalancing has received some attention in the literature, the possibility of *tail misbalancing* has, to our knowledge, remained entirely unexplored.

To address memory misbalancing issues, two broad approaches have been proposed. The first involves *linear filtering* of the data, while the second relies on *non-linear transformations*. The linear filtering approach is advocated by Christensen and Nielsen (2007), Bollerslev et al. (2013), and Andersen and Varneskov (2021), who consider regressions involving fractionally differenced variables. In particular, they study specifications of the form

$$y_t = \hat{\mu} + \hat{\beta} \Delta^{\hat{d}_x} x_{t-1} + \hat{u}_t \text{ and } \Delta^{\hat{d}_y} y_t = \hat{\mu} + \hat{\beta} \Delta^{\hat{d}_x} x_{t-1} + \hat{u}_t,$$

with $\hat{d}_y, \hat{d}_x$ preliminary memory estimators for y_t and x_t respectively.

It is worth noting that while differencing reduces persistence, it does not alter the tail behavior of the series. On the other hand, nonlinear filtering may achieve both memory and tail rebalancing. Marmer (2007), Kasparis (2011), Kasparis et al. (2015), Phillips (2015), and Hu, Kasparis, and Wang (2024), among others, consider possibly nonlinear predictive regressions of the form

$$y_t = \mu + \beta f(x_{t-1}) + u_t,$$

where u_t is a martingale difference error term. As noted in these studies, certain nonlinear transformations—such as reduced-growth, bounded, or integrable functions—can weaken the signal of highly persistent predictors, thereby improving the memory balance between returns and predictors.

To illustrate this effect, suppose that x_t is a finite-variance random walk. Then the following orders of magnitude apply (see Park and Phillips, 1999, 2001):

$$\sum_{t=1}^n |f(x_t)| = \begin{cases} O_p(n^{3/2}), f(x) = x \\ O_p(n^{1+b/2}), f(x) = |x|^b, b > -1 \\ O_p(n \ln(n)), f(x) = \ln |x| \\ O_p(n), f(x) = 1 \{x > 0\} \\ O_p(n^{1/2}), f(x) \text{ integrable} \end{cases}$$

These results demonstrate that certain transformations of random walks can produce sample functionals with weaker signal strength than those associated with stationary processes, for which one typically has $\sum_{t=1}^n |f(x_t)| = O_p(n)$ (see e.g. Giraitis, 1985). Simulation experiments in Kasparis et al. (2015) and Phillips (2015) further show that, for specific choices of f , the sample paths of a martingale difference sequence u_t and those of $f(x_t) + u_t$ are almost indistinguishable, even if x_t is a highly persistent process.

In addition to reducing persistence, transformations with reduced growth rates are readily seen to mitigate heavy-tailed behavior. For instance, suppose that $X \sim \tilde{S}_\alpha$. Then the transformation $X^{(0.5)}$ has finite variance whenever the stability parameter satisfies $\alpha > 1$. Moreover, the logarithmic transformation—widely used in empirical finance— $\ln(1 + |X|)$ produces a random variable with exponentially decaying tails.

Empirical Evidence for the Predictability of Stock Returns. We primarily focus on monthly returns. For this frequency, we consider four types of return series and nine predictive variables. Preliminary results indicate that the most pronounced differences between monthly and quarterly return predictability arise for the book-to-market (BM) ratio and stock variance (svar) predictors. In particular, stock variance exhibits no predictive power at the monthly frequency but displays strong predictability for quarterly returns. The book-to-market ratio appears to have predictive power at both frequencies; however, its predictive strength is substantially greater at the quarterly horizon. For this reason, we restrict attention to the BM and svar predictors in the analysis of quarterly returns.

We begin our analysis by reporting preliminary estimates of the memory and tail parameters. As noted above, the IVX methodology does not rely on such preliminary estimates. Nevertheless, these estimates help illustrate the relevance of our methods for the empirical application considered here. Table 1 reports memory and tail parameter estimates for the return series and the predictors. Specifically, we employ the Local Whittle (LW) and Exact Local Whittle (ELW) estimators using a bandwidth of $n^{0.65}$, where n denotes the sample size, as recommended by Shimotsu and Phillips (2005). For the Hill estimator we consider four bandwidth choices: $k_1 = 0.05n$, $k_2 = 0.1n$, $k_3 = 0.15n$ and $k_4 = 0.2n$ yielding the corresponding estimates $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$, and $\hat{\alpha}_4$. In addition, for each tail estimate we report the implied nonstationarity boundary for the memory parameter, defined as $\hat{d}_{i*} = 1 - 1/\hat{\alpha}_i$, $i = \{1, 2, 3, 4\}$.

Table 1 shows that stock returns exhibit negligible memory and possess finite second moments. The tail index estimates range from approximately 2.3 to 2.8 for monthly returns and are slightly higher for quarterly returns. In contrast, all nine monthly predictors considered display pronounced long-memory behavior, with memory parameter estimates ranging from 0.518 to 1.085. Moreover, for five of the nine predictors the Hill tail index estimates are consistently below two, indicating infinite variance. In all cases, the estimated memory parameters exceed the corresponding estimated nonstationarity thresholds, implying that the predictors exhibit nonstationary long-memory dynamics.

Tables 2–5 report IVX F-statistic values for tests of the predictability hypothesis across various predictors, regression specifications, and weight functions. In particular, Tables 2 and 3 present results for monthly returns. Table 2 focuses on a linear regression specification and considers five choices of the weight function, namely $w(x) = x^{(i)}$, $i = \{1, 0.75, 0.5, 0.25, 0.01\}$.¹⁰ Table 3 reports IVX F-statistics for nonlinear specifications in which both the regression and weight functions are given by $f(x) = w(x) = x^{(i)}$, $i = \{1, 0.75, 0.5, 0.25\}$. Both tables indicate strong evidence of predictability for ntis and inflation. In addition, there is some evidence of predictability for tbl, and to a lesser extent for BM and EP. By contrast, there is no evidence of predictability for widely used predictors such as DY and svar (at least at the monthly frequency). Interestingly, the test statistics for inflation are largest under nonlinear specifications, suggesting the presence of nonlinear predictive effects. Finally, Tables 4 and 5 report results for quarterly returns. As noted above, the analysis at this frequency focuses on two predictors only: BM and svar. Both variables exhibit strong predictive power, with BM significant at the 1% level and svar at the 5% level. Notably, svar displays no predictive power at the monthly frequency.

¹⁰We also experimented with the Cauchy estimator weight $w(x) = x^{(0)}$; however, this choice led to singularity issues for some predictor series. To obtain comparable performance while avoiding singular matrices, we instead employ the weight function $w(x) = x^{(0.01)}$.

Moreover, the test statistics for the svar predictive regression are maximized under nonlinear specifications, once again suggesting the presence of nonlinearities in the predictive relationship.

Table 1: Memory and Tail Estimates (1926–2024)

Panel A: Monthly Returns; 1926:01 – 2024:12

Dependent Variables	LW	ELW	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$
Returns	0.051	0.054	2.697	2.690	2.614	2.365
Returns-Rfree	0.055	0.058	2.774	2.670	2.602	2.348
CRSP_SPvw	-0.012	-0.011	2.658	2.885	2.772	2.567
CRSP_SPvw-Rfree	0.011	0.010	2.696	2.851	2.768	2.548

Panel B: Monthly Predictors; 1926:01 – 2024:12

Predictors	LW	ELW	$\hat{\alpha}_1$	d_{1*}	$\hat{\alpha}_2$	d_{2*}	$\hat{\alpha}_3$	d_{3*}	$\hat{\alpha}_4$	d_{4*}
BM	0.943	0.951	1.707	0.414	1.879	0.468	1.916	0.478	1.812	0.448
tbl	1.080	1.085	1.899	0.473	1.762	0.432	1.527	0.345	1.439	0.305
lty	1.042	1.048	2.628	0.500	2.391	0.500	2.228	0.500	1.972	0.493
ntis	0.937	0.859	2.011	0.500	1.923	0.480	1.901	0.474	1.897	0.473
Inflation	0.516	0.538	2.666	0.500	2.888	0.500	2.761	0.500	2.503	0.500
svar	0.518	0.518	1.469	0.319	1.299	0.230	1.169	0.144	1.144	0.126
DY	0.886	0.906	2.327	0.500	2.373	0.500	2.414	0.500	2.263	0.500
EP	0.698	0.699	2.308	0.500	2.103	0.500	2.129	0.500	2.065	0.500
Dpay	0.657	0.630	1.754	0.430	1.943	0.485	1.968	0.492	1.726	0.421

Panel C: Quarterly Returns; 1926:1 – 2024:4

Dependent Variables	LW	ELW	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\alpha}_3$	$\hat{\alpha}_4$
Returns	-0.154	-0.135	2.639	2.696	2.464	2.291
Returns-Rfree	-0.140	-0.132	2.378	2.551	2.283	2.300
CRSP_SPvw	-0.309	-0.312	2.743	2.940	2.905	2.625
CRSP_SPvw-Rfree	-0.199	-0.204	3.020	2.937	2.680	2.554

Panel D: Quarterly Predictors; 1926:1 – 2024:4

Predictors	LW	ELW	$\hat{\alpha}_1$	d_{1*}	$\hat{\alpha}_2$	d_{2*}	$\hat{\alpha}_3$	d_{3*}	$\hat{\alpha}_4$	d_{4*}
BM	0.739	0.741	1.885	0.470	1.681	0.405	1.787	0.440	1.808	0.447
svar	0.388	0.394	1.369	0.269	1.323	0.244	1.267	0.211	1.204	0.169

Table 2: IVX predictability tests; monthly data; linear fits: $f(x) = x$

$w(x)$	$x^{(1)}$	$x^{(0.75)}$	$x^{(0.5)}$	$x^{(0.25)}$	$x^{(0.01)}$
BM					
Returns	0.283	0.053	0.000	0.005	0.004
Returns-Rfree	0.316	0.045	0.003	0.053	0.096
CRSP_SPvw	3.581*	2.071	1.204	0.779	0.621
CRSP_SPvw-Rfree	3.686*	2.014	1.032	0.522	0.287
tbl					
Returns	0.266	0.089	0	0.097	0.344
Returns-Rfree	4.383**	3.305*	2.014	0.818	0.127
CRSP_SPvw	0.171	0.04	0.006	0.163	0.445
CRSP_SPvw-Rfree	4.071**	3.047*	1.814	0.692	0.085
lty					
Returns	0.321	0.471	0.631	0.733	0.707
Returns-Rfree	3.173*	3.277*	3.1*	2.569	1.807
CRSP_SPvw	0.056	0.093	0.137	0.174	0.187
CRSP_SPvw-Rfree	2.087	2.025	1.773	1.345	0.872
Inflation					
Returns	2.242	2.256	2.232	2.149	1.989
Returns-Rfree	3.346*	3.609**	3.735*	3.646*	3.265*
CRSP_SPvw	2.902*	2.996*	2.922*	2.613	2.062
CRSP_SPvw-Rfree	4.145**	4.534**	4.618**	4.248**	3.364*
ntis					
Returns	6.412**	5.434**	3.704*	1.729	0.256
Returns-Rfree	5.975**	4.920**	3.195*	1.34	0.115
CRSP_SPvw	6.520**	5.653**	3.971**	1.959	0.367
CRSP_SPvw-Rfree	6.068**	5.116**	3.431	1.535	0.189
svar					
Returns	1.443	1.478	1.03	0.26	0.015
Returns-Rfree	0.827	0.836	0.575	0.117	0.047
CRSP_SPvw	0.035	0.061	0.12	0.3	0.704
CRSP_SPvw-Rfree	0.23	0.299	0.362	0.509	0.866
DY					
Returns	0.541	0.318	0.147	0.037	0
Returns-Rfree	0.332	0.195	0.089	0.023	0
CRSP_SPvw	0.406	0.24	0.111	0.028	0
CRSP_SPvw-Rfree	0.231	0.137	0.063	0.016	0
EP					
Returns	3.415*	2.589	1.541	0.486	0
Returns-Rfree	1.802	1.339	0.781	0.241	0
CRSP_SPvw	1.979	1.496	0.885	0.278	0
CRSP_SPvw-Rfree	0.832	0.615	0.356	0.11	0
Dpay					
Returns	0.9	1.592	2.122	2.123	1.353
Returns-Rfree	0.264	0.679	1.063	1.13	0.675
CRSP_SPvw	0.09	0.263	0.369	0.266	0.027
CRSP_SPvw-Rfree	0.02	0.003	0.027	0.01	0.036

Table 3: IVX predictability tests; monthly data; nonlinear fits: $f(x) = w(x)$

$f(x)$	$x^{(1)}$	$x^{(0.75)}$	$x^{(0.5)}$	$x^{(0.25)}$
BM				
Returns	0.283	0.052	0	0.005
Returns-Rfree	0.316	0.045	0.003	0.053
CRSP_SPvw	3.581*	2.064	1.198	0.775
CRSP_SPvw-Rfree	3.686*	2.007	1.027	0.52
tbl				
Returns	0.266	0.089	0	0.097
Returns-Rfree	4.383**	3.304*	2.014	0.818
CRSP_SPvw	0.171	0.04	0.006	0.164
CRSP_SPvw-Rfree	4.071**	3.047*	1.814	0.692
lty				
Returns	0.321	0.471	0.631	0.733
Returns-Rfree	3.173*	3.277*	3.1*	2.57
CRSP_SPvw	0.056	0.093	0.137	0.174
CRSP_SPvw-Rfree	2.087	2.026	1.773	1.345
Inflation				
Returns	2.242	2.256	2.231	2.148
Returns-Rfree	3.346*	3.609*	3.734*	3.644*
CRSP_SPvw	2.902*	2.996*	2.921*	2.612
CRSP_SPvw-Rfree	4.145**	4.535**	4.618**	4.247**
ntis				
Returns	6.412**	5.431**	3.701*	1.727
Returns-Rfree	5.975**	4.918**	3.192*	1.339
CRSP_SPvw	6.520**	5.65**	3.967**	1.957
CRSP_SPvw-Rfree	6.068**	5.114**	3.428*	1.533
svar				
Returns	1.443	1.478	1.03	0.26
Returns-Rfree	0.827	0.836	0.575	0.117
CRSP_SPvw	0.035	0.061	0.12	0.3
CRSP_SPvw-Rfree	0.23	0.299	0.362	0.509
DY				
Returns	0.541	0.318	0.147	0.037
Returns-Rfree	0.332	0.195	0.089	0.023
CRSP_SPvw	0.406	0.24	0.111	0.028
CRSP_SPvw-Rfree	0.231	0.137	0.063	0.016
EP				
Returns	3.415*	2.581	1.527	0.478
Returns-Rfree	1.802	1.336	0.776	0.238
CRSP_SPvw	1.979	1.488	0.874	0.272
CRSP_SPvw-Rfree	0.832	0.612	0.352	0.108
Dpay				
Returns	0.9	1.592	2.131	2.172
Returns-Rfree	0.264	0.679	1.067	1.148
CRSP_SPvw	0.09	0.263	0.369	0.267
CRSP_SPvw-Rfree	0.02	0.003	0.027	0.01

Table 4: IVX predictability tests; quarterly data; linear fits: $f(x) = x$

$w(x)$	$x^{(1)}$	$x^{(0.75)}$	$x^{(0.5)}$	$x^{(0.25)}$	$x^{(0.01)}$
BM					
Returns	0.963	0.305	0.065	0.01	0.011
Returns-Rfree	0.97	0.263	0.025	0.004	0.02
CRSP_SPvw	7.258***	3.95**	2.148	1.293	1.005
CRSP_SPvw-Rfree	7.261***	3.789*	1.877	0.939	0.558
svar					
Returns	0.003	0	0.002	0.012	0.003
Returns-Rfree	0.077	0.092	0.098	0.066	0.011
CRSP_SPvw	3.892**	3.943**	3.34*	2.072	0.881
CRSP_SPvw-Rfree	5.32**	5.333**	4.348**	2.495	0.959

Table 5: IVX predictability tests; quarterly data; nonlinear fits: $f(x) = w(x)$

$f(x)$	$x^{(1)}$	$x^{(0.75)}$	$x^{(0.5)}$	$x^{(0.25)}$
BM				
Returns	0.963	0.304	0.064	0.01
Returns-Rfree	0.97	0.262	0.025	0.004
CRSP_SPvw	7.258***	3.907**	2.111	1.27
CRSP_SPvw-Rfree	7.261***	3.751**	1.847	0.923
svar				
Returns	0.003	0	0.002	0.012
Returns-Rfree	0.077	0.092	0.098	0.066
CRSP_SPvw	3.892**	3.941**	3.337**	2.071
CRSP_SPvw-Rfree	5.32**	5.329**	4.342**	2.493

APPENDIX

10. High Level Conditions

Throughout, $C, C_1, C_2, \dots \in (0, \infty)$ denote generic constants which may take different values at each appearance, even within the same proof.

The following inequality will be utilized repeatedly. For all $\theta > 0$ and some $C_\theta \in (0, \infty)$

$$||Y|^\theta - |X|^\theta| \leq C_\theta \begin{cases} (|X|^{\theta-1}|Y-X| + |Y-X|^\theta), & \theta \geq 1 \\ |Y-X|^\theta, & 0 < \theta < 1 \end{cases}. \quad (42)$$

Below, we consider a set of **high-level (HL)** conditions that closely resemble those in DK2021 and yield limit theory for locally integrable and integrable kernel sample functionals of stochastic arrays $X_t(n)$. These conditions are then shown to hold for HT-WNPs under **Assumptions BFR** and **MI**.

There is one notable difference between the **HL** conditions used here and those considered by DK2021. In this work, we assume that, upon normalization, $X_t(n)$ possesses a uniformly bounded sub- α moment, whereas DK2021 assume at least two moments. Before presenting these conditions, we introduce some notation. Let

$$\Omega_n(\eta) := \{ \{s, t\} \in \mathbb{N} : \lfloor \eta n \rfloor \leq s \leq \lfloor ns \rfloor, \lfloor \eta n \rfloor + s \leq t \leq n \},$$

for each $\eta \in (0, 1/2)$.

Assumption HL (high-level conditions)

HL0 Let $\{X_t(n)\}_{t=1}^n$, $n \in \mathbb{N}$ be a random array and $\{\mathcal{F}_t\}_{t=-\infty}^\infty$ a filtration such that $X_t(n)$ is $\mathcal{F}_t$ -measurable for all t and n .

HL1 $X_t(n) = X_t(n)^+ + X_t(n)^- + R_t(n)$, where $X_t(n)^-$ is $\mathcal{F}_0$ -measurable, and $\sup_{1 \leq t \leq n} P\{\gamma_n^{-1}|R_t(n)| > \epsilon\} \rightarrow 0$ for every $\epsilon > 0$.

HL2 Let $\gamma_n > 0$ a deterministic sequence with $\gamma_n \rightarrow \infty$. There are random variables X^+ and X^- , where X^+ has bounded density Φ_{X^+} such that: for every $\delta \in (0, 1)$ and $\{t_n\}$ with $\lfloor n\delta \rfloor \leq t_n \leq n$

(a) $\gamma_n^{-1}X_{t_n}(n)^+ \xrightarrow{d} X^+$, conditionally on $\mathcal{F}_0$ in the sense that for all bounded and continuous $h : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbf{E} \left[h \left(\gamma_n^{-1} X_{t_n}(n)^+ \right) \mid \mathcal{F}_0 \right] \xrightarrow{p} \int_{\mathbb{R}} h(x) \Phi_{X^+}(x) dx; \text{ and}$$

(b) $\gamma_n^{-1}X_{t_n}(n)^- \xrightarrow{d} X^-$, and $\gamma_n^{-1} [X_n(n)^- - X_{t_n}(n)^-] \xrightarrow{p} 0$.

HL3 $\gamma_t^{-1}X_t(n)$ has Lebesgue density $\mathcal{D}_{n,t}(x)$ such that for some $n_0 \geq t_0 \geq 1$,

$$\sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x \mathcal{D}_{n,t}(x) < \infty.$$

HL4 For every bounded and Lipschitz continuous $g : \mathbb{R} \rightarrow \mathbb{R}$

$$\frac{1}{n} \sum_{t=1}^n g(\gamma_n^{-1} X_t(n)) = \frac{1}{n} \sum_{t=1}^n \mathbf{E} [g(\gamma_n^{-1} X_t(n)) \mid \mathcal{F}_0] + o_p(1).$$

HL5 For all $\alpha' \in (0, \alpha)$

$$\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} |\gamma_n^{-1} X_t(n)|^{\alpha'} < \infty;$$

HL6 $\sup_{n \geq 1} \frac{\gamma_n}{n} \sum_{t=1}^n \gamma_t^{-1} < \infty$.

HL7 Let $X_0(n) = 0$ and $t > s \geq 0$. Conditionally on $\mathcal{F}_s$, $\gamma_{t-s}^{-1}(X_t(n) - X_s(n))$ has density $\mathcal{D}_{t,s,n}(x)$ such that

$$\sup_{n \geq n_0, 0 \leq s < t \leq n, t-s \geq t_0} \sup_x \mathcal{D}_{t,s,n}(x) < \infty.$$

HL8 For all $q_0, q_1 > 0$

$$\lim_{\eta \downarrow 0} \limsup_{n \rightarrow \infty} \sup_{(s,t) \in \Omega_n(\eta)} \sup_{|x| \leq q_0 \eta^{q_1}} |\mathcal{D}_{t,s,n}(x) - \mathcal{D}_{t,s,n}(0)| = 0$$

HL9

- (a) $\lim_{\eta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\gamma_n}{n} \sum_{t=\lfloor (1-\eta)n \rfloor}^n \gamma_t^{-1} = 0$;
- (b) $\lim_{\eta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\gamma_n}{n} \sum_{t=t_0}^{\lfloor \eta n \rfloor} \gamma_t^{-1} = 0$;
- (c) $\limsup_{n \rightarrow \infty} \frac{\gamma_n}{n} \sum_{t=t_0}^n \gamma_t^{-1} < \infty$;
- (d) for each $\eta \in (0, 1)$, there exist $l_0, l_1 > 0$ such that

$$\liminf_{n \rightarrow \infty} \gamma_n^{-1} \inf_{(s,t) \in \Omega_n(\eta)} \gamma_{t-s} \geq \eta^{l_1} / l_0;$$

- (e) $\sup_{n \geq 1} \sup_{1 \leq t \leq n} \gamma_n^{-1} \gamma_t < \infty$.

Conditions **HL0–HL6** facilitate limit theory for locally integrable functionals of HT-WNP arrays upon normalization. For kernel functionals, **HL7–HL9** are required in addition to these conditions. We note that **HL7** is a strengthened version of **HL3**. Limit theory for locally integrable and kernel functionals is provided in **Theorems 17** and **18** below.

*Theorem 17. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lebesgue integrable, and **HL0–HL4**, **HL6** hold. Furthermore, suppose f is bounded or that the following hold:*

- (i) $\int_{\mathbb{R}} |f(x+y)| \Phi(x) dx < \infty$, for all $y \in \mathcal{Y}$, and $P(X^- \in \mathcal{Y}) = 1$;
- (ii) **HL5** holds;
- (iii) for some $q \in (0, \alpha)$, $|f(x)| = O(|x|^q)$, as $|x| \rightarrow \infty$;
- (iv) $\frac{1}{n} \sum_{t=1}^{t_0-1} f(\gamma_n^{-1} X_t(n)) = o_p(1)$, with t_0 as in **HL3**.

Then as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} X_t(n)) \xrightarrow{d} \int_{\mathbb{R}} f(x + X^-) \Phi_{X^+}(x) dx. \quad (43)$$

Remark. If $\tilde{f} : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is such that $\tilde{f}(x) = f(x)$ almost everywhere, and $\frac{1}{n} \sum_{t=1}^{t_0-1} \tilde{f}(\gamma_n^{-1} X_t(n)) = o_p(1)$, with t_0 as in **HL3**. then (43) holds with f replaced by $\tilde{f}$.

Proof of Theorem 17. The proof follows from Theorem 3.1 in DK2021. For convenience we provide a summary of the key steps. Given that f is locally integrable, for each $\varepsilon > 0$, there is a Lipschitz continuous f_ε that approximates f in L_1 . In particular,

$$\int_{|x| \leq \varepsilon^{-1}} |f(x) - f_\varepsilon(x)| dx < \varepsilon \text{ and } f_\varepsilon(x) = 0, |x| > \varepsilon^{-1}.$$

It can be shown that for each $\varepsilon > 0$ and $0 < \delta < 1$

$$\frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} X_t(n)) = G_\varepsilon[\gamma_n^{-1} X_n(n)^-] + R_{n,\varepsilon,\delta} \quad (44)$$

with $G_\varepsilon[\cdot] = \int_{\mathbb{R}} f_\varepsilon(x + \cdot) dF_{X^+}(x)$ continuous and $R_{n,\varepsilon,\delta} = o_p(1)$ as $n \rightarrow \infty$ first, then $\delta \rightarrow 0$, and then $\varepsilon \rightarrow 0$. Moreover, as $n \rightarrow \infty$ we have

$$G_\varepsilon[\gamma_n^{-1} X_n(n)^-] \xrightarrow{d} G_\varepsilon[X^-], \quad (45)$$

and

$$G_\varepsilon[X^-] = \int_{\mathbb{R}} f(x + X^-) \Phi_{X^+}(x) dx + o_{a.s.}(1), \quad (46)$$

as $\varepsilon \rightarrow 0$. Finally, we note that the approximation with respect to ε can be dropped, if f is bounded and Lipschitz. In particular, (44) simplifies to

$$\frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} X_t(n)) = G[\gamma_n^{-1} X_n(n)^-] + R_{n,\delta} \quad (47)$$

with $R_{n,\delta} = o_p(1)$ as $n \rightarrow \infty$ first, then $\delta \rightarrow 0$, where $G[\cdot] = \int_{\mathbb{R}} f(x + \cdot) \Phi_{X^+}(x) dx$. ■

Theorem 18. Let $\{h_n\}$ a positive sequence with $\gamma_n^{-1} h_n + \gamma_n(nh_n)^{-1} \rightarrow 0$. In addition, $\frac{\gamma_n}{h_n n} \sum_{t=1}^{t_0-1} K\left(\frac{X_t(n) - x}{h_n}\right) = o_p(1)$, where t_0 is as in **HL7**. Then under **Assumptions K, HL0–HL4 and HL6–HL9**, as $n \rightarrow \infty$,

$$\frac{\gamma_n}{h_n n} \sum_{t=1}^n K\left(\frac{X_t(n) - x}{h_n}\right) \xrightarrow{d} \Phi_{X^+}(-X^-) \int_{\mathbb{R}} K(u) du.$$

Proof of Theorem 18. The proof is identical to that of Theorem 3.3 in DK2021. ■

11. Proof of Main Limit Theorems

Proof of Theorem 1. In view of **Theorem 17**, it suffices to verify conditions (i), (iii), (iv), of that theorem, together with **HL0–HL6**. We begin by verifying **HL0–HL6**.

HL0-1 : Under **Assumptions INN, BFR and MI**, **HL0** and **HL1** hold trivially, by setting $X_t(n)^+ = x_t^+(n)$, $X_t(n)^- = x_t^-(n)$ and $R_t(n) = 0$ where $x_t^+(n)$, $x_t^-(n)$ are defined in (8), and by taking $\mathcal{F}_0 = \sigma(\xi_0, \xi_{-1}, \dots)$.

HL2 : Let γ_n be as in **Definition 1**, let X^+, X^- be as in **Lemma 1**. By **Definition 2** (α -stable distribution), X^+ has an integrable characteristic function and therefore admits a bounded density. Moreover, **HL2(a)** and the first part of **HL2(b)** follow immediately from **Lemma 1**. Finally, the second part of **HL2(b)** follows from **Lemma FS $_{\alpha}$ M(b)**.

HL3 : The result follows immediately from **Lemma 21**. In particular, that lemma shows that $\gamma_n^{-1}x(n)$ and $\gamma_n^{-1}x_t^+(n)$ have integrable characteristic functions which is sufficient for the result; see the verification of **HL8** in the proof of **Theorem 2** below for more details.

HL4 : This follows from **Lemma 10**.

HL5 : This is a direct consequence of **Lemma FS $_{\alpha}$ M(a)**.

HL6 : It follows from **Definition 1** and the subsequent remarks that γ_n is regularly varying n with index $0 \leq p < 1$. In particular, under **BFR**, $p = 0$, whereas under **MI**, $p = \alpha_{\kappa}\mathcal{H}$, where $0 \leq \alpha_{\kappa} < 1$ and $0 < \mathcal{H} < 1$. Consequently, **HL6** follows from standard arguments for regularly varying sequences; see, e.g., Lemma B.2 of DK2021.

Finally, we verify conditions (i), (iii), (iv) of **Theorem 17**.

Condition (i) : For each $y \in \mathbb{R}$ define $Y = X^+ + y$. Note that Y is α -stable with location parameter y and therefore $\mathbf{E}|Y|^{\alpha'} < \infty$ for all $\alpha' \in (0, \alpha)$ -see e.g. Samorodnitsky and Taqqu (1994; Property 1.2.16. p.18). Furthermore, Φ_Y , the density of Y , exists and is bounded due to the fact that Y has an integrable characteristic function. Then for $m > 0$ large enough,

$$\begin{aligned} \int_{\mathbb{R}} |f(x+y)| \Phi_{X^+}(x) dx &= \mathbf{E}|f(Y)| \leq_{(1)} \int_{|u| \leq m} |f(u)| \Phi_Y(u) du + C\mathbf{E}|Y|^{\alpha'} \\ &\leq \sup_u \Phi_Y(u) \int_{|u| \leq m} |f(u)| du + C\mathbf{E}|Y|^{\alpha'} < \infty, \end{aligned}$$

where $\leq_{(1)}$ follows from condition (ii) of **Theorem 1**.

Condition (iii) : this is identical to condition (ii) of **Theorem 1**.

Condition (iv) : It can be readily seen that condition (iii.a) of **Theorem 1** implies condition (iv).

Moreover, under condition (iii.b), **HL4** holds with $t_0 = 1$. To see this, first recall that

$$x_t = \sum_{k=0}^{\infty} a_{k,t} \xi_{t-k},$$

where

$$a_{k,t}(n) = \sum_{j=0}^{(t-1) \wedge k} \phi_j c_{k-j} = \begin{cases} \sum_{j=0}^k \phi_j(n) c_{k-j} =: a_k & \text{if } 0 \leq k \leq t-1, \\ \sum_{j=0}^{t-1} \phi_j(n) c_{k-j} =: a_{k,t}^- & \text{if } k \geq t, \end{cases}$$

and $\phi_0(n) \neq 0$, for all n . Note that, under **BFR** $\phi_0(n) = \phi_0 \neq 0$ by assumption, whereas under **MI**, $\phi_j(n) = \rho_n^j$. Hence, in the latter case $\phi_0(n) = 1$. In addition,

under **BFR** and **MI**, there is at least one coefficient $\{c_j\}_{j \in \mathbb{Z}_+}$ that is non-zero. Let k^* be the minimal index of the non-zero coefficients. Then for all $t \geq 1$,

$$|a_{k^*,t}(n)| \geq \inf_{n \in \mathbb{N}} |\phi_0(n) c_{k^*}| > 0. \quad (48)$$

Next, note that by **HL3**, $\gamma_t^{-1}x_t$ admits a density $\mathcal{D}_{n,t}$ such that

$$\sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x \mathcal{D}_{n,t}(x) < \infty,$$

for some $n_0 \geq 1$ and $t_0 \geq 1$. The required result will follow if we show that for any $t_0 \geq 1$,

$$\sup_{n \geq n_0, 1 \leq t \leq t_0} \sup_x \mathcal{D}_{n,t}(x) < \infty.$$

Indeed, by (48), for all $1 \leq t \leq t_0$ we get

$$\begin{aligned} \int_{\mathbb{R}} |\mathbf{E}(e^{i\lambda \gamma_t^{-1}x_t})| d\lambda &= \int_{\mathbb{R}} \prod_{k=0}^{\infty} |\psi_{\xi}(\gamma_t^{-1}a_{k,t}(n)\lambda)| d\lambda \\ &\leq \frac{\sup_{1 \leq t \leq t_0} \gamma_t}{\inf_{1 \leq t \leq t_0, n \in \mathbb{N}} |a_{k^*,t}(n)|} \int_{\mathbb{R}} |\psi_{\xi}(\lambda)| d\lambda < \infty, \end{aligned}$$

as required. ■

Proof of Theorem 2. In view of **Theorem 18**, it suffices to show that Conditions **HL0–HL4** and **HL6–HL9** hold. Conditions **HL0–HL6** have been already established in the proof of **Theorem 17**. We shall show that **HL7–HL9** hold.

HL7 : We show that **HL7** holds with $X_t(n) = x_t(n)$, $t \geq 1$. In view of (8) write

$$\begin{aligned} x_t^+(n) &= \sum_{k=0}^{t-1} a_k(n) \xi_{t-k} = \sum_{l=1}^t a_{t-l}(n) \xi_l. \\ [l = t - k \rightarrow k = t - l] \end{aligned}$$

Hence, for $t > s$

$$\begin{aligned} x_t(n) - x_s(n) &= \sum_{l=1}^t a_{t-l}(n) \xi_l - \sum_{l=1}^s a_{s-l}(n) \xi_l + x_t^-(n) - x_s^-(n) \\ &= \sum_{l=s+1}^t a_{t-l}(n) \xi_l + \sum_{l=1}^s [a_{t-l}(n) - a_{s-l}(n)] \xi_l + x_t^-(n) - x_s^-(n) =: \sum_{l=s+1}^t a_{t-l}(n) \xi_l + \mathcal{X}_{t,s}(n). \end{aligned}$$

The first term on the r.h.s. in the last line above is

$$\begin{aligned} \sum_{l=s+1}^t a_{t-l}(n) \xi_l &= \sum_{k=0}^{t-s-1} a_k(n) \xi_{t-k} \stackrel{d}{=} x_{t-s}^+(n). \\ [l = t - k \rightarrow k = t - l] \end{aligned}$$

It follows from **Lemma 21** that $\gamma_t^{-1}x_t^+(n)$ possesses a density, $\mathcal{D}_{t,n}^+(x)$ say, such that $\sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x \mathcal{D}_{t,n}^+(x) < \infty$, for some $n_0 \geq t_0 \geq 1$. In view of the above, conditionally on $\mathcal{F}_s$, $\gamma_{t-s}^{-1}[x_t(n) - x_s(n)]$ has a density $\mathcal{D}_{t,s,n}(x)$ such that $\mathcal{D}_{t,s,n}(x) = \mathcal{D}_{t-s,n}^+(x - \mathcal{X}_{t,s}(n))$ and

$$\sup_{n \geq n_0, 0 \leq s < t \leq n, t-s \geq t_0} \sup_x \mathcal{D}_{t-s,n}^+(x - \mathcal{X}_{t,s}(n)) \leq \sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x \mathcal{D}_{t,n}^+(x) < \infty.$$

HL8 : We begin with some notation. For a function $\varphi(x)$, let $\tilde{\varphi}(\lambda)$ denote its Fourier transform, defined by

$$\tilde{\varphi}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda x} \varphi(x) dx.$$

By Fourier inversion; see, for example, Theorem 5.1 of Lang (1993), we have

$$\varphi(-x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\lambda} \tilde{\varphi}(\lambda) d\lambda,$$

and

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\lambda} \tilde{\varphi}(\lambda) d\lambda.$$

Next, with $\mathcal{D}_{t,n}^+(x)$ as defined in the verification of **HL7** above, we have

$$\int_{\mathbb{R}} |\lambda| |\tilde{\mathcal{D}}_{t,n}^+(\lambda)| d\lambda = \int_{\mathbb{R}} |\lambda| \left| \int_{\mathbb{R}} e^{-i\lambda x} \mathcal{D}_{t,n}^+(x) dx \right| d\lambda = \int_{\mathbb{R}} |\lambda| |\mathbf{E} e^{-i\lambda x_t^+(n)}| d\lambda.$$

Combining the above with **Lemma 21** we obtain

$$\sup_{n \geq n_0, t_0 \leq t \leq n} \int_{\mathbb{R}} |\lambda| |\tilde{\mathcal{D}}_{t,n}^+(\lambda)| d\lambda = \frac{1}{\sqrt{2\pi}} \sup_{n \geq n_0, t_0 \leq t \leq n} \int_{\mathbb{R}} |\lambda| |\mathbf{E} e^{i\lambda x_t^+(n)}| d\lambda < \infty.$$

Since

$$|\lambda x| \wedge 1 \leq (1 + |\lambda|) |x| \wedge (1 + |\lambda|) = (1 + |\lambda|) (|x| \wedge 1),$$

it follows that

$$\begin{aligned} & \sup_{(s,t) \in \Omega_n(\eta)} \sup_{|x| \leq q_0 \eta^{q_1}} |\mathcal{D}_{t,s,n}(x) - \mathcal{D}_{t,s,n}(0)| \\ &= \sup_{(s,t) \in \Omega_n(\eta)} \sup_{|x| \leq q_0 \eta^{q_1}} \left| \mathcal{D}_{t-s,n}^+(x - \mathcal{X}_{t,s}(n)) - \mathcal{D}_{t-s,n}^+(\mathcal{X}_{t,s}(n)) \right| \\ &= \sup_{(s,t) \in \Omega_n(\eta)} \sup_{|x| \leq q_0 \eta^{q_1}} \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i[x - \mathcal{X}_{t,s}(n)]\lambda} \tilde{\mathcal{D}}_{t-s,n}^+(\lambda) d\lambda - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\mathcal{X}_{t,s}(n)\lambda} \tilde{\mathcal{D}}_{t-s,n}^+(\lambda) d\lambda \right| \\ &\leq \frac{2}{\sqrt{2\pi}} \sup_{(s,t) \in \Omega_n(\eta)} \sup_{|x| \leq q_0 \eta^{q_1}} \int_{\mathbb{R}} (|\lambda x| \wedge 1) |\tilde{\mathcal{D}}_{t-s,n}^+(\lambda)| d\lambda \\ &\leq \frac{2}{\sqrt{2\pi}} \sup_{(s,t) \in \Omega_n(\eta)} \sup_{|x| \leq q_0 \eta^{q_1}} (|x| \wedge 1) \int_{\mathbb{R}} (1 + |\lambda|) |\tilde{\mathcal{D}}_{t-s,n}^+(\lambda)| d\lambda \\ &\leq \frac{2}{\sqrt{2\pi}} \sup_{|x| \leq q_0 \eta^{q_1}} (|x| \wedge 1) \sup_{n \geq n_0, t_0 \leq t \leq n} \int_{\mathbb{R}} (1 + |\lambda|) |\tilde{\mathcal{D}}_{t,n}^+(\lambda)| d\lambda \\ &\leq C \sup_{|x| \leq q_0 \eta^{q_1}} (|x| \wedge 1) \rightarrow 0, \end{aligned}$$

as $\eta \rightarrow 0$.

HL9 : Since γ_n is regularly varying, **HL9** follows directly from the proof of Theorem 3.4 in DK2021. ■

Proof of Theorem 3. By **Theorem 1**,

$$\frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1})^2 \xrightarrow{d} \int_{\mathbb{R}} f^2(x) \varrho(x) dx.$$

In view of this, it follows from Wang's (2014) MCLT -cf. Wang (2014; cf. Thm 2.1)- that

$$\begin{aligned} & \left[\frac{1}{\sqrt{n}} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1}) u_t, \frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1})^2 \right] \\ & \xrightarrow{d} \left[MN \left[0, \sigma_u^2 \int_{\mathbb{R}} f^2(x) \varrho(x) dx \right], \int_{\mathbb{R}} f^2(x) \varrho(x) dx \right], \end{aligned} \quad (49)$$

if we show that

$$n^{-1/2} \max_{1 \leq t \leq n} |f(\gamma_n^{-1} x_t)| = o_p(1). \quad (50)$$

Eq. (50) follows from arguments identical to those used for the proof of Thm 4.1 in DK2021. We note that eq. (2.2) and (2.3) in Wang (2014) are not satisfied under the conditions of **Theorem 3**; however, they are unnecessary in the present setting. Indeed, here the mixing variate X^- is the weak limit of an $\mathcal{F}_0$ -measurable term. Moreover, a careful reading of Wang's (2014) derivations shows that (49) can be strengthened to

$$\begin{aligned} & \left[\frac{1}{\sqrt{n}} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1}) u_t, \frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1})^2, \gamma_n^{-1} x_n^- \right] \\ & \xrightarrow{d} \left[MN \left[0, \sigma_u^2 \int_{\mathbb{R}} f^2(x) \varrho(x) dx \right], \int_{\mathbb{R}} f^2(x) \varrho(x) dx, X^- \right]. \end{aligned} \quad (51)$$

Next, note that by (44)-(46) for any $0 < \delta < 1$, $\varepsilon > 0$ there is some continuous $G_\varepsilon[\cdot]$ and $R_{n,\varepsilon,\delta}$ -cf. (44)- such that

$$\frac{1}{n} \sum_{t=1}^n g(\gamma_n^{-1} x_{t-1}) = G_\varepsilon[\gamma_n^{-1} x_n^-] + R_{n,\varepsilon,\delta},$$

and $G_\varepsilon[\gamma_n^{-1} x_n^-] \xrightarrow{d} G_\varepsilon[X^-]$, as $n \rightarrow \infty$, whilst $R_{n,\varepsilon,\delta} = o_p(1)$ as $n \rightarrow \infty$ first, then $\delta \rightarrow 0$, and then $\varepsilon \rightarrow 0$. Furthermore, $G_\varepsilon[X^-] = \int_{\mathbb{R}} g(x) \varrho(x) dx + o_{a.s.}(1)$ as $\varepsilon \rightarrow 0$. In view of this, (51), the continuous mapping theorem, and Billingsley (1968, Thm 4.2) as $n \rightarrow \infty$ first, then $\delta \rightarrow 0$, and then $\varepsilon \rightarrow 0$

$$\begin{aligned} & \left[\frac{1}{\sqrt{n}} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1}) u_t, \frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1})^2, \frac{1}{n} \sum_{t=1}^n g(\gamma_n^{-1} x_{t-1}) \right] \\ & = \left[\frac{1}{\sqrt{n}} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1}) u_t, \frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1})^2, G_\varepsilon[\gamma_n^{-1} x_n^-] + R_{n,\varepsilon,\delta} \right] \\ & = \left[\frac{1}{\sqrt{n}} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1}) u_t, \frac{1}{n} \sum_{t=1}^n f(\gamma_n^{-1} x_{t-1})^2, G_\varepsilon[\gamma_n^{-1} x_n^-] \right] + o_p(1) \quad (n \rightarrow \infty, \delta \rightarrow 0, \varepsilon \rightarrow 0) \\ & \xrightarrow{d} \left[MN \left[0, \sigma_u^2 \int_{\mathbb{R}} f^2(x) \varrho(x) dx \right], \int_{\mathbb{R}} f^2(x) \varrho(x) dx, G_\varepsilon[X^-] \right] \quad (n \rightarrow \infty) \\ & = \left[MN \left[0, \sigma_u^2 \int_{\mathbb{R}} f^2(x) \varrho(x) dx \right], \int_{\mathbb{R}} f^2(x) \varrho(x) dx, \int_{\mathbb{R}} g(x) \varrho(x) dx \right] + o_{a.s.}(1) \quad (\varepsilon \rightarrow 0), \end{aligned}$$

as required. ■

Proof of Theorem 4. The proof is similar to that of **Theorem 3** above. By **Theorem 2**,

$$\frac{\gamma_n}{h_n n} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right) \xrightarrow{d} \varrho(0) \int_{\mathbb{R}} K(u) du.$$

In view of this convergence and Wang (2014; cf. Thm 2.1), we have

$$\begin{aligned} & \left[\sqrt{\frac{\gamma_n}{h_n n}} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right) u_t, \frac{\gamma_n}{h_n n} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right)^2 \right] \\ & \xrightarrow{d} \left[MN \left[0, \sigma_u^2 \varrho(0) \int_{\mathbb{R}} K^2(u) du \right], \varrho(0) \int_{\mathbb{R}} K^2(u) du \right], \end{aligned} \quad (52)$$

provided that

$$\sqrt{\frac{\gamma_n}{h_n n}} \max_{1 \leq t \leq n} \left| K\left(\frac{x_{t-1} - x}{h_n}\right) \right| = o_p(1). \quad (53)$$

The latter approximation follows from an argument similar to the one used for the proof of Thm 4.1 in DK2021. We note that the mixing variate in the mixed Gaussian limit in (52) is the weak limit of $\gamma_n^{-1} x_n^-$, which is $\mathcal{F}_0$ -measurable. Therefore, the requirements in eq. (2.2) and (2.3) of Wang (2014) are unnecessary in the present setting. Furthermore, as mentioned in the previous proof, (52) can be generalized as follows:

$$\begin{aligned} & \left[\sqrt{\frac{\gamma_n}{h_n n}} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right) u_t, \frac{\gamma_n}{h_n n} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right)^2, \gamma_n^{-1} x_n^- \right] \\ & \xrightarrow{d} \left[MN \left[0, \sigma_u^2 \varrho(0) \int_{\mathbb{R}} K^2(u) du \right], \varrho(0) \int_{\mathbb{R}} K^2(u) du, X^- \right]. \end{aligned} \quad (54)$$

Next, let $\varphi_{\eta^2}(x)$ the density of a $N(0, \eta^2)$ variate. It can be shown (see e.g. DK2021, Theorem 3.3) that under our assumptions as $n \rightarrow \infty$ first and then as $\eta \rightarrow 0$

$$\frac{\gamma_n}{h_n n} \sum_{t=1}^n L\left(\frac{x_{t-1} - x}{h_n}\right) = \int_{\mathbb{R}} L(u) du \cdot \frac{1}{n} \sum_{t=1}^n \varphi_{\eta^2}(\gamma_n^{-1} x_{t-1}) + o_p(1)$$

and

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n \varphi_{\eta^2}(\gamma_n^{-1} x_t) & \xrightarrow{d} \int_{\mathbb{R}} \varphi_{\eta^2} \varrho(x) dx, \text{ as } n \rightarrow \infty \\ & \xrightarrow{a.s.} \varrho(0), \text{ as } \eta \rightarrow 0. \end{aligned}$$

More specifically, by (47), noting that φ_{η^2} is bounded and Lipschitz, there is a remainder $R_{n,\delta} = o_p(1)$ as $n \rightarrow \infty$ first followed by $\delta \rightarrow 0$ and a continuous $G_\eta[\cdot] (= \int_{\mathbb{R}} \varphi_{\eta^2}(x + \cdot) dF_{X^+}(x))$ such that

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n \varphi_{\eta^2}(\gamma_n^{-1} x_t) & = G_\eta[\gamma_n^{-1} x_n^-] + R_{n,\delta} \\ & = G_\eta[\gamma_n^{-1} x_n^-] + o_p(1) \quad (n \rightarrow \infty, \delta \rightarrow 0) \\ & \xrightarrow{d} G_\eta[X^-] \quad (n \rightarrow \infty) \\ & = \int_{\mathbb{R}} \varphi_{\eta^2} \varrho(x) dx + o(1) \xrightarrow{a.s.} \varrho(0) \quad (\eta \rightarrow 0). \end{aligned}$$

In view of the above, (54), the continuous mapping theorem, and Billingsley (1968, Thm 4.2) we obtain

$$\begin{aligned}
& \left[\sqrt{\frac{\gamma_n}{h_n n}} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right) u_t, \frac{\gamma_n}{h_n n} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right)^2, \frac{\gamma_n}{h_n n} \sum_{t=1}^n L\left(\frac{x_{t-1} - x}{h_n}\right) \right] \\
&= \left[\sqrt{\frac{\gamma_n}{h_n n}} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right) u_t, \frac{\gamma_n}{h_n n} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right)^2, \int_{\mathbb{R}} L(u) du \cdot \frac{1}{n} \sum_{t=1}^n \varphi_{\eta^2}(\gamma_n^{-1} x_t) \right] + o_p(1) \\
&\quad (n \rightarrow \infty) \\
&= \left[\sqrt{\frac{\gamma_n}{h_n n}} \sum_{t=1}^n K\left(\frac{x_t - x}{h_n}\right) u_t, \frac{\gamma_n}{h_n n} \sum_{t=1}^n K\left(\frac{x_{t-1} - x}{h_n}\right)^2, \int_{\mathbb{R}} L(u) du \cdot G_{\eta}[\gamma_n^{-1} x_n^-] \right] + o_p(1) \\
&\quad (n \rightarrow \infty) \\
&\xrightarrow{d} \left[MN \left[0, \sigma_u^2 \varrho(0) \int_{\mathbb{R}} K^2(u) du \right], \varrho(0) \int_{\mathbb{R}} K^2(u) du, \int_{\mathbb{R}} L(u) du \cdot G_{\eta}[X^-] \right] + o_p(1) \quad (n \rightarrow \infty) \\
&= \left[MN \left[0, \sigma_u^2 \varrho(0) \int_{\mathbb{R}} K^2(u) du \right], \varrho(0) \int_{\mathbb{R}} K^2(u) du, \varrho(0) \int_{\mathbb{R}} L(u) du \right] + o_{a.s.}(1) \quad (\eta \rightarrow 0),
\end{aligned}$$

as required. ■

Proof of Theorem 5. We begin with the proof of part (a). Let

$$S \sim S_{\alpha_u} \left\{ \sigma_u^{\alpha_u} \int_{\mathbb{R}} |f(x)|^{\alpha_u} \varrho(x) dx, \beta_u^* \frac{\int_{\mathbb{R}} \text{sgn}[f(x)] |f(x)|^{\alpha_u} \varrho(x) dx}{\int_{\mathbb{R}} |f(x)|^{\alpha_u} \varrho(x) dx} \right\}.$$

Noting that $\mathbf{E} \exp[i\lambda S] \neq 0$ ($i = \sqrt{-1}$) for all $\lambda \in \mathbb{R}$, the result will follow from Jakubowski (1986), Lemma 1.2 if we show that for each λ

$$\Psi_n(\lambda) := \prod_{t=1}^n \left\{ \mathbf{E} \exp \left[i\lambda \frac{1}{n^{1/\alpha_u}} f\left(\frac{x_{t-1}}{\gamma_n}\right) u_t \right] \mid \mathcal{F}_{t-1} \right\} \xrightarrow{p} \mathbf{E} \exp[i\lambda S].$$

Using the same arguments as those used in the proof of Thm 4.1 in DK2021 we have as $n \rightarrow \infty$

$$\max_{1 \leq t \leq n} |g_{t,n}| := \max_{1 \leq t \leq n} \left| \lambda n^{-1/\alpha_u} f\left(\frac{x_{t-1}}{\gamma_n}\right) \right| = o_p(1). \quad (55)$$

By **Assumption HTE**, we have

$$\ln \mathbf{E}(e^{i\lambda u_t} \mid \mathcal{F}_{t-1}) = -\sigma_u^{\alpha_u} |\lambda|^{\alpha_u} H_u(|\lambda|^{-1}) \left[1 + i\beta_u^* \text{sgn}(\lambda) \tan\left(\frac{\pi\alpha_u}{2}\right) \right] [1 + R_t(\lambda)].$$

In view of condition (ii) of **Assumption HTE** and (55) and where the remainder term satisfies

$$\sup_{1 \leq t \leq n} R_t(g_{t,n}) = o_p(1), \quad (56)$$

as $n \rightarrow \infty$. To see this fix $\varepsilon, \delta > 0$ and write

$$\begin{aligned}
P\left(\sup_{1 \leq t \leq n} |R_t(g_{t,n})| > \varepsilon\right) &\leq P\left(\sup_{1 \leq t \leq n} |g_{t,n}| > \delta\right) \\
&+ P\left(\sup_{1 \leq t \leq n} |R_t(g_{t,n})| > \varepsilon, \sup_{1 \leq t \leq n} |g_{t,n}| \leq \delta\right)
\end{aligned}$$

$$\leq P\left(\sup_{1 \leq t \leq n} |g_{t,n}| > \delta\right) + P\left(\sup_{|\lambda| \leq \delta} \sup_{1 \leq t \leq n} |R_t(\lambda)| > \varepsilon\right).$$

Then (56) follows from the fact that

$$\lim_{n \rightarrow \infty} P\left(\sup_{1 \leq t \leq n} |g_{t,n}| > \delta\right) = 0$$

and

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left(\sup_{|\lambda| \leq \delta} \sup_{1 \leq t \leq n} |R_t(\lambda)| > \varepsilon\right) = 0.$$

In view of the above, as $n \rightarrow \infty$

$$\begin{aligned} \ln \Psi_n(\lambda) &= \sum_{t=1}^n \left\{ -\sigma_u^{\alpha_u} \left| \lambda n^{-1/\alpha_u} f\left(\frac{x_{t-1}}{\gamma_n}\right) \right|^{\alpha_u} H_u \left(\left| \lambda n^{-1/\alpha_u} f\left(\frac{x_{t-1}}{\gamma_n}\right) \right|^{-1} \right) \right. \\ &\quad \cdot \left. \left[1 + i\beta_u^* \operatorname{sgn} \left[\lambda n^{-1/\alpha_u} f\left(\frac{x_{t-1}}{\gamma_n}\right) \right] \tan\left(\frac{\pi\alpha_u}{2}\right) \right] \right\} [1 + R_t(g_{t,n})] \\ &=_{(1)} \sum_{t=1}^n \left\{ -\sigma_u^{\alpha_u} \left| \lambda n^{-1/\alpha_u} f\left(\frac{x_{t-1}}{\gamma_n}\right) \right|^{\alpha_u} H_u(|g_{t,n}|^{-1}) \right. \\ &\quad \left. \left[1 + i\beta_u^* \operatorname{sgn} \left[\lambda n^{-1/\alpha_u} f\left(\frac{x_{t-1}}{\gamma_n}\right) \right] \tan\left(\frac{\pi\alpha_u}{2}\right) \right] [1 + o_p(1)] \right\} \\ &=_{(2)} \sum_{t=1}^n -\sigma_u^{\alpha_u} \left| \lambda n^{-1/\alpha_u} f\left(\frac{x_{t-1}}{\gamma_n}\right) \right|^{\alpha_u} \left[1 + i\beta_u^* \operatorname{sgn} \left[\lambda n^{-1/\alpha_u} f\left(\frac{x_{t-1}}{\gamma_n}\right) \right] \tan\left(\frac{\pi\alpha_u}{2}\right) \right] [1 + o_p(1)] \\ &=_{(3)} -\sigma_u^{\alpha_u} \int_{\mathbb{R}} |\lambda f(x)|^{\alpha_u} \left[1 + i\beta_u^* \operatorname{sgn}[\lambda f(x)] \tan\left(\frac{\pi\alpha_u}{2}\right) \right] \varrho(x) dx [1 + o_p(1)] \\ &= -\sigma_u^{\alpha_u} \int_{\mathbb{R}} |f(x)|^{\alpha_u} \varrho(x) dx |\lambda|^{\alpha_u} \left[1 + \frac{i\beta_u^* \operatorname{sgn}(\lambda) \int_{\mathbb{R}} |f(x)|^{\alpha_u} \operatorname{sgn}[f(x)] \varrho(x) dx \tan\left(\frac{\pi\alpha_u}{2}\right)}{\int_{\mathbb{R}} |f(x)|^{\alpha_u} \varrho(x) dx} \right] + o_p(1), \end{aligned}$$

where $=_{(1)}$ follows from (56), $=_{(2)}$ is due to condition (iii) of **Assumption HTE** and (55), and $=_{(3)}$ is a direct consequence of **Theorem 1**.

The proof of part (b) follows from **Theorem 2** and proceeds by arguments similar to those used above. ■

12. Proof of Main Lemmas; Lemma FS_αM & Weak Convergence

Proof of Lemma 1 (weak convergence). We will only prove the result for the case $\alpha \neq 1$. The proof for $\alpha = 1$ uses arguments similar to those used below. Throughout the proof $\beta^\# := \beta^* \tan(\alpha\pi/2)$.

BFR1. Set

$$G_{nk}^+ := -\sigma^\alpha \left| \frac{\lambda}{\gamma_n} a_k \right|^\alpha H \left(\left| \frac{\lambda}{\gamma_n} a_k \right|^{-1} \right) \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right],$$

and

$$G_{nk}^- := -\sigma^\alpha \left| \frac{\lambda}{\gamma_n} a_{k, [nr]}^- \right|^\alpha H \left(\left| \frac{\lambda}{\gamma_n} a_{k, [nr]}^- \right|^{-1} \right) \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_{k, [nr]}^- \right) \right].$$

Note that by (79) for k large enough ($k \geq k_0$, say)

$$\frac{1}{2}\alpha^\alpha k^{-1/\alpha} < a_k < \frac{3}{2}\alpha^\alpha k^{-1/\alpha}.$$

Hence, for k large and all $n \geq 1$

$$\operatorname{sgn}\left(\frac{\lambda}{\gamma_n}a_k\right) = \operatorname{sgn}\left(\frac{\lambda}{\gamma_n}\right) = \operatorname{sgn}(\lambda).$$

In view of the above,

$$\begin{aligned} \sum_{k=0}^{\lfloor (n-1)r \rfloor} G_{nk}^+ &=_{(1)} -\sigma^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda)\right] \sum_{k=k_0}^{\lfloor (n-1)r \rfloor} \left|\frac{\lambda}{\gamma_n}a_k\right|^\alpha H\left(\left|\frac{\lambda}{\gamma_n}a_k\right|^{-1}\right) + o(1), \\ &=_{(2)} -\alpha^{\alpha+1}\sigma^\alpha |\lambda|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda)\right] + o(1), \end{aligned} \quad (57)$$

where the approximation in $=_{(1)}$ follows from the fact that $\sup_{1 \leq k \leq k_0} \left|\frac{\lambda}{\gamma_n}a_k\right| \rightarrow 0$ as $n \rightarrow \infty$, whilst $=_{(2)}$ follows from arguments similar to those used for deriving (81). Note that the limit term above is independent of r due to the fact $L(x)$ is SV i.e.

$$\gamma_n^\alpha \sim L(\gamma_n n^{1/\alpha}) \sim L(\gamma_n [(n-1)r]^{1/\alpha}). \quad (58)$$

Furthermore, identical arguments as those used above give

$$\sum_{k=0}^{\lfloor (n-1)r \rfloor} |G_{nk}^+| = O(1). \quad (59)$$

Next, we show that for each $0 < r \leq 1$,

$$\sum_{k=\lfloor (n-1)r \rfloor + 1}^{\infty} G_{nk}^- = -\sigma^\alpha \alpha^{\alpha+1} |\lambda|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda)\right] + o(1). \quad (60)$$

Without loss of generality set $r = 1$. As before the limit does not depend on r due to eq. (58) above. Fix $t_0 \geq 1$. Then as $n \rightarrow \infty$ we have

$$\sum_{k=n}^{\infty} G_{nk}^- = \sum_{k=n+t_0}^{\infty} G_{nk}^- + o(1)$$

where the approximation above follows from the fact that

$$\sup_{n+t_0 \leq k \leq n} \left|\frac{\lambda}{\gamma_n}a_{k, \lfloor nr \rfloor}^-\right| \rightarrow 0,$$

as $n \rightarrow \infty$. Next, using the same decomposition as the one in eq. (87) we get for t_0 large enough and as $n \rightarrow \infty$

$$\begin{aligned} \sum_{k=n+t_0}^{\infty} G_{nk}^- &= \sum_{k=n+t_0}^{2(n+t_0)} G_{nk}^- + \sum_{k=2(n+t_0)+1}^{\infty} G_{nk}^- \\ &= \sum_{k=t_0}^{n+2t_0} -\sigma^\alpha \left|\frac{\lambda}{\gamma_n} \sum_{j=k}^{k+n} c_j\right|^\alpha H\left(\left|\frac{\lambda}{\gamma_n} \sum_{j=k}^{k+n} c_j\right|^{-1}\right) \left[1 + i\beta^\# \operatorname{sgn}\left(\frac{\lambda}{\gamma_n} \sum_{j=k}^{k+n} c_j\right)\right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=2(n+t_0)+1}^{\infty} -\sigma^\alpha \left| \frac{\lambda}{\gamma_n} \sum_{j=k-n}^k c_j \right|^\alpha H \left(\left| \frac{\lambda}{\gamma_n} \sum_{j=k-n}^k c_j \right|^{-1} \right) \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} \sum_{j=k-n}^k c_j \right) \right] \\
& =_{(3)} -\sigma^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda) \right] \sum_{k=t_0}^{n+2t_0} \left| \frac{\lambda}{\gamma_n} \sum_{j=k}^{k+n} c_j \right|^\alpha H \left(\left| \frac{\lambda}{\gamma_n} \sum_{j=k}^{k+n} c_j \right|^{-1} \right) \\
& \quad -\sigma^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda) \right] \sum_{k=2(n+t_0)+1}^{\infty} \left| \frac{\lambda}{\gamma_n} \sum_{j=k-n}^k c_j \right|^\alpha H \left(\left| \frac{\lambda}{\gamma_n} \sum_{j=k-n}^k c_j \right|^{-1} \right) \\
& =: -\sigma^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda) \right] (D'_n + D''_n) \\
& =_{(4)} -\sigma^\alpha \alpha^{\alpha+1} |\lambda|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda) \right] + o(1), \tag{61}
\end{aligned}$$

where $=_{(3)}$ follows from **Lemma 6**,¹¹ D'_n, D''_n resemble¹² the terms $C'_{3,n}, C''_{3,n}$ of (87), and $=_{(4)}$ follows from an almost identical argument as the one used for establishing (97). Therefore, (60) holds true. Moreover, using arguments similar to those used above we get

$$\sum_{k=\lfloor (n-1)r \rfloor + 1}^{\infty} |G_{nk}^-| = O(1). \tag{62}$$

Next, consider

$$\mathbf{E} \left(e^{i\lambda x_{\lfloor nr \rfloor} / \gamma_n} \right) = \mathbf{E} \left(e^{i\lambda x_{\lfloor nr \rfloor}^+ / \gamma_n} \right) \cdot \mathbf{E} \left(e^{i\lambda x_{\lfloor nr \rfloor}^- / \gamma_n} \right).$$

We shall show that for each λ as $n \rightarrow \infty$

$$\mathbf{E} \left(e^{i\lambda x_{\lfloor nr \rfloor}^+ / \gamma_n} \right) \rightarrow \mathbf{E} \left(e^{i\lambda X^+} \right),$$

and

$$\mathbf{E} \left(e^{i\lambda x_{\lfloor nr \rfloor}^- / \gamma_n} \right) \rightarrow \mathbf{E} \left(e^{i\lambda X^-} \right).$$

We start with the former. It can be easily checked that

$$\sup_{1 \leq k \leq n} |g_{nk}^+| := \sup_{1 \leq k \leq n} \left| \frac{\lambda}{\gamma_n} a_k \right| = o(1).$$

¹¹Let $I_n(k)$ and $Q_n(k)$ as in **Lemma 6**. By the aforementioned lemma for t_0 large enough we get $0 < I_n(k)/2 < \sum_{j=k-n}^k c_j < 3I_n(k)/2$ for all $k \geq 2(n+t_0)+1$, and $0 < Q_n(k)/2 < \sum_{j=k}^{k+n} c_j < 3Q_n(k)/2$ for all $n+2t_0 \geq k \geq t_0$ and $n \geq t_0$.

¹²Set $R_\alpha(x) = |x|^\alpha H(|x|^{-1})$. Then

$$D'_n = \sum_{k=t_0}^{n+2t_0} R_\alpha \left(\frac{\lambda}{\gamma_n} \sum_{j=k}^{k+n} c_j \right), \text{ whilst } C'_{3,n} = \sum_{k=t_0}^{n+2t_0} R_\alpha \left(\frac{1}{\gamma_n} \sum_{j=k}^{k+n} c_j \right)$$

Moreover,

$$D''_n = \sum_{k=2(n+t_0)+1}^{\infty} R_\alpha \left(\frac{\lambda}{\gamma_n} \sum_{j=k-n}^k c_j \right), \text{ whilst } C''_{3,n} = \sum_{k=2(n+t_0)+1}^{\infty} R_\alpha \left(\frac{1}{\gamma_n} \sum_{j=k-n}^k c_j \right).$$

Next, by virtue of (3) there is some $R(\lambda)$ such that $R(\lambda) \rightarrow 0$ as $\lambda \rightarrow 0$, and

$$\begin{aligned} \mathbf{E} \left(e^{i\lambda x_{[nr]}^+ / \gamma_n} \right) &= \prod_{k=1}^{\lfloor (n-1)r \rfloor} \left| \psi_\xi \left(\frac{\lambda}{\gamma_n} a_k \right) \right| \\ &= \prod_{k=1}^{\lfloor (n-1)r \rfloor} \exp \left\{ G_{nk}^+ \left[1 + R(g_{nk}^+) \right] \right\} = \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk}^+ \left[1 + R(g_{nk}^+) \right] \right\}. \end{aligned}$$

Set $R_n^+ := \sup_{1 \leq k \leq n} |R(g_{nk}^+)|$. It follows from the above that $R_n^+ \rightarrow 0$, as $n \rightarrow \infty$. In view of the latter, (57) and (59) we have

$$\begin{aligned} \mathbf{E} \left(e^{i\lambda x_{[nr]}^+ / \gamma_n} \right) &= \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk}^+ \left[1 + R(g_{nk}^+) \right] \right\} = \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk}^+ \right\} + o(1) \\ &= \exp \left\{ -\sigma^\alpha \alpha^{\alpha+1} \left[1 + i\beta^\# \operatorname{sgn}(\lambda) \right] \right\} + o(1) = \mathbf{E} \left(e^{i\lambda X^+} \right) + o(1), \end{aligned}$$

as required.

Next, it can be easily seen that

$$\sup_{1 \leq k \leq n, 0 \leq r \leq 1} |g_{nkr}^-| := \sup_{1 \leq k \leq n, 0 \leq r \leq 1} \left| \frac{\lambda}{\gamma_n} a_{k, \lfloor nr \rfloor}^- \right| = o(1).$$

Using identical arguments to those used above together with (61) and (62) as $n \rightarrow \infty$ we have

$$\begin{aligned} \mathbf{E} \left(e^{i\lambda x_{[nr]}^- / \gamma_n} \right) &= \prod_{k=\lfloor (n-1)r \rfloor + 1}^{\infty} \psi_\xi \left(\frac{\lambda}{\gamma_n} a_{k, \lfloor nr \rfloor}^- \right) \\ &= \exp \left\{ \sum_{k=\lfloor (n-1)r \rfloor + 1}^{\infty} G_{nk}^- \left[1 + R(g_{nkr}^-) \right] \right\} \\ &= \exp \left\{ -\sigma^\alpha \alpha^{\alpha+1} \left[1 + i\beta^\# \operatorname{sgn}(\lambda) \right] \right\} + o(1) = \mathbf{E} \left(e^{i\lambda X^+} \right) + o(1), \end{aligned}$$

as required.

BFR2. In view of **Lemma 9**, it suffices to show that

$$\mathbf{E} \left(e^{i\lambda \tilde{x}_{[nr]} / \gamma_n} \right) = \exp \left\{ -\sigma^\alpha \alpha |\lambda c_\infty|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda c_\infty) \right] \right\} + o(1),$$

for all $0 < r \leq 1$, where $c_\infty := \sum_{j=0}^{\infty} c_j$. Set

$$G_{nk} := -\sigma^\alpha \left| \frac{\lambda c_\infty}{\gamma_n} k^{-\frac{1}{\alpha}} \right|^\alpha H \left(\left| \frac{\lambda c_\infty}{\gamma_n} k^{-\frac{1}{\alpha}} \right|^{-1} \right) \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda c_\infty}{\gamma_n} k^{-\frac{1}{\alpha}} \right) \right],$$

and

$$g_{nk} := \frac{\lambda c_\infty}{\gamma_n} k^{-\frac{1}{\alpha}}.$$

In view of eq. (3) the characteristic function of $\gamma_n^{-1} \tilde{x}_{[nr]}$ is

$$\mathbf{E} \left(e^{i\lambda \tilde{x}_{[nr]} / \gamma_n} \right) = \prod_{k=1}^{\lfloor (n-1)r \rfloor} \left| \psi_\xi \left(\frac{\lambda c_\infty}{\gamma_n} k^{-\frac{1}{\alpha}} \right) \right|$$

$$= \prod_{k=1}^{\lfloor (n-1)r \rfloor} \exp \{G_{nk} [1 + R(g_{nk})]\} = \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} [1 + R(g_{nk})] \right\}, \quad (63)$$

with $R(g_{nk}) \rightarrow 0$ as $g_{nk} \rightarrow 0$. In particular, we have $R_n := \sup_{1 \leq k \leq n} |R(g_{nk})| \rightarrow 0$ as $n \rightarrow \infty$, due to the fact $\sup_{1 \leq k \leq n} |g_{nk}| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, as $n \rightarrow \infty$ we have

$$\left| \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} R(g_{nk}) \right| \leq R_n \sum_{k=1}^{\lfloor (n-1)r \rfloor} |G_{nk}| = o(1),$$

if we show that $\sum_{k=1}^n |G_{nk}|$ is bounded. It follows from the above that as $n \rightarrow \infty$

$$\mathbf{E} \left(e^{i\lambda \tilde{x}_{\lfloor nr \rfloor} / \gamma_n} \right) = \exp \left\{ \sum_{k=1}^n G_{nk} \right\} + o(1). \quad (64)$$

Next, we show that

$$\sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} = -\sigma^\alpha \alpha |\lambda c_\infty|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda c_\infty) \right] + o(1). \quad (65)$$

An almost identical argument to the one provided below shows that $\sum_{k=1}^n |G_{nk}|$ is bounded. In view of (63)-(65), we have

$$\mathbf{E} \left(e^{i\lambda \tilde{x}_{\lfloor nr \rfloor} / \gamma_n} \right) = \exp \left\{ -\sigma^\alpha \alpha |\lambda c_\infty|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda c_\infty) \right] \right\} + o(1), \quad (66)$$

which is the requisite result. We finish with the proof of (65). Note that for all n ,

$$\sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} = -\sigma^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda c_\infty) \right] \sum_{k=1}^{\lfloor (n-1)r \rfloor} \left| \frac{\lambda c_\infty}{\gamma_n} k^{-\frac{1}{\alpha}} \right|^\alpha H \left(\left| \frac{\lambda c_\infty}{\gamma_n} k^{-\frac{1}{\alpha}} \right|^{-1} \right). \quad (67)$$

Next, as $n \rightarrow \infty$, the sum

$$\begin{aligned} & \sum_{k=1}^{\lfloor (n-1)r \rfloor} \left| \frac{\lambda c_\infty}{\gamma_n} k^{-\frac{1}{\alpha}} \right|^\alpha H \left(\left| \frac{\lambda c_\infty}{\gamma_n} k^{-\frac{1}{\alpha}} \right|^{-1} \right) = \left| \frac{\lambda c_\infty}{\gamma_n} \right|^\alpha \sum_{k=1}^{\lfloor (n-1)r \rfloor} k^{-1} H \left(|c_\infty \lambda|^{-1} \gamma_n k^{\frac{1}{\alpha}} \right). \\ & =_{(1)} \alpha \left| \frac{\lambda c_\infty}{\gamma_n} \right|^\alpha \int_{u=|c_\infty \lambda|^{-1} \gamma_n}^{|c_\infty \lambda|^{-1} \gamma_n (\lfloor (n-1)r \rfloor)^{\frac{1}{\alpha}}} \frac{H(u)}{u} du [1 + o(1)] =_{(2)} \alpha |\lambda c_\infty|^\alpha + o(1), \end{aligned} \quad (68)$$

where $=_{(1)}$ is due to **Lemma 3(i)**, $=_{(2)}$ due to the fact that

$$L \left(\gamma_n n^{\frac{1}{\alpha}} \right) \sim \gamma_n^\alpha,$$

and

$$\int_{u=|c_\infty \lambda|^{-1} \gamma_n}^{|c_\infty \lambda|^{-1} \gamma_n (\lfloor (n-1)r \rfloor)^{\frac{1}{\alpha}}} H(u) u^{-1} du \sim \int_{u=\gamma_n}^{\gamma_n n^{\frac{1}{\alpha}}} \frac{H(u)}{u} du \sim L \left(\gamma_n n^{\frac{1}{\alpha}} \right).$$

To see validity of the latter statement write

$$\begin{aligned} & \frac{L \left(\gamma_n n^{\frac{1}{\alpha}} \right) - L(\gamma_n)}{L \left(\gamma_n n^{\frac{1}{\alpha}} \right)} = 1 - \frac{L(\gamma_n)}{L \left(\gamma_n n^{\frac{1}{\alpha}} \right)} \\ & \sim 1 - \frac{L(\gamma_n)}{\gamma_n^\alpha} \rightarrow 1. \end{aligned}$$

Note that (65) follows from (67) and (68).

MI-SM. From **Lemma FS_αA** we have $|\gamma_n^{-1}x_{\lfloor nr \rfloor}(n)^-| = o_p(1)$ for all $0 < r \leq 1$. Therefore, it suffices to show that

$$\mathbf{E} \exp \left[i\lambda \gamma_n^{-1} x_t(n)^+ \right] = \exp \left\{ -\sigma^\alpha \alpha^{-1} |\lambda c_\infty|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda c_\infty) \right] \right\} + o(1).$$

Set

$$G_{nk} := -\sigma^\alpha \left| \frac{\lambda}{\gamma_n} a_k \right|^\alpha H \left(\left| \frac{\lambda}{\gamma_n} a_k \right|^{-1} \right) \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right],$$

where $a_k(n) := a_k = \sum_{l=0}^k \rho_n^{k-l} c_l$ with $0 < \rho_n^j < 1$ for all $j \geq 1$ and n large enough.

In view of the above, as $n \rightarrow \infty$,

$$\begin{aligned} \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} &=_{(1)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-1} \sum_{k=1}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right] [1 + o(1)] \\ &=_{(2)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-1} \sum_{k=1}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda \rho_n^k c_\infty) \right] [1 + o(1)] \\ &=_{(3)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-1} \sum_{k=1}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda c_\infty) \right] [1 + o(1)] \\ &=_{(4)} -\sigma^\alpha \alpha^{-1} |\lambda c_\infty|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda c_\infty) \right] + o(1), \end{aligned}$$

where $=_{(1)}$ follows from an argument analogous to the one used for the first approximation in (126); the equalities $=_{(2)}$ and $=_{(4)}$ from **Lemma 5**, and $=_{(3)}$ from the fact that, for n sufficiently large, $\operatorname{sgn}(\lambda c_\infty) = \operatorname{sgn}(\lambda \rho_n^k c_\infty)$. Furthermore, it is straightforward to verify that $\sum_{k=1}^n |G_{nk}| = O(1)$ and $\sup_{k \geq 0} |a_k| < \infty$. Hence, in view of **Assumption INN**, there is some $R(x) \rightarrow 0$ as $x \rightarrow 0$ such that

$$\begin{aligned} \mathbf{E} \left(e^{i\lambda x_{\lfloor nr \rfloor}^+ / \gamma_n} \right) &= \prod_{k=1}^{\lfloor (n-1)r \rfloor} \psi_\xi \left(\frac{\lambda}{\gamma_n} a_k \right) \\ &= \prod_{k=1}^{\lfloor (n-1)r \rfloor} \exp \left\{ G_{nk} \left[1 + R \left(\frac{\lambda}{\gamma_n} a_k \right) \right] \right\} = \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} \left[1 + R \left(\frac{\lambda}{\gamma_n} a_k \right) \right] \right\}, \\ &=_{(4)} \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} + o(1) \right\} = \exp \left\{ -\sigma^\alpha \alpha^{-1} |\lambda c_\infty|^\alpha \left[1 + i\beta^\# \operatorname{sgn}(\lambda c_\infty) \right] \right\} + o(1), \end{aligned}$$

where $=_{(4)}$ follows from the fact that

$$\sum_{k=1}^n \left| G_{nk} R \left(\frac{\lambda}{\gamma_n} a_k \right) \right| \leq \sup_k \left| R \left(\frac{\lambda}{\gamma_n} a_k \right) \right| \sum_{k=1}^n |G_{nk}| = o(1),$$

and the proof is complete.

MI-LM ($1/\alpha < m < 1$). In view of (8), we can write

$$\gamma_n^{-1} x_{\lfloor nr \rfloor} = \gamma_n^{-1} x_{\lfloor nr \rfloor}^+ + \gamma_n^{-1} x_{\lfloor nr \rfloor}^-,$$

where $\gamma_n^{-1}x_{[nr]}^- = o_p(1)$ due to **Lemma FS_αM**. Therefore, it suffices to show that for each $0 < r \leq 1$, as $n \rightarrow \infty$,

$$\mathbf{E} \exp \left(\lambda \gamma_n^{-1} x_{[nr]}^+ \right) \rightarrow \exp \left(-|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \mathbb{V}_1 \left[1 + i\beta^\# \operatorname{sgn}(\lambda) \right] \right).$$

Write

$$G_{nk} := -\sigma^\alpha \left| \frac{\lambda}{\gamma_n} a_k \right|^\alpha H \left(\left| \frac{\lambda}{\gamma_n} a_k \right|^{-1} \right) \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right],$$

where as in the previous part i.e. $a_k(n) := a_k = \sum_{l=0}^k \rho_n^{k-l} c_l$. Set

$$q(x) := e^{-x} \int_0^x e^y y^{-m} dy.$$

By Lemma 7.2 of DK2018, as $n \rightarrow \infty$,

$$\sup_{0 \leq k \leq n} \left| \kappa_n^{-(1-m)} \sum_{j=0}^k \rho_n^j c_{k-j} - q(k/\kappa_n) \right| = o(1).$$

In fact, under the additional requirement $c_j = m_0 \cdot j^{-m} [1 + O(j^{-\nu})]$, $0 < \nu < 1 - m$, it is possible to strengthen this result as follows:

$$\sup_{0 \leq k \leq n} \left| \kappa_n^{-(1-m)} \sum_{j=0}^k \rho_n^j c_{k-j} - q(k/\kappa_n) \right| = O(\kappa_n^{-\nu}), \quad (69)$$

as $n \rightarrow \infty$. Let $m_n := \lfloor \kappa_n^p \rfloor$, for some $0 < p < 1$. Moreover, set $\delta_n := \lfloor \kappa_n \varsigma_n \rfloor$, $\varsigma_n = [\ln(1 + \kappa_n)]^2$. By (69), for n large enough,

$$\begin{aligned} \inf_{m_n \leq k \leq \delta_n} \kappa_n^{-(1-m)} a_k(n) &= \inf_{m_n \leq k \leq n} q \left(\frac{k}{\kappa_n} \right) + R_n \\ &\geq_{(1)} q \left(\frac{m_n}{\kappa_n} \right) \wedge q \left(\frac{\delta_n}{\kappa_n} \right) + R_n \geq_{(2)} q \left(\frac{m_n}{\kappa_n} \right) + R_n, \end{aligned}$$

where $R_n = O(\kappa_n^{-\nu})$, and $\geq_{(1)}$ follows from the fact that $q(x)$ attains its minimum on $[m_n, \delta_n]$ on a boundary point of that interval.¹³ Moreover, $\geq_{(2)}$ holds from the fact that as $n \rightarrow \infty$,

$$q \left(\frac{m_n}{\kappa_n} \right) \cdot q \left(\frac{\delta_n}{\kappa_n} \right)^{-1} \rightarrow 0.$$

To see this by **Proposition 2(a,b)**, as $n \rightarrow \infty$,

$$\frac{q \left(\frac{m_n}{\kappa_n} \right)}{q \left(\frac{\delta_n}{\kappa_n} \right)} \sim \frac{\frac{1}{1-m} \kappa_n^{-(1-m)(1-p)}}{\left(\frac{\delta_n}{\kappa_n} \right)^{-m}} \sim \frac{\frac{1}{1-m} \kappa_n^{-(1-m)(1-p)}}{\varsigma_n^{-m}} \rightarrow 0.$$

Next, by **Proposition 2(b)** we have

$$q \left(\frac{m_n}{\kappa_n} \right) = \frac{1}{1-m} \left(\frac{m_n}{\kappa_n} \right)^{1-m} + R_{1n}, \quad R_{1n} = O \left(\left(\frac{m_n}{\kappa_n} \right)^{2-m} \right).$$

¹³Note that the second derivative of $q(x)$ is $q''(x) = -mx^{-m-1} - q'(x)$. Hence, if x_* is a stationary point of $q(x)$ interior in some interval $[a, b]$, we have $q''(x_*) = -mx_*^{-m-1} < 0$. Hence, $q(x_*)$ corresponds to a maximum. In view of this, $q(x)$ attains its minimum on $[a, b]$ at a boundary point of the interval.

Choose ν such that $1 - m > \nu > (1 - p)(1 - m)$. Then, for all $m_n \leq k \leq \delta_n$, and n large enough, we get

$$\begin{aligned}
\kappa_n^{-(1-m)} a_k(n) &\geq q \left(\frac{m_n}{\kappa_n} \right) + R_n = \frac{1}{1-m} \left(\frac{m_n}{\kappa_n} \right)^{1-m} + R_{1n} + R_n \\
&= \frac{1}{1-m} \kappa_n^{-(1-p)(1-m)} + O \left(\kappa_n^{-(1-p)(2-m)} \right) + O \left(\kappa_n^{-\nu} \right) \\
&= \kappa_n^{-(1-p)(1-m)} \left[\frac{1}{1-m} + O \left(\kappa_n^{-1} \right) + O \left(\kappa_n^{(1-p)(1-m)-\nu} \right) \right] \\
&\geq \kappa_n^{-(1-p)(1-m)} \left[\frac{1}{1-m} - \frac{1}{2(1-m)} \right] = \kappa_n^{-(1-p)(1-m)} \frac{1}{2(1-m)} > 0.
\end{aligned}$$

Hence, for all $m_n \leq k \leq \delta_n$, and n large enough,

$$\operatorname{sgn}(\lambda) = \operatorname{sgn} \left(\lambda \frac{\kappa_n^{-(1-p)(1-m)}}{2(1-m)} \right) = \operatorname{sgn} \left(\lambda \kappa_n^{-(1-m)} a_k(n) \right) = \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k(n) \right). \quad (70)$$

In view of the above for each $0 < r \leq 1$, as $n \rightarrow \infty$,

$$\begin{aligned}
\sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} &=_{(1)} -\sigma^\alpha |\lambda|^\alpha H(m_n) \gamma_n^{-\alpha} \sum_{k=1}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right] + o(1) \\
&=_{(2)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-b} \sum_{k=1}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right] [1 + o(1)] \\
&=_{(3)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-b} \sum_{k=m_n}^{\delta_n} |a_k|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right] [1 + o(1)] \\
&=_{(4)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-b} \sum_{k=m_n}^{\delta_n} |a_k|^\alpha [1 + i\beta^\# \operatorname{sgn}(\lambda)] [1 + o(1)] \\
&=_{(5)} -|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \mathbb{V}_1 [1 + i\beta^\# \operatorname{sgn}(\lambda)] + o(1), \quad (71)
\end{aligned}$$

where $=_{(1)}$ follows from analogous arguments as those used for the proof of (140), $=_{(2)}$ follows from the fact that $H(m_n) \gamma_n^{-\alpha} \sim \kappa_n^{-b}$, $=_{(3)}$ is due to **Lemma 15** and **Lemma 17**, $=_{(4)}$ is a direct consequence of (70), and $=_{(5)}$ follows from **Lemma 17**. Moreover, arguments analogous to those used for the proof of (71) show that

$$\sum_{k=1}^n |G_{nk}| = O(1).$$

Next, note that $\gamma_n^{-1} \sup_{1 \leq k \leq n} |a_k| \rightarrow 0$, as $n \rightarrow \infty$, cf. (138). In view of **Assumption INN**, there is some $R(x) \rightarrow 0$ as $x \rightarrow 0$ such that the characteristic function of $x_{\lfloor nr \rfloor}^+ / \gamma_n$

$$\begin{aligned}
\mathbf{E} \left(e^{i\lambda x_{\lfloor nr \rfloor}^+ / \gamma_n} \right) &= \prod_{k=1}^{\lfloor (n-1)r \rfloor} \psi_\xi \left(\frac{\lambda}{\gamma_n} a_k \right) \\
&= \prod_{k=1}^{\lfloor (n-1)r \rfloor} \exp \left\{ G_{nk} \left[1 + R \left(\frac{\lambda}{\gamma_n} a_k \right) \right] \right\} = \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} \left[1 + R \left(\frac{\lambda}{\gamma_n} a_k \right) \right] \right\},
\end{aligned}$$

$$=_{(6)} \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} + o(1) \right\} =_{(7)} \exp \left\{ -|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \mathbb{V}_1(\alpha, m) \left[1 + i\beta^\# \operatorname{sgn}(\lambda) \right] \right\} + o(1),$$

where $=_{(6)}$ follows from the fact that as $n \rightarrow \infty$, $\sup_{1 \leq k \leq n} \left| R\left(\frac{\lambda}{\gamma_n} a_k\right) \right| = o(1)$, and

$$\sum_{k=1}^n \left| G_{nk} R\left(\frac{\lambda}{\gamma_n} a_k\right) \right| \leq \sup_k \left| R\left(\frac{\lambda}{\gamma_n} a_k\right) \right| \sum_{k=1}^n |G_{nk}| = o(1).$$

Finally, $=_{(7)}$ is a direct consequence of (71).

MI-NM ($1 < m < 1 + 1/\alpha$). In view of (8), we can write

$$\gamma_n^{-1} x_{\lfloor nr \rfloor} = \gamma_n^{-1} x_{\lfloor nr \rfloor}^+ + \gamma_n^{-1} x_{\lfloor nr \rfloor}^-$$

where $\gamma_n^{-1} x_{\lfloor nr \rfloor}^- = 0$, by **Assumption MI-NM**. It suffices to show that for each $0 < r \leq 1$, as $n \rightarrow \infty$,

$$\mathbf{E} \exp \left(\lambda \gamma_n^{-1} x_{\lfloor nr \rfloor}^+ \right) \rightarrow \exp \left(-|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \mathbb{V}_2 \left[1 + i\beta^\# \mathbb{V}_3 \operatorname{sgn}(m_0) \operatorname{sgn}(\lambda) \right] \right).$$

Write

$$G_{nk} := -\sigma^\alpha \left| \frac{\lambda}{\gamma_n} a_k \right|^\alpha H \left(\left| \frac{\lambda}{\gamma_n} a_k \right|^{-1} \right) \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right],$$

where recall that $a_k(n) := a_k = \sum_{l=0}^k \rho_n^{k-l} c_l$. Moreover,

$$p(x) := x^{1-m} - e^{-x} \int_{y=0}^x e^y y^{1-m} dy,$$

and by (193), as $n \rightarrow \infty$,

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left| \frac{1}{\kappa_n^{1-m}} a_k - \frac{m_0 p(k/\kappa_n)}{1-m} \right|^\alpha = o(1). \quad (72)$$

Let $\delta_n := \lfloor \kappa_n \varsigma_n \rfloor$, $\varsigma_n = \lfloor \ln(1 + \kappa_n) \rfloor^2$. For each $0 < r \leq 1$, as $n \rightarrow \infty$,

$$\begin{aligned} & \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} =_{(1)} -\sigma^\alpha |\lambda|^\alpha \gamma_n^{-\alpha} \sum_{k=1}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right] \\ & =_{(2)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-b} \sum_{k=1}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right] [1 + o(1)] \\ & =_{(3)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-b} \sum_{k=1}^{\delta_n} |a_k|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\gamma_n} a_k \right) \right] [1 + o(1)] \\ & =_{(4)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-1} \sum_{k=1}^{\delta_n} \left| \frac{m_0 p(k/\kappa_n)}{1-m} \right|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\frac{\lambda}{\kappa_n^{1-m}} a_k \right) \right] [1 + o(1)] \\ & =_{(5)} -\sigma^\alpha |\lambda|^\alpha \kappa_n^{-1} \sum_{k=1}^{\delta_n} \left| \frac{m_0 p(k/\kappa_n)}{1-m} \right|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\lambda \frac{m_0 p(k/\kappa_n)}{1-m} \right) \right] [1 + o(1)] \\ & =_{(6)} -|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \int_0^\infty \left| \frac{p(x)}{1-m} \right|^\alpha \left[1 + i\beta^\# \operatorname{sgn} \left(\lambda \frac{m_0 p(x)}{1-m} \right) \right] dx + o(1) \end{aligned}$$

$$\begin{aligned}
&= -|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \left[\int_0^\infty \left| \frac{p(x)}{1-m} \right|^\alpha dx + i\beta^\# \int_0^\infty \left| \frac{p(x)}{1-m} \right|^\alpha \operatorname{sgn} \left(\lambda \frac{m_0 p(x)}{1-m} \right) dx \right] + o(1) \\
&= -|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \int_0^\infty \left| \frac{p(x)}{1-m} \right|^\alpha dx \left[1 + i\beta^\# \frac{\int_0^\infty |p(x)|^\alpha \operatorname{sgn} \left(\lambda \frac{p(x)}{1-m} \right) dx}{\int_0^\infty |p(x)|^\alpha dx} \right] + o(1) \\
&= -|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \int_0^\infty \left| \frac{p(x)}{1-m} \right|^\alpha dx \left[1 - i\beta^\# \operatorname{sgn}(m_0) \frac{\int_0^\infty |p(x)|^\alpha \operatorname{sgn}(p(x)) dx}{\int_0^\infty |p(x)|^\alpha dx} \operatorname{sgn}(\lambda) \right] + o(1) \\
&= -|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \mathbb{V}_2 \left[1 - i\beta^\# \operatorname{sgn}(m_0) \mathbb{V}_3 \operatorname{sgn}(\lambda) \right] + o(1), \tag{73}
\end{aligned}$$

where $=_{(1)}$ follows from (150), $=_{(2)}$ the fact that $\gamma_n^\alpha \sim \kappa_n^b \ell_1(\kappa_n) \sim \kappa_n^b$, $=_{(3)}$ is a consequence of **Lemma 15**, $=_{(4)}$ is due to **Corollary 1**, and $=_{(6)}$ follows from **Corollary 2**. Finally, we show that $=_{(5)}$ holds. Set

$$K_{n,k} := \left\{ 1 \leq k \leq \delta_n : \operatorname{sgn} \left(\frac{1}{\kappa_n^{1-m}} a_k \right) \neq \operatorname{sgn}(p(k/\kappa_n)) \right\}.$$

As $n \rightarrow \infty$, we have

$$\begin{aligned}
&\kappa_n^{-1} \left| \sum_{k=1}^{\delta_n} |p(k/\kappa_n)|^\alpha \left[\operatorname{sgn} \left(\frac{1}{\kappa_n^{1-m}} a_k \right) - \operatorname{sgn}(p(k/\kappa_n)) \right] \right| \\
&\leq \kappa_n^{-1} \sum_{k=1}^{\delta_n} |p(k/\kappa_n)|^\alpha \left| \operatorname{sgn} \left(\frac{1}{\kappa_n^{1-m}} a_k \right) - \operatorname{sgn}(p(k/\kappa_n)) \right| \\
&= \kappa_n^{-1} \sum_{k \in K_n} |p(k/\kappa_n)|^\alpha \left| \operatorname{sgn} \left(\frac{1}{\kappa_n^{1-m}} a_k \right) - \operatorname{sgn}(p(k/\kappa_n)) \right| \\
&\leq 2\kappa_n^{-1} \sum_{k=1}^{\delta_n} \left| p(k/\kappa_n) - \frac{1}{\kappa_n^{1-m}} a_k \right|^\alpha = o(1),
\end{aligned}$$

where the last approximation follows from (72).

Next, note that an argument analogous to that used for (73) gives

$$\sum_{k=1}^n |G_{nk}| = O(1). \tag{74}$$

Furthermore,

$$\gamma_n^{-1} \sup_{1 \leq k \leq n} |a_k| = \gamma_n^{-1} \sup_{1 \leq k \leq n} \left| \sum_{l=0}^k \rho_n^{k-l} c_l \right| \leq \gamma_n^{-1} \sum_{l=0}^\infty |c_l| \rightarrow 0,$$

as $n \rightarrow \infty$. In view of **Assumption INN**, there is some $R(x) \rightarrow 0$ as $x \rightarrow 0$ such that the characteristic function of $x_{\lfloor nr \rfloor}^+ / \gamma_n$

$$\begin{aligned}
&\mathbf{E} \left(e^{i\lambda x_{\lfloor nr \rfloor}^+ / \gamma_n} \right) = \prod_{k=1}^{\lfloor (n-1)r \rfloor} \psi_\xi \left(\frac{\lambda}{\gamma_n} a_k \right) \\
&= \prod_{k=1}^{\lfloor (n-1)r \rfloor} \exp \left\{ G_{nk} \left[1 + R \left(\frac{\lambda}{\gamma_n} a_k \right) \right] \right\} = \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} \left[1 + R \left(\frac{\lambda}{\gamma_n} a_k \right) \right] \right\},
\end{aligned}$$

$$= \exp \left\{ \sum_{k=1}^{\lfloor (n-1)r \rfloor} G_{nk} + o(1) \right\} = -|\lambda|^\alpha \sigma^\alpha |m_0|^\alpha \mathbb{V}_2 \left[1 - i\beta^\# \operatorname{sgn}(m_0) \mathbb{V}_3 \operatorname{sgn}(\lambda) \right] + o(1),$$

where we have used (73) and the fact that $\sup_{1 \leq k \leq n} \left| R\left(\frac{\lambda}{\gamma_n} a_k\right) \right| = o(1)$. Notice that the latter together with (74) implies that

$$\sum_{k=1}^n \left| G_{nk} R\left(\frac{\lambda}{\gamma_n} a_k\right) \right| \leq \sup_k \left| R\left(\frac{\lambda}{\gamma_n} a_k\right) \right| \sum_{k=1}^n |G_{nk}| = o(1).$$

This completes the proof. ■

Proof of Lemma FS_αM. (a) We start with part (a). We shall prove the result first under **BFR1**, then under **BFR2** and finally under **MI**. Before we present the proofs, we provide a preliminary result that will be used throughout. Recall that $h(x)$ SV such that (see Remark 1)

$$H(x) = \begin{cases} h(x), & \alpha \in (0, 2) \\ -\int_0^x s^2 d\left(\frac{h(s)}{s^2}\right), & \alpha = 2 \end{cases},$$

where in the latter case $h(x) = o(H(x))$. To see this note that

$$\begin{aligned} \int_0^x s^2 d\left(\frac{h(s)}{s^2}\right) &= h(x) - h(0) - \int_0^x \frac{h(s)}{s^2} ds^2 \\ &= h(x) - h(0) - 2 \int_0^x \frac{h(s)}{s} ds, \end{aligned}$$

with $\int_0^x \frac{h(s)}{s} ds$ SV (see e.g. BGT 1987; Proposition 1.5.7). Now by BGT 1987, Propoposition 1.5.9a, $\lim_{x \rightarrow \infty} \int_0^x \frac{h(s)}{s} ds / h(x) = \infty$. Hence, $h(x) = o(H(x))$. Therefore, for $\alpha \in (0, 2]$

$$h(x) = O(H(x)), \text{ as } x \rightarrow \infty. \quad (75)$$

BFR1. Let $\varepsilon \in (0, \alpha)$ and

$$A_{n,t} := \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{t-1} a_k \xi_{t-k} \right|^{\alpha-\varepsilon} \text{ and } B_{n,t} := \mathbf{E} \left| \gamma_n^{-1} \sum_{k=t}^{\infty} a_{k,t}^- \xi_{t-k} \right|^{\alpha-\varepsilon},$$

Then

$$\mathbf{E} \left| \gamma_n^{-1} x_t \right|^{\alpha-\varepsilon} = \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{\infty} a_{k,t}^- \xi_{t-k} \right|^{\alpha-\varepsilon} \leq 2(A_{n,t} + B_{n,t}).$$

Therefore, it suffices to show that $\sup_{1 \leq t \leq n} A_{n,t}, \sup_{1 \leq t \leq n} B_{n,t} < \infty$.

First, we consider $A_{n,t}$. Let

$$S_{n,k} := \gamma_n |a_k|^{-1}.$$

Similarly to Astrauskas (1983), set

$$\bar{\xi}_{n,k} := \xi_k 1\{|\xi_k| \geq S_{n,k}\} \text{ and } \underline{\xi}_{n,k} := \xi_k 1\{|\xi_k| < S_{n,k}\}.$$

Without loss of generality, for $\alpha \in (1, 2]$ we assume that $\mathbf{E} \bar{\xi}_{n,k} = \mathbf{E} \underline{\xi}_{n,k} = 0$. This convention can be dispensed with if we instead set

$$\bar{\xi}_{n,k} = \xi_k 1\{|\xi_k| \geq S_{n,k}\} - \mathbf{E}[\xi_k 1\{|\xi_k| \geq S_{n,k}\}],$$

$$\underline{\xi}_{n,k} = \xi_k 1\{|\xi_k| < S_{n,k}\} - \mathbf{E}[\xi_k 1\{|\xi_k| < S_{n,k}\}]. \quad (76)$$

For more details see footnote 14. Fix $k_0 \geq 1$. Note that by **Assumption INN** we have $\mathbf{E}|\xi_k|^{\alpha-\varepsilon} < \infty$ -see e.g. Thm 2.6.4 in Ibragimov and Linnik (1971). Hence, as $n \rightarrow \infty$ we get

$$\begin{aligned} A_{n,t} &= \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{t-1} a_k \xi_k \right|^{\alpha-\varepsilon} \leq 2\mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{k_0-1} a_k \xi_k \right|^{\alpha-\varepsilon} + 2\mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \xi_k \right|^{\alpha-\varepsilon} \\ &\leq 2k_0 \gamma_n^{-1} \sum_{k=0}^{k_0-1} |a_k|^{\alpha-\varepsilon} \mathbf{E} |\xi_k|^{\alpha-\varepsilon} + 2\mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \xi_k \right|^{\alpha-\varepsilon} = 2\mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \xi_k \right|^{\alpha-\varepsilon} + o(1) \\ &\leq 4\mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \bar{\xi}_{n,k} \right|^{\alpha-\varepsilon} + 4\mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \underline{\xi}_{n,k} \right|^{\alpha-\varepsilon} + o(1) \\ &\leq 4\mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \bar{\xi}_{n,k} \right|^{\alpha-\varepsilon} + 4 \left\{ \mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \underline{\xi}_{n,k} \right|^{\alpha+\varepsilon} \right\}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} + o(1) \\ &\leq_{(1)} 4 \left[\gamma_n^{-(\alpha-\varepsilon)} \sum_{k=k_0}^n |a_k|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} + \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=k_0}^n |a_k|^{\alpha+\varepsilon} \mathbf{E} |\underline{\xi}_{n,k}|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right] \\ &=: 4 \left[A_{1,n} + A_{2,n}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right], \end{aligned}$$

where $\leq_{(1)}$ follows either from Loève's or the Bahr-Esseen¹⁴ inequalities depending on the value of the exponents (see p. 133 in Astrauskas, 1983). By Lemma 1 in Astrauskas (1983), for $\alpha \in (0, 2]$ as $S_{n,k} \rightarrow \infty$,

$$\mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} \asymp S_{n,k}^{-\varepsilon} h(S_{n,k}) \text{ and } \mathbf{E} |\underline{\xi}_{n,k}|^{\alpha+\varepsilon} \asymp S_{n,k}^{\varepsilon} h(S_{n,k}). \quad (78)$$

Note that under **BFR1**,

$$|a_k| = \left| \sum_{j=0}^k c_{k-j} \right| \leq \sum_{j=0}^{\infty} |c_j| =: \epsilon < \infty.$$

Hence, as $n \rightarrow \infty$

$$\inf_{0 \leq k \leq n} S_{n,k} = \gamma_n \inf_{0 \leq k \leq n} |a_k|^{-1} = \gamma_n \frac{1}{\sup_{0 \leq k \leq n} |a_k|} \geq \gamma_n \epsilon^{-1} \rightarrow \infty.$$

Therefore, the approximations in (78) hold uniformly in $0 \leq k \leq n$.

¹⁴We remark that if we set $\bar{\xi}_{n,k}$ and $\underline{\xi}_{n,k}$ as in (76), then

$$\mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} \leq 2\mathbf{E} |\xi_k 1\{|\xi_k| \geq S_{n,k}\}|^{\alpha-\varepsilon} \text{ and } \mathbf{E} |\underline{\xi}_{n,k}|^{\alpha+\varepsilon} \leq 4\mathbf{E} |\xi_k 1\{|\xi_k| < S_{n,k}\}|^{\alpha+\varepsilon}.$$

In addition, using Lemma 1 in Astrauskas (1983), as $S_{n,k} \rightarrow \infty$,

$$\mathbf{E} |\xi_k 1\{|\xi_k| \geq S_{n,k}\}|^{\alpha-\varepsilon} \asymp S_{n,k}^{-\varepsilon} h(S_{n,k}) \text{ and } \mathbf{E} |\xi_k 1\{|\xi_k| < S_{n,k}\}|^{\alpha+\varepsilon} \asymp S_{n,k}^{\varepsilon} h(S_{n,k}). \quad (77)$$

Set $R_{-\alpha}(x) := (x)^{-\alpha} H(x)$, $x > 0$. In view of the above and (75), as $n \rightarrow \infty$

$$\begin{aligned} A_{1,n} &= \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=k_0}^n |a_k|^{\alpha-\varepsilon} \mathbf{E} \left| \bar{\xi}_{n,k} \right|^{\alpha-\varepsilon} = C_1 \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=k_0}^n |a_k|^{\alpha-\varepsilon} S_{n,k}^{-\varepsilon} h(S_{n,k}) [1 + o(1)] \\ &\leq C \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^\alpha H(S_{n,k}) [1 + o(1)] = C \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^\alpha H(\gamma_n |a_k|^{-1}) [1 + o(1)] \\ &= C \sum_{k=k_0}^n R_{-\alpha}(\gamma_n |a_k|^{-1}) [1 + o(1)] =: C A_{3,n} [1 + o(1)]. \end{aligned}$$

Similarly,

$$A_{2,n} = \gamma_n^{-(\alpha+\varepsilon)} \sum_{k=k_0}^n |a_k|^{\alpha+\varepsilon} \mathbf{E} \left| \underline{\xi}_{n,k} \right|^{\alpha+\varepsilon} \leq C A_{3,n} [1 + o(1)].$$

Set $I_k := \int_{k+1}^{\infty} x^{-(1+\frac{1}{\alpha})} dx = \alpha(k+1)^{-1/\alpha}$. Euler summation (see e.g. Magdalinos, 2012 or **Lemma 3** here) gives

$$\left| \sum_{r=k+1}^{\infty} r^{-(1+\frac{1}{\alpha})} - \int_{k+1}^{\infty} x^{-(1+\frac{1}{\alpha})} dx \right| \leq k^{-(1+\frac{1}{\alpha})}.$$

Noting that

$$a_k = \sum_{r=0}^k c_r = - \sum_{r=k+1}^{\infty} c_r,$$

and $c_r \sim r^{-(1+\frac{1}{\alpha})}$, for k large enough and $\eta/2 > 0$ arbitrarily small we get

$$\begin{aligned} \left| \sum_{r=k+1}^{\infty} c_r - I_k \right| &\leq \sum_{r=k+1}^{\infty} \left| c_r - r^{-(1+\frac{1}{\alpha})} \right| + \sum_{r=k+1}^{\infty} \left| r^{-(1+\frac{1}{\alpha})} - I_k \right| \\ &\leq \sum_{r=k+1}^{\infty} \left| c_r r^{(1+\frac{1}{\alpha})} - 1 \right| r^{-(1+\frac{1}{\alpha})} + k^{-(1+\frac{1}{\alpha})} \\ &\leq (\eta/2) \sum_{r=k+1}^{\infty} r^{-(1+\frac{1}{\alpha})} + k^{-(1+\frac{1}{\alpha})} \leq (\eta/2) \sum_{r=k+1}^{\infty} \left| r^{-(1+\frac{1}{\alpha})} - I_k \right| + (\eta/2) I_k + k^{-(1+\frac{1}{\alpha})} \\ &\leq (\eta/2 + 1) k^{-(1+\frac{1}{\alpha})} + (\eta/2) I_k = O\left(k^{-(1+\frac{1}{\alpha})}\right) + (\eta/2) I_k. \end{aligned}$$

Hence, as $k \rightarrow \infty$

$$|a_k/I_k - 1| = \left| \frac{a_k - I_k}{I_k} \right| \leq O(k^{-1}) + \eta/2 < \eta.$$

Thus, as $k \rightarrow \infty$ we have

$$|a_k| \sim \alpha k^{-1/\alpha} \quad \text{and} \quad |a_k|^\alpha \sim \alpha^\alpha k^{-1}. \quad (79)$$

Let $\eta > 0$ arbitrarily small and $k_0 \geq 1$ large enough such that $\left| \frac{|a_k|^\alpha}{\alpha^\alpha k^{-1}} - 1 \right| < \eta$, for all $k \geq k_0$, which possible because of (79). Hence, as $n \rightarrow \infty$ first and then as $\eta \rightarrow 0$

$$|A_{4,n} - A_{5,n}| := \left| \sum_{k=k_0}^n \gamma_n^{-\alpha} |a_k|^\alpha H\left(\gamma_n k^{\frac{1}{\alpha}}\right) - \alpha^\alpha \sum_{k=k_0}^n \gamma_n^{-\alpha} k^{-1} H\left(\gamma_n k^{\frac{1}{\alpha}}\right) \right|$$

$$\begin{aligned}
&\leq \alpha^\alpha \sum_{k=k_0}^n \left| \frac{|a_k|^\alpha}{\alpha^\alpha k^{-1}} - 1 \right| \gamma_n^{-\alpha} k^{-1} H\left(\gamma_n k^{\frac{1}{\alpha}}\right) \leq \eta A_{5,n} \\
&=_{(1)} \eta \alpha^{\alpha+1} (1 + o(1)) \xrightarrow{\eta \rightarrow 0} o(1),
\end{aligned} \tag{80}$$

where $=_{(1)}$ follows from (68). Next, using (79) again, for k large enough ($k \geq k_0$ say) we get $|k^{-\frac{1}{\alpha}} |a_k|^{-1} - \alpha^{-1}| \leq \delta$, for all $\delta \in (0, \alpha^{-1})$. Therefore, by the uniform convergence of SV functions (e.g. BGT 1987, Theorem 1.5.2), for all $\eta > 0$ and $k \geq k_0$, as $n \rightarrow \infty$

$$\left| \frac{H\left(\gamma_n |a_k|^{-1}\right)}{H\left(\gamma_n k^{\frac{1}{\alpha}}\right)} - 1 \right| \leq \sup_{|y - \alpha^{-1}| \leq \delta} \left| \frac{H\left(\gamma_n k^{\frac{1}{\alpha}} y\right)}{H\left(\gamma_n k^{\frac{1}{\alpha}}\right)} - 1 \right| < \eta.$$

Hence, as $n \rightarrow \infty$ first and then as $\eta \rightarrow 0$

$$\begin{aligned}
|A_{3,n} - A_{4,n}| &:= \left| A_{3,n} - \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^\alpha H\left(\gamma_n k^{\frac{1}{\alpha}}\right) \right| \\
&\leq \sum_{k=k_0}^n \left| \frac{H\left(\gamma_n |a_k|^{-1}\right)}{H\left(\gamma_n k^{\frac{1}{\alpha}}\right)} - 1 \right| \gamma_n^{-\alpha} |a_k|^\alpha H\left(\gamma_n k^{\frac{1}{\alpha}}\right) \leq \eta A_{4,n} \xrightarrow{\eta \rightarrow 0} 0,
\end{aligned}$$

where the last approximation follows from the fact that $A_{4,n}$ is bounded as demonstrated above -cf. (68) and (80). The preceding arguments show that

$$A_{1,n} = A_{5,n} + o(1) = \alpha^{\alpha+1} + o(1), \tag{81}$$

and hence $A_{1,n}$ is bounded. Identical arguments also show that $A_{2,n} = \alpha^{\alpha+1} + o(1)$. Therefore, we have shown $\sup_{1 \leq t \leq n} A_{n,t} < \infty$.

Next, consider $B_{n,t}$. Fix $\eta > 0$ and choose t_0 large enough such that $|c_t t^{(1+\frac{1}{\alpha})} - 1| < \eta$ for $t \geq t_0$. Write

$$\begin{aligned}
B_{n,t} &= \mathbf{E} \left| \gamma_n^{-1} \sum_{k=t}^{\infty} a_{k,t}^- \xi_k \right|^{\alpha-\varepsilon} \\
&\leq 2 \left\{ \mathbf{E} \left| \gamma_n^{-1} \sum_{k=t}^{t_0+t-1} a_{k,t}^- \xi_k \right|^{\alpha-\varepsilon} + \mathbf{E} \left| \gamma_n^{-1} \sum_{k=t_0+t}^{\infty} a_{k,t}^- \xi_k \right|^{\alpha-\varepsilon} \right\} := 2 (B'_{n,t} + B''_{n,t}).
\end{aligned}$$

Recall that $a_{k,t}^- = \sum_{j=0}^{t-1} c_{k-j}$ (with $k \geq t$), and $a_k = \sum_{j=0}^k c_{k-j}$. As $n \rightarrow \infty$, the first term

$$\begin{aligned}
B'_{n,t} &\leq \mathbf{E} \left| \gamma_n^{-1} \sum_{k=t}^{t_0+t-1} a_{k,t}^- \xi_k \right|^{\alpha-\varepsilon} \leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=t}^{t_0+t-1} |a_{k,t}^-|^{\alpha-\varepsilon} \\
&\leq \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=t}^{t_0+t} \left(\sum_{j=0}^t |c_{k-j}| \right)^{\alpha-\varepsilon} = \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=t}^{t_0+t} \left(\sum_{j=k-t}^k |c_j| \right)^{\alpha-\varepsilon} \\
&\leq t_0 \gamma_n^{-(\alpha-\varepsilon)} \left(\sum_{j=0}^{\infty} |c_j| \right)^{\alpha-\varepsilon} = o(1),
\end{aligned} \tag{82}$$

uniformly in $1 \leq t \leq n$.

Next, we consider $B''_{n,t}$. Recall that $t_0 \in \mathbb{N}$. First, we shall show that as $n \rightarrow \infty$

$$\sup_{1 \leq t \leq t_0} B''_{n,t} = O(1), \quad (83)$$

and then

$$\sup_{t_0 \leq t \leq n} B''_{n,t} = O(1). \quad (84)$$

We start with the proof of (83). For $t \leq t_0$, by the Loève or the Bahr-Esseen inequality and Fatou's Lemma

$$\begin{aligned} B''_{n,t} &= \gamma_n^{-(\alpha-\varepsilon)} \mathbf{E} \lim_{M \rightarrow \infty} \left| \sum_{k=t_0+t}^M \left(\sum_{j=0}^{t-1} c_{k-j} \right) \xi_k \right|^{\alpha-\varepsilon} \\ &\leq \gamma_n^{-(\alpha-\varepsilon)} \liminf_{M \rightarrow \infty} \mathbf{E} \left| \sum_{k=t_0+t}^M \left(\sum_{j=0}^{t-1} c_{k-j} \right) \xi_k \right|^{\alpha-\varepsilon} \\ &\leq \gamma_n^{-(\alpha-\varepsilon)} \liminf_{M \rightarrow \infty} \sum_{k=t_0+t}^M \left| \sum_{j=0}^{t-1} c_{k-j} \right|^{\alpha-\varepsilon} \mathbf{E} |\xi_k|^{\alpha-\varepsilon} \\ &\leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=t_0+t}^{\infty} \left| \sum_{j=0}^{t-1} c_{k-j} \right|^{\alpha-\varepsilon} \\ &\leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^{\infty} \sum_{j=0}^{t_0} |c_{k-j}|^{\alpha-\varepsilon} \leq C t_0 \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^{\infty} |c_k|^{\alpha-\varepsilon} = o(1), \end{aligned}$$

which establishes (83).

Next, we show (84). Write¹⁵

$$B''_{n,t} = \frac{\gamma_t}{\gamma_n} \mathbf{E} \left| \gamma_t^{-1} \sum_{k=t_0+t}^{\infty} a_{k,t}^- \xi_k \right|^{\alpha-\varepsilon} =: \frac{\gamma_t}{\gamma_n} C_t.$$

We shall show C_t is bounded. Furthermore, without loss of generality we assume that γ_t is increasing. This implies that γ_t/γ_n is bounded. Notice that

$$\gamma_t \sim L(\gamma_t t^{1/\alpha})^{1/\alpha},$$

where L is increasing and $\gamma_t t^{1/\alpha}$ is regularly varying of positive index which has an increasing asymptotic equivalent (e.g. Theorem 1.5.3 in BGT, 1987). Set $T_{k,t} := \gamma_t |a_{k,t}^-|^{-1}$. Note that as $t \rightarrow \infty$

$$\begin{aligned} T_t &:= \inf_{t \leq k} T_{k,t} = \frac{\gamma_t}{\sup_{t \leq k} |a_{k,t}^-|} \\ &= \frac{\gamma_t}{\sup_{t \leq k} \left| \sum_{j=0}^{t-1} c_{k-j} \right|} \geq \frac{\gamma_t}{\sum_{j=0}^{\infty} |c_j|} \rightarrow \infty. \end{aligned} \quad (85)$$

¹⁵Note that due to **Assumption INN**,

$$\mathbf{E} \left| \gamma_t^{-1} \sum_{k=t_0+t}^{\infty} a_{k,t}^- \xi_{t-k} \right|^{\alpha-\varepsilon} = \mathbf{E} \left| \gamma_t^{-1} \sum_{k=t_0+t}^{\infty} a_{k,t}^- \xi_k \right|^{\alpha-\varepsilon}.$$

Hence, as $t \rightarrow \infty$, $T_{k,t} \rightarrow \infty$, uniformly in $k \geq t$. Furthermore, set

$$\bar{\xi}_{k,t} := \xi_k 1\{|\xi_k| \geq T_{k,t}\} \text{ and } \underline{\xi}_{k,t} := \xi_k 1\{|\xi_k| < T_{k,t}\}.$$

In view of the above and Lemma 1 in Astrauskas (1983), for $k \geq t$ and $\alpha \in (0, 2]$ as $t \rightarrow \infty$ (and hence $T_{k,t} \rightarrow \infty$) we have

$$\mathbf{E} \left| \bar{\xi}_{k,t} \right|^{\alpha-\varepsilon} \asymp T_{k,t}^{-\varepsilon} h(T_{k,t}) \text{ and } \mathbf{E} \left| \underline{\xi}_{k,t} \right|^{\alpha+\varepsilon} \asymp T_{k,t}^{\varepsilon} h(T_{k,t}). \quad (86)$$

Set $\bar{a}_k := \sum_{j=0}^k |c_{k-j}|$. As before, without loss of generality, assume that $\mathbf{E} \bar{\xi}_{k,t} = \mathbf{E} \underline{\xi}_{k,t} = 0$. Consider

$$\begin{aligned} 2^{-1} C_t &\leq \mathbf{E} \left| \gamma_t^{-\alpha} \sum_{k=t+t_0}^{\infty} a_{k,t}^- \bar{\xi}_{k,t} \right|^{\alpha-\varepsilon} + \left[\mathbf{E} \left| \gamma_t^{-\alpha} \sum_{k=t+t_0}^{\infty} a_{k,t}^- \underline{\xi}_{k,t} \right|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\ &\leq_{(2)} \gamma_t^{-(\alpha-\varepsilon)} \sum_{k=t+t_0}^{\infty} |a_{k,t}^-|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{k,t}|^{\alpha-\varepsilon} + \left[\gamma_t^{-(\alpha+\varepsilon)} \sum_{k=t+t_0}^{\infty} |a_{k,t}^-|^{\alpha+\varepsilon} \mathbf{E} |\underline{\xi}_{k,t}|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\ &=: C_{1,t} + C_{2,t}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}. \end{aligned}$$

In view of (75) and (86), as $t \rightarrow \infty$, the first term

$$\begin{aligned} C_{1,t} &= \gamma_t^{-(\alpha-\varepsilon)} \sum_{k=t+t_0}^{\infty} |a_{k,t}^-|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{k,t}|^{\alpha-\varepsilon} = C \gamma_t^{-(\alpha-\varepsilon)} \sum_{k=t+t_0}^{\infty} |a_{k,t}^-|^{\alpha-\varepsilon} T_{k,t}^{-\varepsilon} h(T_{k,t}) [1 + o(1)] \\ &\leq C \sum_{k=t+t_0}^{\infty} |\gamma_t^{-1} a_{k,t}^-|^{\alpha-\varepsilon} \left(\gamma_t^{-1} |a_{k,t}^-| \right)^{\varepsilon} H \left(\gamma_t |a_{k,t}^-|^{-1} \right) [1 + o(1)] =: C \cdot C_{3,t} [1 + o(1)]. \end{aligned}$$

Similarly, as $t \rightarrow \infty$

$$C_{2,t} = C \cdot C_{3,t} [1 + o(1)].$$

Write

$$C_{3,t} = \sum_{k=t+t_0}^{\infty} |\gamma_t^{-1} a_{k,t}^-|^{\alpha} H \left(\gamma_t |a_{k,t}^-|^{-1} \right) = \sum_{k=t+t_0}^{\infty} \left| \gamma_t^{-1} \sum_{j=0}^t c_{k-j} \right|^{\alpha} H \left(\gamma_t \left| \sum_{j=0}^t c_{k-j} \right|^{-1} \right).$$

Let $0 < \delta_1 \leq 1 < \delta_2 \leq 2$, and $1 \geq \delta := \delta_2 - \delta_1$. In addition, $t' := t + t_0$. For the purposes of part (b), we consider the following generalization of $C_{3,t}$

$$\begin{aligned} C_{3,t}^* &:= \sum_{k=\delta_1 t+t_0}^{\infty} \left| \gamma_t^{-1} \sum_{j=0}^{\delta_1 t} c_{k-j} \right|^{\alpha} H \left(\gamma_t \left| \sum_{j=0}^{\delta_1 t} c_{k-j} \right|^{-1} \right) \\ &= \left(\sum_{k=\delta_1 t+t_0}^{\delta_2 t'} + \sum_{k=\delta_2 t'+1}^{\infty} \right) \left| \gamma_t^{-1} \sum_{j=0}^{\delta_1 t} c_{k-j} \right|^{\alpha} H \left(\gamma_t \left| \sum_{j=0}^{\delta_1 t} c_{k-j} \right|^{-1} \right) \\ &= \sum_{k=t_0}^{\delta t + \delta_2 t_0} \left| \gamma_t^{-1} \sum_{j=k}^{k+\delta_1 t} c_j \right|^{\alpha} H \left(\gamma_t \left| \sum_{j=k}^{k+\delta_1 t} c_j \right|^{-1} \right) \end{aligned}$$

$$+ \sum_{k=\delta_2 t'+1}^{\infty} \left| \gamma_t^{-1} \sum_{j=k-\delta_1 t}^k c_j \right|^\alpha H \left(\gamma_t \left| \sum_{j=k-\delta_1 t}^k c_j \right|^{-1} \right) =: C'_{3,t} + C''_{3,t}. \quad (87)$$

Define

$$Q_t(k) := \int_k^{k+\delta_1 t} x^{-(1+\frac{1}{\alpha})} dx = \alpha \left[k^{-1/\alpha} - (k + \delta_1 t)^{-1/\alpha} \right]$$

and

$$I_t(k) := \int_{k-\delta_1 t}^k x^{-(1+\frac{1}{\alpha})} dx = \alpha \left[(k - \delta_1 t)^{-1/\alpha} - k^{-1/\alpha} \right], \quad k > \delta_1 t.$$

Recall that $R_{-\alpha}(x) = x^{-\alpha} H(x)$, $x > 0$ and let $\underline{R}_{-\alpha}(x)$ be decreasing such that $R_{-\alpha}(x) \sim \underline{R}_{-\alpha}(x)$, see e.g. BGT 1987 Thm1.5.3. Let $\eta \in (0, 1)$ and choose t_0 large enough such that

$$\left| \frac{Q_t(k)}{\sum_{j=k}^{k+\delta_1 t} c_j} - 1 \right| < \eta,$$

for all $t_0 \leq k \leq \delta t + \delta_2 t_0$, which is possible due to **Lemma 6(aii)**. Using **Lemma 6(b)** and the uniform convergence of regularly functions (e.g. BGT 1987; Thm 1.5.2) we have as $t \rightarrow \infty$

$$\begin{aligned} & \left| C'_{3,t} - \sum_{k=t_0}^{\delta t + \delta_2 t_0} \left| \gamma_t^{-1} \sum_{j=k}^{k+\delta_1 t} c_j \right|^{-\alpha} H \left(\gamma_t Q_t(k)^{-1} \right) \right| \\ & \leq \sum_{k=t_0}^{\delta t + \delta_2 t_0} \left| \frac{R_{-\alpha} \left(\gamma_t \left| \sum_{j=k}^{k+\delta_1 t} c_j \right|^{-1} \right)}{R_{-\alpha} \left(\gamma_t Q_t(k)^{-1} \right)} - \left[\frac{\gamma_t \left| \sum_{j=k}^{k+\delta_1 t} c_j \right|^{-1}}{\gamma_t Q_t(k)^{-1}} \right]^{-\alpha} \right| R_{-\alpha} \left(\gamma_t Q_t(k)^{-1} \right) \\ & \leq \sup_{1-\eta < y < 1+\eta} \left| \frac{R_{-\alpha} (y \gamma_t Q_t(k)^{-1})}{R_{-\alpha} (\gamma_t Q_t(k)^{-1})} - y^{-\alpha} \right| \sum_{k=t_0}^{\delta t + \delta_2 t_0} R_{-\alpha} \left(\gamma_t Q_t(k)^{-1} \right) \\ & = o(1) \cdot \sum_{k=t_0}^{\delta t + \delta_2 t_0} R_{-\alpha} \left(\gamma_t Q_t(k)^{-1} \right). \end{aligned} \quad (88)$$

Similarly, using **Lemma 6(aii,b)**, for t_0 large enough and as $t \rightarrow \infty$ we get

$$\begin{aligned} & \left| \sum_{k=t_0}^{\delta t + \delta_2 t_0} \left| \gamma_t^{-1} \sum_{j=k}^{k+\delta_1 t} c_j \right|^{-\alpha} H \left(\gamma_t Q_t(k)^{-1} \right) - \sum_{k=t_0}^{\delta t + \delta_2 t_0} R_{-\alpha} \left(\gamma_t Q_t(k)^{-1} \right) \right| \\ & \leq \sum_{k=t_0}^{\delta t + \delta_2 t_0} \left| \left(\frac{\gamma_t \left| \sum_{j=k}^{k+\delta_1 t} c_j \right|^{-1}}{\gamma_t Q_t(k)^{-1}} \right)^{-\alpha} - 1 \right| R_{-\alpha} \left(\gamma_t Q_t(k)^{-1} \right) \\ & \leq o(1) \cdot \sum_{k=t_0}^{\delta t + \delta_2 t_0} R_{-\alpha} \left(\gamma_t Q_t(k)^{-1} \right). \end{aligned} \quad (89)$$

Combining (88)-(89),

$$\left| C'_{3,t} - \sum_{k=t_0}^{\delta t + \delta_2 t_0} R_{-\alpha} \left(\gamma_t Q_t(k)^{-1} \right) \right| \leq o(1) \cdot \sum_{k=t_0}^{\delta t + \delta_2 t_0} R_{-\alpha} \left(\gamma_t Q_t(k)^{-1} \right),$$

for t_0 large enough and as $t \rightarrow \infty$. Furthermore, using the fact that $R_{-\alpha}(x) \sim \underline{R}_{-\alpha}(x)$ and **Lemma 6(b)**, we get as $t \rightarrow \infty$

$$\left| \sum_{k=t_0}^{\delta t + \delta_2 t_0} R_{-\alpha}(\gamma_t Q_t(k)^{-1}) - \sum_{k=t_0}^{\delta t + \delta_2 t_0} \underline{R}_{-\alpha}(\gamma_t Q_t(k)^{-1}) \right| \leq o(1) \cdot \sum_{k=t_0}^{\delta t + \delta_2 t_0} \underline{R}_{-\alpha}(\gamma_t Q_t(k)^{-1}),$$

which in turn implies that for t_0 large enough,

$$\left| C'_{3,t} - \sum_{k=t_0}^{\delta t + \delta_2 t_0} R_{-\alpha}(\gamma_t Q_t(k)^{-1}) \right| \leq o(1) \cdot \sum_{k=t_0}^{\delta t + \delta_2 t_0} \underline{R}_{-\alpha}(\gamma_t Q_t(k)^{-1}), \quad (90)$$

where the $o(1)$ term is negligible as $t \rightarrow \infty$. Set $t' := t + t_0$. Using **Lemma 6(ai,b)** and a similar argument to the one that yields (90) we also get for t_0 large enough

$$\left| C''_{3,t} - \sum_{k=\delta_2 t' + 1}^{\infty} R_{-\alpha}(\gamma_t I_t(k)^{-1}) \right| \leq o(1) \cdot \sum_{k=\delta_2 t' + 1}^{\infty} \underline{R}_{-\alpha}(\gamma_t I_t(k)^{-1}). \quad (91)$$

Next, we show that

$$\sum_{k=t_0}^{\delta t + \delta_2 t_0} \underline{R}_{-\alpha}(\gamma_t Q_t(k)^{-1}) = O(1), \quad (92)$$

and

$$\sum_{k=\delta_2 t' + 1}^{\infty} \underline{R}_{-\alpha}(\gamma_t I_t(k)^{-1}) = O(1), \quad (93)$$

i.e. the l.h.s. terms of (90) and (91) are $O(1)$. Noting that $\underline{R}_{-\alpha}(\gamma_t Q_t(k)^{-1})$, $\underline{R}_{-\alpha}(\gamma_t I_t(k)^{-1})$ are decreasing¹⁶ in k , Euler summation gives as $t \rightarrow \infty$

$$\begin{aligned} \sum_{k=t_0}^{\delta t + \delta_2 t_0} \underline{R}_{-\alpha}(\gamma_t Q_t(k)^{-1}) &= \int_{t_0}^{\delta t + \delta_2 t_0} \underline{R}_{-\alpha}(\gamma_t Q_t(y)^{-1}) dy + O(\underline{R}_{-\alpha}(\gamma_t Q_t(t_0)^{-1})) \\ &= \int_{t_0}^{\delta t + \delta_2 t_0} \underline{R}_{-\alpha}(\gamma_t Q_t(y)^{-1}) dy + o(1) \\ &\stackrel{(2)}{=} \int_{t_0}^{\delta t + \delta_2 t_0} R_{-\alpha}(\gamma_t Q_t(y)^{-1}) dy [1 + o(1)] + o(1), \end{aligned} \quad (94)$$

where $\stackrel{(2)}{=}$ follows from the fact that $R_{-\alpha}(x) \sim \underline{R}_{-\alpha}(x)$. Similarly, as $t \rightarrow \infty$

$$\begin{aligned} \sum_{k=\delta_2 t' + 1}^{\infty} \underline{R}_{-\alpha}(\gamma_t I_{k,t}^{-1}) &= \int_{\delta_2 t' + 1}^{\infty} \underline{R}_{-\alpha}(\gamma_t I_t(y)^{-1}) dy + O(\underline{R}_{-\alpha}(\gamma_t I_t(\delta_2 t' + 1)^{-1})) \\ &= \int_{\delta_2 t' + 1}^{\infty} \underline{R}_{-\alpha}(\gamma_t I_t(y)^{-1}) dy + o(1) \\ &\quad [x = y - \delta_1 t \Leftrightarrow y = x + \delta_1 t] \\ &= \int_{\delta t + \delta_2 t_0 + 1}^{\infty} \underline{R}_{-\alpha}(\gamma_t Q_t(x)^{-1}) dx + o(1) \\ &= \int_{\delta t + \delta_2 t_0 + 1}^{\infty} R_{-\alpha}(\gamma_t Q_t(x)^{-1}) dx [1 + o(1)] + o(1). \end{aligned} \quad (95)$$

¹⁶See the last footnote in the proof of **Lemma 6**.

In view of (94) and (95), (92) and (93) will follow if we show that as $t \rightarrow \infty$

$$\int_{t_0}^{\infty} R_{-\alpha} \left(\gamma_t Q_t(x)^{-1} \right) dx = O(1). \quad (96)$$

In fact, the above together with (88)-(91) and (94)-(95) imply that as $t \rightarrow \infty$,

$$C'_{3,t} = \alpha^{\alpha+1} + o(1) \text{ and } C''_{3,t} = o(1). \quad (97)$$

We proceed with the proof of (96). We start with the latter. Consider

$$\begin{aligned} & \int_{t_0}^{\infty} R_{-\alpha} \left(\gamma_t Q_t(x)^{-1} \right) dx \\ &= \int_{t_0}^{\infty} \left\{ \gamma_t \alpha^{-1} \left[u^{-1/\alpha} - (u + \delta_1 t)^{-1/\alpha} \right]^{-1} \right\}^{-\alpha} H \left(\gamma_t \alpha^{-1} \left[u^{-1/\alpha} - (u + \delta_1 t)^{-1/\alpha} \right]^{-1} \right) du \\ &= \int_{t_0}^{\infty} \left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} \left[\left(\frac{u}{\delta_1 t} \right)^{-1/\alpha} - \left(\frac{u}{\delta_1 t} + 1 \right)^{-1/\alpha} \right]^{-1} \right\}^{-\alpha} \\ &\quad \times H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} \left[\left(\frac{u}{\delta_1 t} \right)^{-1/\alpha} - \left(\frac{u}{\delta_1 t} + 1 \right)^{-1/\alpha} \right]^{-1} \right) du \quad \left[z = \frac{u}{\delta_1 t} \rightarrow du = \delta_1 t dz \right] \\ &= \delta_1 t \int_{z=t_0/\delta_1 t}^{\infty} \left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} \left[z^{-1/\alpha} - (z+1)^{-1/\alpha} \right]^{-1} \right\}^{-\alpha} \\ &\quad \times H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} \left[z^{-1/\alpha} - (z+1)^{-1/\alpha} \right]^{-1} \right) dz \\ &=: \delta_1 t \int_{z=t_0/\delta_1 t}^{\infty} \left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right\}^{-\alpha} H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right) dz \\ &= \delta_1 t \int_{z=t_0/\delta_1 t}^{\infty} R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right) dz, \end{aligned}$$

where $g(z) := \left[z^{-1/\alpha} - (z+1)^{-1/\alpha} \right]^{-1}$. Now

$$\begin{aligned} \frac{g(z)}{z^{1/\alpha}} &= z^{-1/\alpha} \left[z^{-1/\alpha} - (z+1)^{-1/\alpha} \right]^{-1} \\ &= \left[1 - (1+1/z)^{-1/\alpha} \right]^{-1} = \frac{1}{1 - \frac{1}{(1+1/z)^{1/\alpha}}} > 1, \end{aligned} \quad (98)$$

for all $z > 0$. Further, it can be easily seen from the above that $g(z)/z^{1/\alpha} \rightarrow 1$ as $z \rightarrow 0$. Hence,

$$\left| \left\{ g(z)/z^{1/\alpha} \right\}^{-\alpha} - 1 \right| < \varepsilon, \text{ for } 0 < z < \delta_\varepsilon. \quad (99)$$

Without loss of generality choose $\delta_\varepsilon < \delta/\delta_1$. Using similar arguments¹⁷ as those used for (68) we have

$$\delta_1 t \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) dz = \alpha^{\alpha+1} + o(1), \quad (100)$$

¹⁷Note that

$$\begin{aligned} & t \int_{z=t_0/t}^{\delta_\varepsilon} R_{-\alpha} \left(\gamma_t \alpha^{-1} t^{1/\alpha} z^{1/\alpha} \right) dz \\ & \left[u = \gamma_t \alpha^{-1} t^{1/\alpha} z^{1/\alpha} \Rightarrow z = \left(\gamma_t \alpha^{-1} t^{1/\alpha} \right)^{-\alpha} u^\alpha \Rightarrow dz = \alpha \left(\gamma_t \alpha^{-1} t^{1/\alpha} \right)^{-\alpha} u^{\alpha-1} du \right] \end{aligned}$$

as $t \rightarrow \infty$. In view of (100), for $\eta > 0$ arbitrarily small and t large enough, the difference,

$$\begin{aligned}
& \delta_1 t \left| \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right) dz - \left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right\}^{-\alpha} H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) dz \right| \\
&= \delta_1 t \left| \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} \left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right\}^{-\alpha} H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right) \right. \\
&\quad \left. - \left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right\}^{-\alpha} H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) dz \right| \\
&\leq \delta_1 t \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} \left| \left[\frac{\left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right\}^{-\alpha} H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right)}{\left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right\}^{-\alpha} H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right)} - \frac{\left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right\}^{-\alpha}}{\left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right\}^{-\alpha}} \right] \right. \\
&\quad \left. \times \left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right\}^{-\alpha} H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) \right| dz \\
&= \delta_1 t \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} \left| \left[\frac{R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \frac{g(z)}{z^{1/\alpha}} \right)}{R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right)} - \frac{\{g(z)\}^{-\alpha}}{\{z^{1/\alpha}\}^{-\alpha}} \right] R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) \right| dz \\
&\leq \delta_1 t \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} \sup_{1 \leq y < \infty} \left| \frac{R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} y \right)}{R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right)} - y^{-\alpha} \right| R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) dz \\
&\leq_{(3)} \eta \delta_1 t \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) dz \leq C \eta, \tag{101}
\end{aligned}$$

where $\leq_{(3)}$ follows from (98), the fact that

$$\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} \inf_{t_0/\delta_1 t \leq z < \infty} z^{1/\alpha} \rightarrow \infty,$$

and the uniform convergence of regularly varying functions -see e.g. BGT 1987, Thm 1.5.2. Next, for $\varepsilon > 0$ arbitrarily small and t large enough, the difference

$$\begin{aligned}
& \delta_1 t \left| \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} \left[\left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right\}^{-\alpha} H \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) - R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) \right] dz \right| \\
&\leq \delta_1 t \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} \left| \frac{\left\{ \gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right\}^{-\alpha}}{\left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right)^{-\alpha}} - 1 \right| R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) dz \\
&= \delta_1 t \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} \left| \frac{\{g(z)\}^{-\alpha}}{(z^{1/\alpha})^{-\alpha}} - 1 \right| R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) dz \\
&\quad = \alpha \left(\gamma_t \alpha^{-1} t^{1/\alpha} \right)^{-\alpha} t \int_{u=\gamma_t \alpha^{-1} t^{1/\alpha} (t_0/t)^{1/\alpha}}^{\gamma_t \alpha^{-1} t^{1/\alpha} \delta_\varepsilon^{1/\alpha}} R_{-\alpha}(u) u^{\alpha-1} du \\
&\quad = \alpha \gamma_t^{-\alpha} \alpha^\alpha \int_{u=\gamma_t \alpha^{-1} t^{1/\alpha} (t_0/t)^{1/\alpha}}^{\gamma_t \alpha^{-1} t^{1/\alpha} \delta_\varepsilon^{1/\alpha}} H(u) u^{-1} du \\
&\quad = \gamma_t^{-\alpha} \alpha^{\alpha+1} \int_{u=\gamma_t \alpha^{-1} t_0^{1/\alpha}}^{\gamma_t \alpha^{-1} t^{1/\alpha} \delta_\varepsilon^{1/\alpha}} H(u) u^{-1} du = \alpha^{\alpha+1} + o(1).
\end{aligned}$$

$$\leq \varepsilon \delta_1 t \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} z^{1/\alpha} \right) dz, \quad (102)$$

where the last inequality follows from (99). In view of (100)-(102), as $t \rightarrow \infty$ we get

$$\begin{aligned} & \int_{t_0}^{\delta_\varepsilon t} R_{-\alpha} \left(\gamma_t Q_t(x)^{-1} \right) dx \\ &= \delta_1 t \int_{z=t_0/\delta_1 t}^{\delta_\varepsilon} R_{-\alpha} \left(\gamma_t \alpha^{-1} (t \delta_1)^{1/\alpha} g(z) \right) dz = O(1). \end{aligned} \quad (103)$$

Next, we show that

$$\int_{\delta_\varepsilon t}^{\infty} R_{-\alpha} \left(\gamma_t Q_t(x)^{-1} \right) dx = o(1), \quad (104)$$

as $t \rightarrow \infty$. Note that (96) follows immediately from (103) and (104).

There is no loss of generality assuming that $R_{-\alpha}(x)$ is decreasing -see the arguments that lead to (95) and Theorem 1.5.3 BGT 1987. Using the mean value theorem

$$z^{-1/\alpha} - (z+1)^{-1/\alpha} = \alpha^{-1} (z + \bar{z})^{-1-\frac{1}{\alpha}} = \alpha^{-1} (z + \bar{z})^{-\frac{1+\alpha}{\alpha}},$$

where $0 \leq \bar{z} \leq 1$. Hence, $g(z) \geq \alpha z^{\frac{1+\alpha}{\alpha}}$. In view of this and setting $\bar{\delta} := \alpha \delta_1^{\frac{1}{\alpha}} (\delta_\varepsilon / \delta_1)^{\frac{1+\alpha}{\alpha}}$ we get as $t \rightarrow \infty$,

$$\begin{aligned} & \int_{\delta_\varepsilon t}^{\infty} R_{-\alpha} \left(\gamma_t Q_t(x)^{-1} \right) dx = \delta_1 t \int_{\delta_\varepsilon / \delta_1}^{\infty} R_{-\alpha} \left(\gamma_t (t \delta_1)^{1/\alpha} g(z) \right) dz \\ & \leq \delta_1 t \int_{\delta_\varepsilon / \delta_1}^{\infty} R_{-\alpha} \left(\alpha \gamma_t (t \delta_1)^{1/\alpha} z^{\frac{1+\alpha}{\alpha}} \right) dz \\ & \left[\begin{aligned} u &= \alpha \gamma_t (t \delta_1)^{1/\alpha} z^{\frac{1+\alpha}{\alpha}} \rightarrow \left(\alpha^{-1} \gamma_t^{-1} (t \delta_1)^{-1/\alpha} u \right)^{\frac{\alpha}{1+\alpha}} = z \\ &\rightarrow \frac{\alpha}{1+\alpha} \left(\alpha^{-1} \gamma_t^{-1} (t \delta_1)^{-1/\alpha} \right)^{\frac{\alpha}{1+\alpha}} u^{-\frac{1}{\alpha+1}} = dz/du \end{aligned} \right] \\ &= \alpha^{-\frac{\alpha}{1+\alpha}} \frac{\alpha}{1+\alpha} \delta_1 t \left(\gamma_t^{-1} (t \delta_1)^{-1/\alpha} \right)^{\frac{\alpha}{1+\alpha}} \int_{\bar{\delta} \gamma_t t^{1/\alpha}}^{\infty} u^{-\frac{1}{\alpha+1}} R_{-\alpha}(u) du \\ &= \alpha^{-\frac{\alpha}{1+\alpha}} \frac{\alpha}{1+\alpha} \delta_1 t \gamma_t^{-\frac{\alpha}{1+\alpha}} (t \delta_1)^{-\frac{1}{1+\alpha}} \int_{\bar{\delta} \gamma_t t^{1/\alpha}}^{\infty} u^{-\left(\frac{\alpha^2}{\alpha+1}+1\right)} H(u) du \\ &\sim_{(4)} \alpha^{-\frac{\alpha}{1+\alpha}} \frac{\alpha}{1+\alpha} \frac{\alpha+1}{\alpha^2} \delta_1 t \gamma_t^{-\frac{\alpha}{1+\alpha}} (t \delta_1)^{-\frac{1}{1+\alpha}} \left(\bar{\delta} \gamma_t t^{1/\alpha} \right)^{-\left(\frac{\alpha^2}{\alpha+1}\right)} H\left(\bar{\delta} \gamma_t t^{1/\alpha}\right) \\ &= C t \gamma_t^{-\alpha} t^{-\frac{1}{1+\alpha}} t^{-\left(\frac{\alpha}{\alpha+1}\right)} H\left(\bar{\delta} \gamma_t t^{1/\alpha}\right) = C \gamma_t^{-\alpha} H\left(\bar{\delta} \gamma_t t^{1/\alpha}\right) \stackrel{(5)}{=} o(1), \end{aligned}$$

where $\sim_{(4)}$ and $\stackrel{(5)}{=}$ follow from BGT 1987 Proposition 1.5.10 and 1.5.9a respectively.

BFR2. Let $\varepsilon \in (0, \alpha)$. Consider the term

$$\begin{aligned} & \mathbf{E} \left| \gamma_n^{-1} x_t \right|^{\alpha-\varepsilon} = \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{\infty} a_{k,t} \xi_{t-k} \right|^{\alpha-\varepsilon} \\ & \leq 2 \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{t-1} a_k \xi_{t-k} \right|^{\alpha-\varepsilon} + 2 \mathbf{E} \left| \gamma_n^{-1} \sum_{k=t}^{\infty} a_{k,t} \xi_{t-k} \right|^{\alpha-\varepsilon} \\ & =: 2 [A_{n,t} + B_{n,t}]. \end{aligned}$$

Recall that $a_k = \sum_{j=0}^k \phi_j c_{k-j}$. Set $S_{n,k} := \gamma_n |a_k|^{-1}$, and note that

$$|a_k| \leq \sup_{j \geq 0} |\phi_j| \sum_{j=0}^{\infty} |c_j| =: \epsilon < \infty.$$

Hence,

$$\inf_{0 \leq k \leq n} S_{n,k} = \gamma_n \inf_{0 \leq k \leq n} |a_k|^{-1} = \gamma_n \frac{1}{\sup_{0 \leq k \leq n} |a_k|} \geq \gamma_n \epsilon^{-1} \rightarrow \infty. \quad (105)$$

Next, set

$$\bar{\xi}_{n,k} := \xi_k 1\{|\xi_k| \geq S_{n,k}\} \text{ and } \underline{\xi}_{n,k} := \xi_k 1\{|\xi_k| < S_{n,k}\}.$$

As before, and without loss of generality we assume that $\mathbf{E} \bar{\xi}_{n,k} = 0$ and $\mathbf{E} \underline{\xi}_{n,k} = 0$, when $\alpha \in (1, 2]$. Let k_0 be some fixed integer. As $n \rightarrow \infty$, we have

$$\begin{aligned} A_{n,t} &= \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{t-1} a_k \xi_k \right|^{\alpha-\varepsilon} \leq 2\mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{k_0-1} a_k \xi_k \right|^{\alpha-\varepsilon} + 2\mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \xi_k \right|^{\alpha-\varepsilon} \\ &\leq 2\gamma_n^{-1} \sum_{k=0}^{k_0-1} |a_k|^{\alpha-\varepsilon} \mathbf{E} |\xi_k|^{\alpha-\varepsilon} + 2\mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \xi_k \right|^{\alpha-\varepsilon} \\ &\leq 4\mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \bar{\xi}_{n,k} \right|^{\alpha-\varepsilon} + 4 \left\{ \mathbf{E} \left| \gamma_n^{-1} \sum_{k=k_0}^{t-1} a_k \underline{\xi}_{n,k} \right|^{\alpha+\varepsilon} \right\}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} + o(1) \\ &\leq_{(1)} C \left\{ \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=k_0}^n |a_k|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} + \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=k_0}^n |a_k|^{\alpha+\varepsilon} \mathbf{E} |\underline{\xi}_{n,k}|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right\} + o(1) \\ &=: C \left\{ A_{1,n} + [A_{2,n}]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right\} + o(1). \end{aligned} \quad (106)$$

where $\leq_{(1)}$ follows either from Loève's or the Bahr-Esseen inequalities depending on the value of the exponents (see p. 133 in Astrauskas, 1983).

By Lemma 1 in Astrauskas (1983), for $\alpha \in (0, 2]$ as $S_{n,k} \rightarrow \infty$,

$$\mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} \asymp S_{n,k}^{-\varepsilon} h(S_{n,k}) \text{ and } \mathbf{E} |\underline{\xi}_{n,k}|^{\alpha+\varepsilon} \asymp S_{n,k}^{\varepsilon} h(S_{n,k}). \quad (107)$$

In view of (105) and (107) as $n \rightarrow \infty$

$$\begin{aligned} A_{1,n} &= C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=k_0}^n |a_k|^{\alpha-\varepsilon} S_{n,k}^{-\varepsilon} h(S_{n,k}) [1 + o(1)] \\ &= C \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^{\alpha} h(S_{n,k}) [1 + o(1)] \leq_{(2)} C \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^{\alpha} H(S_{n,k}) [1 + o(1)] \\ &= C \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^{\alpha} H(\gamma_n |a_k|^{-1}) [1 + o(1)] =: C \cdot A_{3,n} [1 + o(1)], \end{aligned} \quad (108)$$

where $\leq_{(2)}$ follows from (75). Similarly, as $n \rightarrow \infty$

$$A_{2,n} \leq C \cdot A_{3,n} [1 + o(1)]. \quad (109)$$

Without loss of generality set $c_\infty = 1$. Fix $\varepsilon \in (0, 1)$. By **Lemma 8**, and **Assumption BFR2** for k large enough, $k \geq k_0$ say, we have

$$\left| c_\infty k^{-\frac{1}{\alpha}} |a_k|^{-1} - 1 \right| \leq \varepsilon. \quad (110)$$

In view of this, and the uniform convergence of regularly varying functions (e.g. BGT 1987; Thm 1.5.2) as $n \rightarrow \infty$

$$\begin{aligned} & \sup_{k \geq k_0} \left| \frac{[|a_k|^{-1} k^{-1/\alpha}]^{-\alpha} H(\gamma_n |a_k|^{-1})}{H(\gamma_n k^{\frac{1}{\alpha}})} - [|a_k|^{-1} k^{-1/\alpha}]^{-\alpha} \right| \\ & \leq \sup_{k \geq k_0} \sup_{y \in [1-\varepsilon, \infty)} \left| \frac{(\gamma_n k^{\frac{1}{\alpha}} y)^{-\alpha} H(\gamma_n k^{\frac{1}{\alpha}} y)}{(\gamma_n k^{\frac{1}{\alpha}})^{-\alpha} H(\gamma_n k^{\frac{1}{\alpha}})} - y^{-\alpha} \right| = o(1). \end{aligned} \quad (111)$$

Hence,

$$\begin{aligned} & \left| \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^\alpha H(\gamma_n |a_k|^{-1}) - \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^\alpha H(\gamma_n k^{\frac{1}{\alpha}}) \right| \\ & \leq \sum_{k=k_0}^n \left| \frac{|a_k|^\alpha H(\gamma_n |a_k|^{-1})}{H(\gamma_n k^{\frac{1}{\alpha}})} - |a_k|^\alpha \right| \gamma_n^{-\alpha} H(\gamma_n k^{\frac{1}{\alpha}}) \\ & = \sum_{k=k_0}^n \left| \frac{|a_k|^\alpha [k^{-1/\alpha}]^{-\alpha} H(\gamma_n |a_k|^{-1})}{H(\gamma_n k^{\frac{1}{\alpha}})} - |a_k|^\alpha [k^{-1/\alpha}]^{-\alpha} \right| [\gamma_n k^{1/\alpha}]^{-\alpha} H(\gamma_n k^{\frac{1}{\alpha}}) \\ & \leq_{(3)} o(1) \cdot \sum_{k=k_0}^n \gamma_n^{-\alpha} k^{-1} H(\gamma_n k^{\frac{1}{\alpha}}) =_{(4)} o(1) \cdot (\alpha + o(1)) = o(1), \end{aligned} \quad (112)$$

where $\leq_{(3)}$ and $=_{(4)}$ follow from (111) and (68) respectively. Next, fix $\varepsilon > 0$ and let k large enough, $k \geq k_0$ say, such that $||a_k|^\alpha k - 1| < \varepsilon$, which is possible due to **Assumption BFR2**. As $n \rightarrow \infty$ first and then as $\varepsilon \rightarrow 0$,

$$\begin{aligned} & \left| \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^\alpha H(\gamma_n k^{\frac{1}{\alpha}}) - \gamma_n^{-\alpha} \sum_{k=k_0}^n k^{-1} H(\gamma_n k^{\frac{1}{\alpha}}) \right| \\ & \leq C \gamma_n^{-\alpha} \sum_{k=k_0}^n \left| \frac{|a_k|^\alpha}{k^{-1}} - 1 \right| k^{-1} H(\gamma_n k^{\frac{1}{\alpha}}) \leq C \varepsilon \gamma_n^{-\alpha} \sum_{k=k_0}^n k^{-1} H(\gamma_n k^{\frac{1}{\alpha}}). \end{aligned} \quad (113)$$

Therefore, combining (112) and (113) as $n \rightarrow \infty$ we get

$$A_{3,n} = \gamma_n^{-\alpha} \sum_{k=k_0}^n |a_k|^\alpha H(\gamma_n |a_k|^{-1}) \leq C \gamma_n^{-\alpha} \sum_{k=k_0}^n k^{-1} H(\gamma_n k^{1/\alpha}) + o(1). \quad (114)$$

Using (106), (108), (109), and (114) we get as $n \rightarrow \infty$

$$\begin{aligned} & \sup_{1 \leq t \leq n} A_{n,t} \leq o(1) + C \gamma_n^{-\alpha} \sum_{k=k_0}^n k^{-1} H(\gamma_n k^{\frac{1}{\alpha}}) [1 + o(1)] \\ & + C \left\{ \gamma_n^{-\alpha} \sum_{k=k_0}^n k^{-1} H(\gamma_n k^{\frac{1}{\alpha}}) [1 + o(1)] \right\}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \end{aligned}$$

$$=_{(5)} C \left(\alpha + \alpha^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right) + o(1),$$

where $=_{(5)}$ follows from (68).

Next, we show that $B_{n,t}$ is uniformly bounded for $1 \leq t \leq n$. Write

$$B_{n,t} = \mathbf{E} \left| \gamma_n^{-\alpha} \sum_{k=t}^{\infty} a_{k,t} \xi_{t-k} \right|^{\alpha-\varepsilon} = \mathbf{E} \left| \gamma_n^{-\alpha} \sum_{k=t}^{\infty} a_{k,t} \xi_k \right|^{\alpha-\varepsilon}.$$

Recall that $a_{k,t} = \sum_{j=0}^{(k-1) \wedge t} \phi_j c_{k-j}$ and $a_k = \sum_{j=0}^k \phi_j c_{k-j}$. Furthermore, set $\bar{a}_k = \sum_{j=0}^k |\phi_j c_{k-j}|$, and note that $|a_k|, |a_{k,t}| \leq \bar{a}_k$. Consider

$$\begin{aligned} \mathbf{E} \left| \gamma_n^{-\alpha} \sum_{k=t}^{\infty} a_{k,t} \xi_k \right|^{\alpha-\varepsilon} &\leq 2 \mathbf{E} \left| \gamma_n^{-\alpha} \sum_{k=t}^{\infty} a_{k,t} \bar{\xi}_{n,k} \right|^{\alpha-\varepsilon} + 2 \left[\mathbf{E} \left| \gamma_n^{-\alpha} \sum_{k=t}^{\infty} a_{k,t} \xi_{n,k} \right|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\ &\leq_{(6)} 2 \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=t}^{\infty} |a_{k,t}|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} + 2 \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=t}^{\infty} |a_{k,t}|^{\alpha+\varepsilon} \mathbf{E} |\xi_{n,k}|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}, \end{aligned}$$

where $\leq_{(6)}$ follows from a similar argument to the one used for $\leq_{(1)}$. Recall that (Assumption **SM**) $\sum_{j=0}^{\infty} |c_j|^\nu < \infty$ where $\nu = 1 \wedge \tau$, $\tau \in (0, \alpha)$. Choose $\varepsilon > 0$ such that $\alpha - \nu - \varepsilon > 0$. Once more, using (105), (107) and (75), as $n \rightarrow \infty$ the first term above is

$$\begin{aligned} \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=t}^{\infty} |a_{k,t}|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} &= C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=t}^{\infty} |a_{k,t}|^{\alpha-\varepsilon} S_{n,k}^{-\varepsilon} h(S_{n,k}) [1 + o(1)] \\ &\leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=t}^{\infty} |a_{k,t}|^{\alpha-\varepsilon} \gamma_n^{-\varepsilon} |a_k|^\varepsilon H(\gamma_n |a_k|^{-1}) [1 + o(1)] \\ &= C \gamma_n^{-\alpha} \sum_{k=t}^{\infty} |a_{k,t}|^{\alpha-\nu-\varepsilon} |a_{k,t}|^\nu |a_k|^\varepsilon H(\gamma_n |a_k|^{-1}) [1 + o(1)] \\ &\leq C \gamma_n^{-\alpha} \sum_{k=t}^{\infty} (\bar{a}_k)^{\alpha-\nu-\varepsilon} |a_{k,t}|^\nu (\bar{a}_k)^\varepsilon H(\gamma_n |a_k|^{-1}) [1 + o(1)] \\ &= C \gamma_n^{-\alpha} \sum_{k=t}^{\infty} (\bar{a}_k)^{\alpha-\nu} |a_{k,t}|^\nu H(\gamma_n |a_k|^{-1}) [1 + o(1)] =: C \cdot B_{1,n}(t) [1 + o(1)]. \end{aligned}$$

Similarly,

$$\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=t}^{\infty} |a_{k,t}|^{\alpha+\varepsilon} \mathbf{E} |\xi_{n,k}|^{\alpha+\varepsilon} \leq C \cdot B_{1,n}(t) [1 + o(1)].$$

Next, note that

$$\begin{aligned} B_{1,n}(t) &\leq \gamma_n^{-\alpha} \sum_{k=t}^{\infty} (\bar{a}_k)^{\alpha-\nu} H(\gamma_n |a_k|^{-1}) \left(\sum_{j=0}^{t-1} |\phi_j c_{k-j}| \right)^\nu \\ &\leq \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} (\bar{a}_k)^{\alpha-\nu} H(\gamma_n |a_k|^{-1}) |c_{k-j}|^\nu, \end{aligned}$$

where recall that by definition $0 < \nu \leq 1$.

Next, choose $0 < \delta < \alpha - \nu$. Without loss of generality assume that $\sum_{j=0}^{\infty} |c_j| = 1$ (and recall that we have already assumed that $\sum_{j=0}^{\infty} c_j = 1$). For $0 < \eta < 1$ arbitrarily small let $t_0 \in \mathbb{N}$ large enough such that

$$\left| \frac{a_k}{k^{-1/\alpha}} - 1 \right|, \left| \frac{k^{-1/\alpha}}{a_k} - 1 \right|, \left| \frac{\bar{a}_k}{k^{-1/\alpha}} - 1 \right| < \eta,$$

for all $k \geq t_0$, which is possible due to **Lemma 8**. Then as $n \rightarrow \infty$ we have

$$\begin{aligned} \sup_{1 \leq t \leq t_0} B_{1,n}(t) &\leq \gamma_n^{-\alpha} \sum_{j=0}^{t_0} |\phi_j|^\nu \sum_{k=1}^{\infty} (\bar{a}_k)^{\alpha-\nu} H(\gamma_n |a_k|^{-1}) |c_{k-j}|^\nu \\ &\leq_{(7)} C \gamma_n^{-\alpha} H(\gamma_n) \sum_{j=0}^{t_0} \sum_{k=1}^{\infty} (\bar{a}_k)^{\alpha-\nu} |c_{k-j}|^\nu \max \{ |a_k|^\delta, |a_k|^{-\delta} \} \\ &=_{(8)} C \gamma_n^{-\alpha} H(\gamma_n) \sum_{j=0}^{t_0} \sum_{k=t_0+1}^{\infty} (\bar{a}_k)^{\alpha-\nu} |c_{k-j}|^\nu \max \{ |a_k|^\delta, |a_k|^{-\delta} \} + o(1) \\ &\leq (1 + \eta^\delta) C \gamma_n^{-\alpha} H(\gamma_n) \sum_{j=0}^{t_0} \sum_{k=t_0+1}^{\infty} (\bar{a}_k)^{\alpha-\nu} |c_{k-j}|^\nu \max \{ k^{-\delta/\alpha}, k^{\delta/\alpha} \} + o(1) \\ &\leq (1 + \eta^\delta) (1 + \eta^{\alpha-\nu}) C \gamma_n^{-\alpha} H(\gamma_n) \sum_{j=0}^{t_0} \sum_{k=t_0+1}^{\infty} (k^{-(\alpha-\nu)/\alpha}) |c_{k-j}|^\nu k^{\delta/\alpha} + o(1) \\ &= C \gamma_n^{-\alpha} H(\gamma_n) \sum_{j=0}^{t_0} \sum_{k=t_0+1}^{\infty} \left(k^{\frac{-(\alpha-\nu)+\delta}{\alpha}} \right) |c_{k-j}|^\nu + o(1) \\ &= C \gamma_n^{-\alpha} H(\gamma_n) \sum_{j=0}^{t_0} \sum_{k=t_0+1-j}^{\infty} (k+j)^{\frac{-(\alpha-\nu)+\delta}{\alpha}} |c_k|^\nu + o(1) \\ &\leq C \gamma_n^{-\alpha} H(\gamma_n) \sum_{k=1}^{\infty} k^{\frac{-(\alpha-\nu)+\delta}{\alpha}} |c_k|^\nu + o(1) \\ &\leq C \gamma_n^{-\alpha} H(\gamma_n) + o(1) =_{(9)} o(1), \end{aligned}$$

where $\leq_{(7)}$ follows from Potter's Theorem (see e.g. BGT 1987; Thm 1.5.6), and $=_{(8)}$, $=_{(9)}$ because $\gamma_n^\alpha \sim L(n^{1/\alpha} \gamma_n)$ and $H(\gamma_n)/L(n^{1/\alpha} \gamma_n) \rightarrow 0$. To see the validity of the last statement recall that $\lim_{x \rightarrow \infty} L(x)/H(x) = \infty$ -cf. BGT 1987; Proposition 1.5.9a. In view of this as $n \rightarrow \infty$

$$\frac{H(\gamma_n)}{L(n^{1/\alpha} \gamma_n)} = \frac{L(\gamma_n)}{L(n^{1/\alpha} \gamma_n)} \frac{H(\gamma_n)}{L(\gamma_n)} \leq \frac{H(\gamma_n)}{L(\gamma_n)} \rightarrow 0,$$

where we have used the fact that $L(x)$ is increasing. In view of the above, as $n \rightarrow \infty$

$$\sup_{1 \leq t \leq t_0} B_{1,n}(t) = o(1). \quad (115)$$

Next, we show that

$$\sup_{t_0 \leq t \leq n} B_{1,n}(t) = O(1). \quad (116)$$

Set $R_{\nu-\alpha}(x) := x^{\nu-\alpha}H(x)$, $x > 0$. Note that $R_{\nu-\alpha}(x)$ is regularly varying of index $\nu-\alpha < 0$. Further, without loss of generality we assume that $R_{\nu-\alpha}(x)$ is decreasing¹⁸. Recall that

$$B_{1,n}(t) \leq \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} (\bar{a}_k)^{\alpha-\nu} H(\gamma_n |a_k|^{-1}) |c_{k-j}|^\nu =: B_{2,n}(t). \quad (117)$$

It is easy to see that by **Lemma 8**, for k large enough $\bar{a}_k/|a_k|$ is bounded. Hence, for $t \geq t_0$ as $n \rightarrow \infty$

$$\begin{aligned} & \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} (\bar{a}_k)^{\alpha-\nu} \left[H(\gamma_n |a_k|^{-1}) - H(\gamma_n k^{1/\alpha}) \right] |c_{k-j}|^\nu \\ & \leq \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} |c_l|^\nu \left| \frac{(\bar{a}_k^{-1})^{-(\alpha-\nu)} H(\gamma_n |a_k|^{-1})}{[k^{1/\alpha}]^{-(\alpha-\nu)} H(\gamma_n k^{1/\alpha})} - \frac{(\bar{a}_k^{-1})^{-(\alpha-\nu)}}{(k^{1/\alpha})^{-(\alpha-\nu)}} \right| [k^{1/\alpha}]^{-(\alpha-\nu)} H(\gamma_n k^{1/\alpha}) \\ & = \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} |c_l|^\nu \frac{(\bar{a}_k^{-1})^{-(\alpha-\nu)}}{(|a_k^{-1}|)^{-(\alpha-\nu)}} \left| \frac{(|a_k^{-1}|)^{-(\alpha-\nu)} h(\gamma_n |a_k|^{-1})}{[k^{1/\alpha}]^{-(\alpha-\nu)} h(\gamma_n k^{1/\alpha})} - \frac{(|a_k^{-1}|)^{-(\alpha-\nu)}}{(k^{1/\alpha})^{-(\alpha-\nu)}} \right| \\ & \quad \cdot [k^{1/\alpha}]^{-(\alpha-\nu)} H(\gamma_n k^{1/\alpha}) \\ & \leq C \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} |c_l|^\nu \left| \frac{(|a_k^{-1}|)^{-(\alpha-\nu)} H(\gamma_n |a_k|^{-1})}{[k^{1/\alpha}]^{-(\alpha-\nu)} H(\gamma_n k^{1/\alpha})} - \frac{(|a_k^{-1}|)^{-(\alpha-\nu)}}{(k^{1/\alpha})^{-(\alpha-\nu)}} \right| [k^{1/\alpha}]^{-(\alpha-\nu)} H(\gamma_n k^{1/\alpha}) \\ & \leq C \sup_{k \geq t_0} \sup_{|y-1| < \eta} \left| \frac{(\gamma_n k^{1/\alpha} y)^{-(\alpha-\nu)} H(\gamma_n k^{1/\alpha} y)}{(\gamma_n k^{1/\alpha})^{-(\alpha-\nu)} H(\gamma_n k^{1/\alpha})} - y^{-(\alpha-\nu)} \right| \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} |c_l|^\nu [k^{1/\alpha}]^{-(\alpha-\nu)} H(\gamma_n k^{1/\alpha}) \\ & \leq_{(10)} o(1) \cdot \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} |c_l|^\nu [k^{1/\alpha}]^{-(\alpha-\nu)} H(\gamma_n k^{1/\alpha}), \end{aligned} \quad (118)$$

where $\leq_{(10)}$ follows from an almost identical argument as the one used in (111). Next, arguments similar to those used in (113) show that for $t \geq t_0$,

$$\begin{aligned} & \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} \left| (k^{-1/\alpha})^{\alpha-\nu} - (\bar{a}_k)^{\alpha-\nu} \right| H(\gamma_n k^{1/\alpha}) |c_{k-j}|^\nu \\ & \leq \eta \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{k=t}^{\infty} (k^{-1/\alpha})^{\alpha-\nu} H(\gamma_n k^{1/\alpha}) |c_{k-j}|^\nu =: \eta B_{3,n}(t). \end{aligned} \quad (119)$$

Combining (118)-(119) we have for $t \geq t_0$,

$$|B_{2,n}(t) - B_{3,n}(t)| \leq \eta B_{3,n}(t). \quad (120)$$

In view of this and (117), (116) will follow if we show that

$$\sup_{t_0 \leq t \leq n} B_{3,n}(t) = O(1),$$

¹⁸Note that by BGT 1987 (Thm 1.5.3) regularly varying functions have monotone asymptotic equivalents, i.e. there a decreasing $\underline{R}_{\nu-\alpha}(x)$ such that $R_{\nu-\alpha}(x) \sim \underline{R}_{\nu-\alpha}(x)$, as $x \rightarrow \infty$.

which what we set out doing next.

In view of (68) and the fact that $R_{\nu-\alpha}(x)$ is decreasing and $\phi_j \sim j^{-1/\alpha}$, it follows that, for all $1 \leq t \leq n$ as $n \rightarrow \infty$,

$$\begin{aligned} B_{3,n}(t) &= \gamma_n^{-\alpha} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{l=t-j}^{\infty} \left((l+j)^{-1/\alpha} \right)^{\alpha-\nu} H \left(\gamma_n^{1/\alpha} (l+j)^{1/\alpha} \right) |c_l|^\nu \\ &= \gamma_n^{-\nu} \sum_{j=0}^{t-1} |\phi_j|^\nu \sum_{l=t-j}^{\infty} R_{\nu-\alpha} \left(\gamma_n (l+j)^{1/\alpha} \right) |c_l|^\nu \\ &\leq C \gamma_n^{-\nu} \sum_{j=0}^{t-1} |\phi_j|^\nu R_{\nu-\alpha} \left(\gamma_n j^{1/\alpha} \right) \sum_{l=t-j}^{\infty} |c_l|^\nu \end{aligned} \quad (121)$$

$$\begin{aligned} &\leq C \gamma_n^{-\nu} \sum_{j=0}^{t-1} |\phi_j|^\nu R_{\nu-\alpha} \left(\gamma_n j^{1/\alpha} \right) \sum_{l=0}^{\infty} |c_l|^\nu \\ &\leq C \gamma_n^{-\nu} \sum_{j=1}^{t-1} j^{-\nu/\alpha} R_{\nu-\alpha} \left(\gamma_n j^{1/\alpha} \right) [1 + o(1)] \\ &= C \gamma_n^{-\alpha} \sum_{j=1}^n j^{-1} H \left(\gamma_n j^{1/\alpha} \right) [1 + o(1)] = C\alpha + o(1), \end{aligned} \quad (122)$$

as required.

MI-SM. Recall that

$$a_{k,t} = \sum_{j=0}^{(t-1) \wedge k} \phi_j c_{k-j} = \begin{cases} \sum_{j=0}^k \phi_j c_{k-j} =: a_k & \text{if } 0 \leq k \leq t-1, \\ \sum_{j=0}^{t-1} \phi_j c_{k-j} =: a_{k,t}^- & \text{if } k \geq t. \end{cases}$$

with

$$\phi_j := \phi_j(n) = \rho_n^j.$$

Next, we introduce some notation. Let $\epsilon > 0$ such that

$$\alpha - \epsilon > 1, \text{ if } \alpha \in (1, 2],$$

and

$$\alpha - \epsilon \geq \nu, \text{ if } \alpha \in (0, 1],$$

with ν given in **Assumption MI-SM**. In addition, set $\alpha^* := \alpha - \epsilon$, Therefore, by **Assumption MI-SM**

$$\sum_{l=0}^{\infty} |c_l|^{\alpha^*} < \infty.$$

Choose n large enough such that $0 < \rho_n^j < 1$ for all $j \geq 1$. Hence, for n large enough,

$$|a_k| \leq \sum_{j=0}^k |\phi_j c_{k-j}| \leq \sum_{j=0}^{\infty} |c_j|.$$

which gives

$$\inf_{k \geq 1} |a_k^{-1}| \geq 1 / \sum_{j=0}^{\infty} |c_j| =: \eta > 0,$$

for n large. Set

$$A_{1,n} := \gamma_n^{-\alpha} \sum_{k=1}^n |a_k|^\alpha H(\gamma_n |a_k|^{-1}).$$

First, we shall show that

$$\sup_{n \geq 1, 1 \leq t \leq n} A_n(t) := \sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{t-1} a_k \xi_{t-k} \right|^{\alpha-\varepsilon} < \infty, \quad (123)$$

for all $\varepsilon > 0$ such that $\alpha - \varepsilon > 0$. Note that (123) together with the Lyapunov inequality imply that

$$\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{t-1} a_k \xi_{t-k} \right|^{\alpha'} < \infty,$$

for all $\alpha' \leq \alpha - \varepsilon$.

In view of the fact that $\gamma_n \inf_{k \geq 1} |a_k^{-1}| \rightarrow \infty$ and, using the same arguments as those used for establishing (109), we get

$$\sup_{1 \leq t \leq n} A_n(t) := \sup_{1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{t-1} a_k \xi_{t-k} \right|^{\alpha-\varepsilon} \leq C \left[A_{1,n} + A_{1,n}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right] [1 + o(1)], \quad (124)$$

for n large enough. Next, we show that $A_{1,n}$ is bounded.

Set $\beta_k := a_k^{-1}$. As $n \rightarrow \infty$, the difference

$$\begin{aligned} & \left| \sum_{k=1}^n |\gamma_n \beta_k|^{-\alpha} H(|\gamma_n \beta_k|) - \sum_{k=1}^n |\gamma_n \beta_k|^{-\alpha} H(\gamma_n) \right| \\ & \leq \sum_{k=1}^n \left| \frac{|\gamma_n \beta_k|^{-\varepsilon} H(|\gamma_n \beta_k|)}{\gamma_n^{-\varepsilon} H(\gamma_n)} - |\beta_k|^{-\varepsilon} \right| H(\gamma_n) \gamma_n^{-\alpha} |\beta_k|^{-\alpha+\varepsilon} \\ & \leq \sup_{1 \leq k \leq n} \left| \frac{|\gamma_n \beta_k|^{-\varepsilon} H(|\gamma_n \beta_k|)}{\gamma_n^{-\varepsilon} H(\gamma_n)} - |\beta_k|^{-\varepsilon} \right| H(\gamma_n) \sum_{k=1}^n \gamma_n^{-\alpha} |\beta_k|^{-\alpha+\varepsilon} \\ & \leq \sup_{\eta \leq y < \infty} \left| \frac{(\gamma_n y)^{-\varepsilon} H(|\gamma_n y|)}{\gamma_n^{-\varepsilon} H(\gamma_n)} - y^{-\varepsilon} \right| H(\gamma_n) \sum_{k=1}^n \gamma_n^{-\alpha} |\beta_k|^{-\alpha+\varepsilon} \\ & =_{(1)} o(1) \cdot H(\gamma_n) \sum_{k=1}^n \gamma_n^{-\alpha} |\beta_k|^{-\alpha+\varepsilon} \sim_{(2)} o(1) \cdot \kappa_n^{-1} \sum_{k=1}^n |\beta_k|^{-\alpha+\varepsilon} =_{(3)} o(1), \end{aligned}$$

where $=_{(1)}$ follows from BGT 1987, Thm 1.5.2, $\sim_{(2)}$ from **Definition 1 (MI)** with $b = 1$, and $=_{(3)}$ from the fact that

$$S_n := \kappa_n^{-1} \sum_{k=1}^n |\beta_k|^{-\alpha+\varepsilon} = O(1). \quad (125)$$

A proof of (125) is provided below. In view the above and **Lemma 5**, as $n \rightarrow \infty$ we get

$$\begin{aligned} A_{1,n} &= H(\gamma_n) \sum_{k=1}^n \left| \gamma_n^{-1} a_k \right|^\alpha + o(1) \\ &= \kappa_n^{-1} \sum_{k=1}^n |a_k|^\alpha [1 + o(1)] + o(1) = \alpha^{-1} \left| \sum_{l=0}^{\infty} c_l \right|^\alpha + o(1), \end{aligned} \quad (126)$$

where we have used the fact that $H(\gamma_n)/\gamma_n^\alpha \sim \kappa_n^{-1}$. Note that (126) implies that $\sup_{1 \leq t \leq n} A_n(t) = O(1)$.

Next, we show that S_n is bounded. First, suppose that $\alpha \in (0, 1]$. Recall that in this case $\alpha^* = \alpha - \epsilon \geq \nu$, which implies that $\sum_{l=0}^{\infty} |c_l|^{\alpha^*} < \infty$. Using the fact that **[Check again: need to replace $a_k = \sum_{l=1}^k \rho_n^{k-l} c_l$, by $a_k = \sum_{l=0}^k \rho_n^{k-l} c_l$]** $a_k = \sum_{l=1}^k \rho_n^{k-l} c_l$ we get

$$\begin{aligned} S_n &= \kappa_n^{-1} \sum_{k=1}^n |a_k|^{\alpha^*} \leq \kappa_n^{-1} \sum_{k=1}^n \left(\sum_{l=0}^k |\rho_n^{k-l} c_l| \right)^{\alpha^*} \\ &\leq \kappa_n^{-1} \sum_{k=1}^n \sum_{l=0}^k |\rho_n^{k-l} c_l|^{\alpha^*} =: S'_n. \end{aligned}$$

Next,

$$\begin{aligned} S'_n &= \kappa_n^{-1} \sum_{l=0}^n \sum_{k=l \vee 1}^n \rho_n^{\alpha^*(k-l)} |c_l|^{\alpha^*} \leq \kappa_n^{-1} \sum_{l=0}^n \rho_n^{-\alpha^* l} |c_l|^{\alpha^*} \sum_{k=l}^n \rho_n^{\alpha^* k} \\ &= \kappa_n^{-1} \sum_{l=0}^n \rho_n^{-\alpha^* l} |c_l|^{\alpha^*} \left[\sum_{k=1}^n \rho_n^{\alpha^* k} - \sum_{k=1}^{l-1} \rho_n^{\alpha^* k} \right] \\ &= \kappa_n^{-1} \sum_{l=0}^n \rho_n^{-\alpha^* l} |c_l|^{\alpha^*} \left[\rho_n^{\alpha^*} \frac{1 - \rho_n^{\alpha^* n}}{1 - \rho_n^{\alpha^*}} - \rho_n^{\alpha^*} \frac{1 - \rho_n^{\alpha^*(l-1)}}{1 - \rho_n^{\alpha^*}} \right] \\ &= \frac{\rho_n^{\alpha^*}}{1 - \rho_n^{\alpha^*}} \kappa_n^{-1} \sum_{l=0}^n \rho_n^{-\alpha^* l} |c_l|^{\alpha^*} \left[\rho_n^{\alpha^*(l-1)} - \rho_n^{\alpha^* n} \right] \\ &= \frac{1}{1 - \rho_n^{\alpha^*}} \kappa_n^{-1} \sum_{l=0}^n |c_l|^{\alpha^*} - \frac{\rho_n^{\alpha^*}}{1 - \rho_n^{\alpha^*}} \kappa_n^{-1} \sum_{l=0}^n \rho_n^{\alpha^*(n-l)} |c_l|^{\alpha^*} \\ &= \frac{1}{\alpha^*} \left[\sum_{l=0}^n |c_l|^{\alpha^*} - \sum_{l=0}^n \rho_n^{\alpha^*(n-l)} |c_l|^{\alpha^*} \right] [1 + o(1)], \\ &\leq \frac{1}{\alpha^*} \left[\sum_{l=0}^{\infty} |c_l|^{\alpha^*} + \sum_{l=0}^n \rho_n^{\alpha^*(n-l)} |c_l|^{\alpha^*} \right] [1 + o(1)] = O(1), \end{aligned} \tag{127}$$

where we have used the fact that $1/(1 - \rho_n^{\alpha^*}) \kappa_n \rightarrow 1/\alpha^*$.¹⁹ It is straightforward to verify that the second sum in (127) is bounded; however, for the subsequent analysis we prove the stronger claim that it is asymptotically negligible. Let m_n an increasing

¹⁹By the mean value theorem,

$$f(x) = 1 + f'(\bar{x})x, \quad |\bar{x}| \leq |x|$$

with $f(x) = (1 - x)^{\alpha^*} \Rightarrow f'(x) = -\alpha^*(1 - x)^{\alpha^*-1}$. Hence,

$$1 - f(x) = -f'(\bar{x})x = \alpha^*(1 - \bar{x})^{\alpha^*-1}x = \alpha^* \frac{1}{(1 - \bar{x})^{1-\alpha^*}} x$$

Therefore,

$$\frac{x}{1 - f(x)} = \frac{x(1 - \bar{x})^{1-\alpha^*}}{\alpha^* x} \stackrel{\kappa_n^{-1} = x}{=} \frac{\kappa_n^{-1}(1 - \bar{x})^{1-\alpha^*}}{\alpha^* \kappa_n^{-1}} \rightarrow \frac{1}{\alpha^*},$$

where we have used the fact that $\bar{x} \leq \kappa_n^{-1} \rightarrow 0$.

integer valued sequence such that $m_n^{-1} + m_n \kappa_n^{-1} \rightarrow 0$. Noting that for n large enough $0 < \rho_n < 1$, the term

$$\begin{aligned} \sum_{l=0}^n \rho_n^{\alpha^*(n-l)} |c_l|^{\alpha^*} &= \left(\sum_{l=0}^{m_n} + \sum_{l=m_n+1}^n \right) \rho_n^{\alpha^*(n-l)} |c_l|^{\alpha^*} \\ &\leq \sum_{l=0}^{m_n} \rho_n^{\alpha^*(n-m_n)} |c_l|^{\alpha^*} + \sum_{l=m_n+1}^n \rho_n^{\alpha^*(n-l)} |c_l|^{\alpha^*} \\ &\leq \rho_n^{\alpha^*(n-m_n)} \sum_{l=0}^{\infty} |c_l|^{\alpha^*} + \sum_{l=m_n+1}^n |c_l|^{\alpha^*} \rightarrow 0, \end{aligned} \quad (128)$$

as required. The above show that (125) holds for $\alpha \in (0, 1]$.

Next, suppose that $\alpha \in (1, 2]$. Recall that $\alpha^* = \alpha - \epsilon$. Hence, in this case $\alpha^* > 1$. Set $A_{k,n} := \sum_{l=1}^k |\rho_n^{k-l} c_l|$, $\bar{A}_{k,n} := A_{k,n} 1\{A_{k,n} \geq 1\}$ and $\underline{A}_{k,n} := A_{k,n} 1\{A_{k,n} < 1\}$. Hence,

$$\begin{aligned} S_n &= \kappa_n^{-1} \sum_{k=1}^n |a_k|^{\alpha^*} \leq \kappa_n^{-1} \sum_{k=1}^n \left(\sum_{l=0}^k |\rho_n^{k-l} c_l| \right)^{\alpha^*} \\ &= \kappa_n^{-1} \sum_{k=1}^n A_{k,n}^{\alpha^*} \leq 2\kappa_n^{-1} \sum_{k=1}^n (\bar{A}_{k,n}^{\alpha^*} + \underline{A}_{k,n}^{\alpha^*}) \leq 2\kappa_n^{-1} \sum_{k=1}^n (\bar{A}_{k,n}^2 + \underline{A}_{k,n}) \\ &\leq 2\kappa_n^{-1} \sum_{k=1}^n (A_{k,n}^2 + A_{k,n}). \end{aligned} \quad (129)$$

Now, the arguments that lead to (127), also give $\kappa_n^{-1} \sum_{k=1}^n A_{k,n} = O(1)$. Furthermore, $\kappa_n^{-1} \sum_{k=1}^n A_{k,n}^2 = O(1)$ by DK2018 eq. (46), (47). Hence, (125) also holds for $\alpha \in (1, 2]$.

The arguments shown above imply that for all $\alpha' \in (0, \alpha)$

$$\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{-1} x_t^+(n) \right|^{\alpha'} < \infty.$$

Next, we show that

$$\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{-1} x_t^-(n) \right|^{\alpha'} < \infty.$$

Notice that (recall that $a_{k,t}^- = \sum_{l=1}^{t-1} \rho_n^{k-l} c_l = \sum_{j=0}^t \rho_n^j c_{k-j}$, $k \geq t$)

$$\begin{aligned} \inf_{1 \leq t \leq n, k \geq t} |a_{k,t}^-|^{-1} &= \frac{1}{\sup_{1 \leq t \leq n, k \geq t} |a_{k,t}^-|} \geq \frac{1}{\sup_{1 \leq t \leq n, k \geq t} \sum_{j=0}^t |\rho_n^j c_{k-j}|} \\ &\geq 1 / \sum_{j=0}^{\infty} |c_j| =: \eta > 0. \end{aligned}$$

Hence, $\gamma_n \inf_{1 \leq t \leq n, k \geq t} |a_{k,t}^-|^{-1} \rightarrow \infty$. Consequently, by arguments analogous to those used to establish (109), we obtain, as $n \rightarrow \infty$,

$$\sup_{1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{-1} x_t^-(n) \right|^{\alpha-\epsilon} \leq \left[\sup_{1 \leq t \leq n} B_n(t) + \left(\sup_{1 \leq t \leq n} B_n(t) \right)^{\frac{\alpha-\epsilon}{\alpha+\epsilon}} \right] [1 + o(1)], \quad (130)$$

where

$$B_n(t) := \sum_{k=t}^{\infty} \left| \frac{\lambda}{\gamma_n} a_{k,t}^- \right|^\alpha H \left(\left| \gamma_n^{-1} a_{k,t}^- \right|^{-1} \right).$$

Therefore, it suffices to show that $\sup_{1 \leq t \leq n} B_n(t) = O(1)$. Set $\beta_{k,t}^{-1} := a_{k,t}^-$. As $n \rightarrow \infty$,

$$\begin{aligned} & |B_n(t) - B_n(t)'| \\ &:= \left| B_n(t) - \sum_{k=t}^{\infty} \left| \gamma_n^{-1} a_{k,t}^- \right|^\alpha H(\gamma_n) \right| \\ &= \left| \sum_{k=t}^{\infty} \left| \gamma_n^{-1} a_{k,t}^- \right|^\alpha H \left(\left| \gamma_n^{-1} a_{k,t}^- \right|^{-1} \right) - \sum_{k=t}^{\infty} \left| \gamma_n^{-1} a_{k,t}^- \right|^\alpha H(\gamma_n) \right| \\ &\leq \sum_{k=t}^{\infty} \left| \left| \gamma_n \beta_{k,t} \right|^{-\alpha} H \left(\left| \gamma_n \beta_{k,t} \right|^{-1} \right) - \left| \gamma_n \beta_{k,t} \right|^{-\alpha} H(\gamma_n) \right| \\ &\leq \sum_{k=t}^{\infty} \left| \frac{\left| \gamma_n \beta_{k,t} \right|^{-\epsilon} H \left(\left| \gamma_n \beta_{k,t} \right| \right)}{\left| \gamma_n \right|^{-\epsilon} H(\gamma_n)} - \left| \beta_{k,t} \right|^{-\epsilon} \right| \gamma_n^{-\alpha} \left| \beta_{k,t} \right|^{-\alpha+\epsilon} H(\gamma_n) \\ &\leq \sup_{\eta \leq y < \infty} \left| \frac{\left| \gamma_n y \right|^{-\epsilon} H \left(\left| \gamma_n y \right| \right)}{\left| \gamma_n \right|^{-\epsilon} H(\gamma_n)} - y^{-\epsilon} \right| \gamma_n^{-\alpha} \sum_{k=t}^{\infty} \left| \beta_{k,t} \right|^{-\alpha+\epsilon} H(\gamma_n) \\ &\leq o(1) \cdot \gamma_n^{-\alpha} \sum_{k=t}^{\infty} \left| \beta_{k,t} \right|^{-\alpha+\epsilon} H(\gamma_n) \sim o(1) \cdot \kappa_n^{-1} \sum_{k=t}^{\infty} \left| \beta_{k,t} \right|^{-\alpha+\epsilon} =: o(1) \cdot T_n(t). \end{aligned} \quad (131)$$

In view of (130)-(131), Next, we show that $\sup_{1 \leq t \leq n} T_n(t) = O(1)$. First, suppose that $\alpha \in (0, 1]$. We have (recall that $\alpha^* = \alpha - \epsilon$)

$$\begin{aligned} T_n(t) &= \kappa_n^{-1} \sum_{k=t}^{\infty} \left| a_{k,t}^- \right|^{\alpha^*} \leq \kappa_n^{-1} \sum_{k=t}^{\infty} \sum_{j=0}^{t-1} \left| \rho_n^j c_{k-j} \right|^{\alpha^*} \\ &\leq \kappa_n^{-1} \sum_{k=0}^{\infty} \sum_{j=0}^{k \wedge n} \left| \rho_n^j c_{k-j} \right|^{\alpha^*} = \kappa_n^{-1} \sum_{k=0}^{n-1} \sum_{j=0}^k \left| \rho_n^j c_{k-j} \right|^{\alpha^*} + \kappa_n^{-1} \sum_{k=n}^{\infty} \sum_{j=0}^n \left| \rho_n^j c_{k-j} \right|^{\alpha^*} \\ &= S'_n + \kappa_n^{-1} \sum_{k=n}^{\infty} \left(\sum_{j=0}^{[n/2]} + \sum_{j=[n/2]+1}^n \right) \left| \rho_n^j c_{k-j} \right|^{\alpha^*} \\ &\leq O(1) + \sum_{k=[n/2]}^{\infty} |c_k|^{\alpha^*} \kappa_n^{-1} \sum_{j=0}^{[n/2]} \rho_n^{\alpha^* j} + \kappa_n^{-1} \sum_{j=[n/2]+1}^n \rho_n^{\alpha^* j} \sum_{k=0}^{\infty} |c_k|^{\alpha^*} = O(1), \end{aligned} \quad (132)$$

where we have used the facts that $S'_n = O(1)$, $\kappa_n^{-1} \sum_{j=0}^{[n/2]} \rho_n^{\alpha^* j} = O(1)$, and $\kappa_n^{-1} \sum_{j=[n/2]+1}^n \rho_n^{\alpha^* j} = o(1)$ (for the latter two see Lemma 6.4 in DK2018).

Next, consider the case $\alpha \in (1, 2]$. First, note that (132) holds true if we set $\alpha^* = 1$. Set $A_{tkn} := \sum_{j=0}^{t-1} \left| \rho_n^j c_{k-j} \right|$ and $\bar{A}_{tkn} := A_{tkn} 1 \{A_{tkn} > 1\}$, $\underline{A}_{tkn} := A_{tkn} 1 \{A_{tkn} \leq 1\}$. The term

$$\begin{aligned} T_n(t) &\leq \kappa_n^{-1} \sum_{k=t}^{\infty} (A_{tkn})^{\alpha^*} = \kappa_n^{-1} \sum_{k=t}^{\infty} (\bar{A}_{tkn} + \underline{A}_{tkn})^{\alpha^*} \\ &\leq 2\kappa_n^{-1} \sum_{k=t}^{\infty} \left[(\bar{A}_{tkn})^{\alpha^*} + (\underline{A}_{tkn})^{\alpha^*} \right] \leq 2\kappa_n^{-1} \sum_{k=t}^{\infty} \left[(\bar{A}_{tkn})^2 + \underline{A}_{tkn} \right] \\ &\leq 2\kappa_n^{-1} \sum_{k=t}^{\infty} \left[A_{tkn}^2 + A_{tkn} \right]. \end{aligned} \quad (133)$$

It follows from (132), with $\alpha^* = 1$, that $\kappa_n^{-1} \sup_{1 \leq t \leq n} \sum_{k=t}^{\infty} A_{tkn} = O(1)$. Furthermore, $\kappa_n^{-1} \sup_{1 \leq t \leq n} \sum_{k=t}^{\infty} A_{tkn}^2 = O(1)$ by DK2018 (proof of Proposition 2.2).

Combining (131)-(133) we have

$$\sup_{1 \leq t \leq n} B_n(t) = \sup_{1 \leq t \leq n} B_n(t)' + o(1) = O(1), \quad (134)$$

where the last equality follows from identical arguments as those used in (132) and (133) with α^* replaced by α .

MI-LM/NM. Next, we show the finite sub- α moments property under **MI-LM** and **MI-NM**. We start with some notation. Set

$$m_n := \gamma_n \kappa_n^{-(1-m)} \left(\Longleftrightarrow m_n / \gamma_n = \kappa_n^{-1+m} \right) \quad (135)$$

In addition, recall that (**Definition 1**)

$$\gamma_n = \ell_1(\kappa_n)^{1/\alpha} \kappa_n^{b/\alpha} \text{ where } \ell_1(x) \sim H(x^{1/\alpha} \ell_1(x)^{1/\alpha}),$$

and

$$a_{k,t} = \sum_{j=0}^{(t-1) \wedge k} \phi_j c_{k-j} = \begin{cases} \sum_{j=0}^k \phi_j c_{k-j} =: a_k & \text{if } 0 \leq k \leq t-1, \\ \sum_{j=0}^{t-1} \phi_j c_{k-j} =: a_{k,t}^- & \text{if } k \geq t. \end{cases}$$

with

$$\phi_j := \phi_j(n) = \rho_n^j.$$

Remark 20.

(a) Note that the above imply that $m_n \sim \ell_1(\kappa_n)^{1/\alpha} \kappa_n^{1/\alpha}$. To see this write

$$\begin{aligned} m_n &= \gamma_n \kappa_n^{-(1-m)} \sim \ell_1(\kappa_n)^{1/\alpha} \kappa_n^{b/\alpha} \kappa_n^{-(1-m)} \\ &= \ell_1(\kappa_n)^{1/\alpha} \kappa_n^{\frac{1}{\alpha}(1+\alpha-\alpha m)} \kappa_n^{-(1-m)} = \ell_1(\kappa_n)^{1/\alpha} \kappa_n^{\left(\frac{1}{\alpha}+1-m\right)} \kappa_n^{-(1-m)} \\ &= \ell_1(\kappa_n)^{1/\alpha} \kappa_n^{1/\alpha}. \end{aligned}$$

(b) In addition, we have the following relationship that will be utilized later. As $n \rightarrow \infty$,

$$H(m_n) \sim H(\kappa_n^{1/\alpha} \ell_1(\kappa_n)^{1/\alpha}) \sim \ell_1(\kappa_n).$$

MI-LM ($m \in (1/\alpha, 1)$). Set

$$q(x) := e^{-x} \int_0^x e^y y^{-m} dy.$$

By Lemma 7.2 of DK2018,

$$\sup_{k_0 \leq k \leq n} \left| \kappa_n^{-(1-m)} \sum_{j=0}^k \rho_n^j c_{k-j} - q(k/\kappa_n) \right| = o(1), \quad (136)$$

as $n \rightarrow \infty$. Hence,

$$\sup_{0 \leq k \leq n} \kappa_n^{-(1-m)} \left| \sum_{j=0}^k \rho_n^j c_{k-j} \right| = \sup_{0 \leq k \leq n} q(k/\kappa_n) + o(1). \quad (137)$$

In view of the fact²⁰ that $\sup_{x \geq 0} q(x) \leq C_0$, for some $C_0 \in (0, \infty)$, we get

$$\sup_{0 \leq k \leq n} q(k/\kappa_n) \leq C_0.$$

Next, note that (see **Remark 20**) $m_n = \gamma_n \kappa_n^{-(1-m)} \sim \kappa_n^{1/\alpha} \ell_1(\kappa_n)^{1/\alpha} \rightarrow \infty$. Set $S_{n,k} := \gamma_n |a_k|^{-1}$, and $S_{n,k}^* := m_n^{-1} S_{n,k}$. In view of the above,

$$\begin{aligned} \inf_{1 \leq k \leq n} S_{n,k} &= m_n \inf_{1 \leq k \leq n} (\kappa_n^{1-m} \gamma_n^{-1} S_{n,k}) = m_n \inf_{1 \leq k \leq n} S_{n,k}^* \\ &= m_n \frac{1}{\kappa_n^{-(1-m)} \sup_{1 \leq k \leq n} |a_k|} \geq m_n (2C_0)^{-1} \rightarrow \infty, \end{aligned} \quad (138)$$

as $n \rightarrow \infty$.

Set

$$A(n) := \frac{1}{\gamma_n^\alpha} \sum_{k=1}^{n-1} |a_k|^\alpha H\left(|\gamma_n a_k^{-1}|\right) = \sum_{k=1}^{n-1} |S_{n,k}|^{-\alpha} H(|S_{n,k}|). \quad (139)$$

In view of (138), and using the same arguments as those used for establishing (124) we get

$$\sup_{1 \leq t \leq n} A_{n,t} := \sup_{1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{t-1} a_k \xi_{t-k} \right|^{\alpha-\varepsilon} \leq C \left[A(n) + A(n)^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right] [1 + o(1)],$$

for all $\varepsilon > 0$ such that $\alpha - \varepsilon > 0$ and n large enough. Hence, it suffices to show that $A(n) = O(1)$, which is what we set out to do next.

Let $\epsilon > 0$ and $b_\epsilon := 1 + (\alpha - \epsilon) - (\alpha - \epsilon)m$ such that (225) is satisfied. For n large enough the difference

$$\begin{aligned} \left| A(n) - \sum_{k=1}^{n-1} |S_{n,k}|^{-\alpha} H(m_n) \right| &= \left| \sum_{k=1}^{n-1} |S_{n,k}|^{-\alpha} H(S_{n,k}) - \sum_{k=1}^{n-1} |S_{n,k}|^{-\alpha} H(m_n) \right| \\ &\leq \sum_{k=1}^{n-1} \left\{ \left| \frac{|m_n \kappa_n^{1-m} \gamma_n^{-1} S_{n,k}|^{-\epsilon} H(|m_n \kappa_n^{1-m} \gamma_n^{-1} S_{n,k}|)}{m_n^{-\epsilon} H(m_n)} - | \kappa_n^{1-m} \gamma_n^{-1} S_{n,k} |^{-\epsilon} \right| \right. \\ &\quad \left. \cdot \kappa_n^{\epsilon(1-m)} H(m_n) \gamma_n^{-\alpha} | \gamma_n^{-1} S_{n,k} |^{-\alpha+\epsilon} \right\} \\ &\leq \sum_{k=1}^{n-1} \left| \frac{|m_n S_{n,k}^*|^{-\epsilon} H(|m_n S_{n,k}^*|)}{m_n^{-\epsilon} H(m_n)} - |S_{n,k}^*|^{-\epsilon} \right| \kappa_n^{\epsilon(1-m)} H(m_n) \gamma_n^{-\alpha} | \gamma_n^{-1} S_{n,k} |^{-\alpha+\epsilon} \\ &\leq \sup_{(2C_0)^{-1} \leq y < \infty} \left| \frac{(m_n y)^{-\epsilon} H(|m_n y|)}{m_n^{-\epsilon} H(m_n)} - y^{-\epsilon} \right| \kappa_n^{\epsilon(1-m)} H(m_n) \sum_{k=1}^{n-1} \gamma_n^{-\alpha} |a_k^{-1}|^{-\alpha+\epsilon} \\ &=_{(1)} o(1) \cdot H(m_n) \kappa_n^{\epsilon(1-m)} \sum_{k=1}^{n-1} \gamma_n^{-\alpha} |a_k|^{-\alpha+\epsilon} \end{aligned}$$

²⁰Notice that

$$0 \leq q(x) = \frac{\int_0^x e^y y^{-m} dy}{\int_0^x e^y dy + 1} = \frac{\int_0^1 e^y y^{-m} dy + \int_1^x e^y y^{-m} dy}{\int_0^x e^y dy + 1} \leq \frac{e \int_0^1 y^{-m} dy}{\int_0^x e^y dy + 1} + 1 < \infty.$$

$$\sim o(1) \cdot \frac{H(m_n)}{H(m_n)} \frac{1}{\kappa_n^{b_\epsilon}} \sum_{k=1}^{n-1} |a_k|^{\alpha-\epsilon} =_{(2)} o(1), \quad (140)$$

where $=_{(1)}$ follows from BGT 1987, Thm 1.5.2. In addition, for $=_{(2)}$ we have used the fact that (cf. **Remark 20**)

$$\begin{aligned} \kappa_n^{\epsilon(1-m)} \gamma_n^{-\alpha} &\sim \kappa_n^{\epsilon(1-m)} \kappa_n^{-b} H(m_n)^{-1} = \kappa_n^{\epsilon(1-m)} \kappa_n^{-(1+\alpha-\alpha m)} H(m_n)^{-1} \\ &= \kappa_n^{-(1+(\alpha-\epsilon)-(\alpha-\epsilon)m)} H(m_n)^{-1} = \kappa_n^{-b_\epsilon} H(m_n)^{-1}, \end{aligned}$$

and

$$\frac{1}{\kappa_n^{b_\epsilon}} \sum_{k=1}^{n-1} |a_k|^{\alpha-\epsilon} = |m_0|^{\alpha-\epsilon} \mathbb{V}_1(\alpha - \epsilon) + o(1),$$

with $\mathbb{V}_1$ given in **Lemma 17** (see **Lemma 15** and **Lemma 17** for more details). In view the above, as $n \rightarrow \infty$

$$\begin{aligned} A(n) &= H(m_n) \gamma_n^{-\alpha} \sum_{k=1}^n |a_k|^\alpha + o(1) \\ &\sim \frac{H(m_n)}{H(m_n)} \frac{1}{\kappa_n^b} \sum_{k=1}^n |a_k|^\alpha = |m_0|^\alpha \mathbb{V}_1(\alpha) + o(1). \end{aligned} \quad (141)$$

Let

$$B_{n,t} := \mathbf{E} \left| \gamma_n^{-1} \sum_{k=t}^{\infty} a_{k,t}^- \xi_{t-k} \right|^{\alpha-\epsilon} = \mathbf{E} \left| \gamma_n^{-1} \sum_{k=t}^{\infty} a_{k,t}^- \xi_k \right|^{\alpha-\epsilon}, \quad (142)$$

for all $\epsilon > 0$ such that $\alpha - \epsilon > 0$. We shall show that $\sup_{1 \leq t \leq n} B_{n,t} = O(1)$, which will conclude the proof this Lemma under **Assumption M1-LM**. Set

$$S_{n,n \wedge k} := \left(\sum_{j=0}^{n \wedge k} |\rho_n^j c_{k-j}| \right)^{-1}.$$

The following calculations are similar to those that lead to (138). First, notice that

$$\kappa_n^{m-1} \sup_{k \geq 0} \sum_{j=0}^{n \wedge k} |\rho_n^j c_{k-j}| \leq \kappa_n^{m-1} \sup_{0 \leq k \leq n} \sum_{j=0}^k |\rho_n^j c_{k-j}| + \kappa_n^{m-1} \sup_{k > n} \sum_{j=0}^n |\rho_n^j c_{k-j}|.$$

By (136), the first term above satisfies (137). Let $\delta_n > 0$ be an integer valued sequence such that $\delta_n^{-1} \kappa_n \ln(\kappa_n^m) = o(1)$. This implies that $\kappa_n^m \rho_n^{\delta_n} = o(1)$ (cf. DK2018, Lemma 6.4). In view if this, the second term

$$\begin{aligned} \kappa_n^{m-1} \sup_{k > n} \sum_{j=0}^n |\rho_n^j c_{k-j}| &\leq \kappa_n^{m-1} \left[\sup_{k > n} \sum_{j=0}^{\delta_n} |\rho_n^j c_{k-j}| + \sup_{k > n} \sum_{j=\delta_n}^n |\rho_n^j c_{k-j}| \right] \\ &= \kappa_n^{m-1} \left[\sup_{k > n} \sum_{j=k-\delta_n}^k |\rho_n^{k-j} c_j| + \sup_{k \geq 0} |c_k| \sum_{j=\delta_n}^n |\rho_n^j| \right] \\ &\leq \kappa_n^{m-1} \left[\sup_{j > n-\delta_n} |c_j| \sup_{k > n} \sum_{j=k-\delta_n}^k |\rho_n^{k-j}| + \sup_{k \geq 0} |c_k| \sum_{j=\delta_n}^n |\rho_n^j| \right] \\ &= \kappa_n^{m-1} \left[\sup_{j > n-\delta_n} |c_j| \sum_{j=0}^{\delta_n} |\rho_n^j| + C \sum_{j=\delta_n}^n |\rho_n^j| \right] = \kappa_n^{m-1} \left[\sup_{j > n-\delta_n} |c_j| O(\kappa_n) + O(\kappa_n \rho_n^{\delta_n}) \right] \end{aligned}$$

$$= \sup_{j > n - \delta_n} |c_j| O(\kappa_n^m) + O(\kappa_n^m \rho_n^{\delta_n}) \leq C (n - \delta_n)^{-m} \kappa_n^m + o(1) = o(1).$$

Hence, similarly to (138) as $n \rightarrow \infty$

$$\gamma_n \inf_{k \geq 0} S_{n,n \wedge k} = \frac{\gamma_n}{\sup_{k \geq 0} \sum_{j=0}^{n \wedge k} |\rho_n^j c_{k-j}|} \geq (2C_0)^{-1} m_n \rightarrow \infty, \quad (143)$$

for some $C_0 < \infty$. Next, set

$$\bar{\xi}_{n,k} := \xi_k 1\{|\xi_k| \geq \gamma_n S_{n,n \wedge k}\} \quad \text{and} \quad \underline{\xi}_{n,k} := \xi_k 1\{|\xi_k| < \gamma_n S_{n,n \wedge k}\}.$$

As before, without loss of generality we will use the convention that $\mathbf{E} \bar{\xi}_{n,k} = \mathbf{E} \underline{\xi}_{n,k} = 0$ for $1 < \alpha \leq 2$. It follows from (143) and Astrauskas (1983; Lemma 1) that as $n \rightarrow \infty$

$$\mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} \asymp S_{n,k}^{-\varepsilon} h(S_{n,n \wedge k}) \quad \text{and} \quad \mathbf{E} |\underline{\xi}_{n,k}|^{\alpha+\varepsilon} \asymp S_{n,k}^{\varepsilon} h(S_{n,n \wedge k}), \quad (144)$$

uniformly in $k \geq 0$. Similar arguments as those used before, see e.g. (109), give

$$B_{n,t} \leq C \left[B_{1n,t} + B_{2n,t}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right],$$

with

$$B_{1n,t} := \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=t}^{\infty} |a_{k,t}^-|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon},$$

and

$$B_{2n,t} := \gamma_n^{-(\alpha+\varepsilon)} \sum_{k=t}^{\infty} |a_{k,t}^-|^{\alpha+\varepsilon} \mathbf{E} |\underline{\xi}_{n,k}|^{\alpha+\varepsilon}.$$

Using (75), (143), and (144) we get

$$\begin{aligned} \sup_{1 \leq t \leq n} B_{1n,t} &\leq C \gamma_n^{-(\alpha-\varepsilon)} \sup_{1 \leq t \leq n} \sum_{k=t}^{\infty} |a_{k,t}^-|^{\alpha-\varepsilon} S_{n,n \wedge k}^{-\varepsilon} H(S_{n,n \wedge k}) [1 + o(1)] \\ &= C \gamma_n^{-(\alpha-\varepsilon)} \sup_{1 \leq t \leq n} \sum_{k=t}^{\infty} \left| \sum_{j=0}^{t-1} \rho_n^j c_{k-j} \right|^{\alpha-\varepsilon} S_{n,n \wedge k}^{-\varepsilon} H(S_{n,n \wedge k}) [1 + o(1)] \\ &\leq C \gamma_n^{-(\alpha-\varepsilon)} \sup_{1 \leq t \leq n} \sum_{k=t}^{\infty} \left(\sum_{j=0}^{t-1} |\rho_n^j c_{k-j}| \right)^{\alpha-\varepsilon} S_{n,n \wedge k}^{-\varepsilon} H(S_{n,n \wedge k}) [1 + o(1)] \\ &= C \sup_{1 \leq t \leq n} \sum_{k=t}^{\infty} \left(\gamma_n^{-1} \sum_{j=0}^{t-1} |\rho_n^j c_{k-j}| \right)^{\alpha-\varepsilon} S_{n,n \wedge k}^{-\varepsilon} H(S_{n,n \wedge k}) [1 + o(1)] \\ &\leq C \sup_{1 \leq t \leq n} \sum_{k=t}^{\infty} S_{n,n \wedge k}^{-\alpha+\varepsilon} S_{n,n \wedge k}^{-\varepsilon} H(S_{n,n \wedge k}) [1 + o(1)] \\ &= C \sup_{1 \leq t \leq n} \sum_{k=t}^{\infty} S_{n,n \wedge k}^{-\alpha} H(S_{n,n \wedge k}) [1 + o(1)] \leq C \sum_{k=0}^{\infty} S_{n,n \wedge k}^{-\alpha} H(S_{n,n \wedge k}) [1 + o(1)]. \quad (145) \end{aligned}$$

The following arguments are similar to those that lead to (140) above. Let ϵ and b_ϵ as in (140). Then as $n \rightarrow \infty$ the difference

$$\left| \sum_{k=0}^{\infty} S_{n,n \wedge k}^{-\alpha} H(S_{n,n \wedge k}) - \sum_{k=0}^{\infty} S_{n,n \wedge k}^{-\alpha} H(m_n) \right|$$

$$\begin{aligned}
&\leq \sum_{k=0}^{\infty} \left\{ \left| \frac{(m_n \kappa_n^{1-m} \gamma_n^{-1} S_{n,n \wedge k})^{-\epsilon} H(m_n \kappa_n^{1-m} \gamma_n^{-1} S_{n,n \wedge k})}{m_n^{-\epsilon} H(m_n)} - (\kappa_n^{1-m} \gamma_n^{-1} S_{n,n \wedge k})^{-\epsilon} \right| \right. \\
&\quad \left. \kappa_n^{\epsilon(1-m)} (\gamma_n^{-1} S_{n,n \wedge k})^{-\alpha+\epsilon} \gamma_n^{-\alpha} H(m_n) \right\} \\
&\leq \sup_{(2C_0)^{-1} \leq y < \infty} \left| \frac{(m_n y)^{-\epsilon} H(m_n y)}{m_n^{-\epsilon} H(m_n)} - y^{-\epsilon} \right| \sum_{k=0}^{\infty} \kappa_n^{\epsilon(1-m)} (\gamma_n^{-1} S_{n,n \wedge k})^{-\alpha+\epsilon} \gamma_n^{-\alpha} H(m_n) \\
&= \sup_{(2C_0)^{-1} \leq y < \infty} \left| \frac{(m_n y)^{-\epsilon} H(m_n y)}{m_n^{-\epsilon} H(m_n)} - y^{-\epsilon} \right| \frac{H(m_n)}{\kappa_n^{b_\epsilon} H(m_n)} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n \wedge k} |\rho_n^j c_{k-j}| \right)^{-\alpha+\epsilon} [1 + o(1)] \\
&= o(1) \cdot \frac{1}{\kappa_n^{b_\epsilon}} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n \wedge k} |\rho_n^j c_{k-j}| \right)^{\alpha-\epsilon} [1 + o(1)] =_{(3)} o(1), \tag{146}
\end{aligned}$$

where $=_{(3)}$ is due to the fact that

$$\begin{aligned}
&\frac{1}{\kappa_n^{b_\epsilon}} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n \wedge k} |\rho_n^j c_{k-j}| \right)^{\alpha-\epsilon} \leq \frac{1}{\kappa_n^{b_\epsilon}} \sum_{k=0}^n \left(\sum_{j=0}^k |\rho_n^j c_{k-j}| \right)^{\alpha-\epsilon} \\
&+ \frac{1}{\kappa_n^{b_\epsilon}} \sum_{k=n}^{\infty} \left(\sum_{j=0}^n |\rho_n^j c_{k-j}| \right)^{\alpha-\epsilon} =_{(4)} \mathbb{V}_1(\alpha - \epsilon) + o(1) = O(1). \tag{147}
\end{aligned}$$

We note that $=_{(4)}$ is a direct consequence of **Lemma 15** and **Lemma 17**. Combining (145)-(147) we have

$$\sup_{1 \leq t \leq n} B_{1n,t} \leq C \sum_{k=0}^{\infty} S_{n,n \wedge k}^{-\alpha} H(S_{n,n \wedge k}) [1 + o(1)]$$

as

$$\begin{aligned}
&= C \sum_{k=0}^{\infty} S_{n,n \wedge k}^{-\alpha} H(m_n) [1 + o(1)] \leq \frac{1}{\kappa_n^b} \sum_{k=0}^n \left(\sum_{j=0}^k |\rho_n^j c_{k-j}| \right)^{\alpha} \\
&+ \frac{1}{\kappa_n^b} \sum_{k=n}^{\infty} \left(\sum_{j=0}^n |\rho_n^j c_{k-j}| \right)^{\alpha} =_{(5)} |m_0|^\alpha \mathbb{V}_1(\alpha) + o(1) = O(1), \tag{148}
\end{aligned}$$

where $=_{(5)}$ is again a consequence of **Lemma 15** and **Lemma 17**.

Finally, identical arguments to those used above show that

$$\sup_{1 \leq t \leq n} B_{2n,t} = O(1).$$

Hence, the proof is complete. $\blacksquare$

MI-NM ($m \in (1, 1 + 1/\alpha)$). Noting that $0 < \rho_n < 1$ for all n large enough we have as $n \rightarrow \infty$

$$\gamma_n \inf_{1 \leq k \leq n} |a_k|^{-1} = \gamma_n \frac{1}{\sup_{1 \leq k \leq n} |a_k|} \geq \frac{\gamma_n}{\sum_{j=0}^{\infty} |c_j|} \rightarrow \infty. \tag{149}$$

As in the previous part, set

$$A(n) := \frac{1}{\gamma_n^\alpha} \sum_{k=1}^{n-1} |a_k|^\alpha H(|\gamma_n a_k^{-1}|).$$

Noting that $\gamma_n \inf_{1 \leq k \leq n} |a_k^{-1}| \rightarrow \infty$ and, using the same arguments as those used for in the previous parts (see e.g., (106) and (109)), we get

$$\sup_{1 \leq t \leq n} A_{n,t} := \sup_{1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{t-1} a_k \xi_{t-k} \right|^{\alpha-\varepsilon} \leq C \left[A(n) + A(n)^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right] [1 + o(1)],$$

for all $\varepsilon > 0$ such that $\alpha - \varepsilon > 0$. Therefore, it suffices to show that $A(n) = O(1)$, which is what we set out to do next.

As $n \rightarrow \infty$, the difference

$$\begin{aligned} & \left| A(n) - \frac{1}{\gamma_n^\alpha} \sum_{k=1}^{n-1} |a_k|^\alpha \right| \\ & \leq \sup_{1 \leq k \leq n} \left| H(|\gamma_n a_k^{-1}|) - 1 \right| \frac{1}{\gamma_n^\alpha} \sum_{k=1}^{n-1} |a_k|^\alpha =_{(1)} o(1) \cdot \frac{1}{\gamma_n^\alpha} \sum_{k=1}^{n-1} |a_k|^\alpha \\ & =_{(2)} o(1) \cdot \{|m_0|^\alpha \mathbb{V}_2(\alpha, m) [1 + o(1)]\} = o(1), \end{aligned} \quad (150)$$

where $=_{(1)}$ is due to **Assumption NM**, and (149), whilst $=_{(2)}$ follows from **Lemma 18**. Hence,

$$A(n) = \frac{1}{\gamma_n^\alpha} \sum_{k=1}^{n-1} |a_k|^\alpha + o(1) = |m_0|^\alpha \mathbb{V}_2(\alpha, m) + o(1),$$

and the proof is complete.

(b) Next, we show part (b). Recall that under **BFR1** $a_{k,t}^- = \sum_{j=0}^{t-1} c_{k-j}$. Write

$$\begin{aligned} x_t^- &= \sum_{k=t}^{\infty} a_{k,t}^- \xi_{t-k} = \sum_{k=t}^{\infty} \sum_{j=0}^{t-1} c_{k-j} \xi_{t-k} \\ & [-i = t - k \iff k = t + i], [l = t - j \iff j = t - l] \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{t-1} c_{t+i-j} \xi_{-i} = \sum_{i=0}^{\infty} \sum_{l=0}^t c_{i+l} \xi_{-i}. \end{aligned}$$

Hence,

$$\begin{aligned} x_t^- - x_s^- &= \sum_{i=0}^{\infty} \sum_{l=0}^t c_{i+l} \xi_{-i} - \sum_{i=0}^{\infty} \sum_{l=0}^s c_{i+l} \xi_{-i} \\ &= \sum_{i=0}^{\infty} \sum_{l=s+1}^t c_{i+l} \xi_{-i} = \sum_{i=0}^{\infty} \sum_{j=i+s+1}^{i+t} c_j \xi_{-i} \stackrel{d}{=} \sum_{i=0}^{\infty} \sum_{j=i+s+1}^{i+t} c_j \xi_i \\ & [i = k - t \iff k = i + t] \\ &= \sum_{k=t}^{\infty} \sum_{j=k-(t-s)+1}^k c_j \xi_{k-t} \stackrel{d}{=} \sum_{k=t}^{\infty} \sum_{j=k-(t-s)+1}^k c_j \xi_k. \end{aligned}$$

Set $\varepsilon > 0$ such that $\alpha - \varepsilon > 0$. Further, $0 < s < r \leq 1$. In view of the above

$$\mathbf{E} \left| \gamma_n^{-1} (x_{[nr]}^- - x_{[ns]}^-) \right|^{\alpha-\varepsilon} = \mathbf{E} \left| \gamma_n^{-1} \sum_{k=[nr]}^{\infty} \sum_{j=k-([nr]-[ns])+1}^k c_j \xi_k \right|^{\alpha-\varepsilon}$$

$$=: \mathbf{E} \left| \gamma_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\infty} b_{k,n} \xi_k \right|^{\alpha-\varepsilon}.$$

Next, set $T_{n,t} := \gamma_n |b_{k,n}|^{-1}$. Note that as $n \rightarrow \infty$

$$T_n := \inf_{k \geq 1} T_{k,n} = \frac{\gamma_n}{\sup_{1 \leq k} |b_{k,n}|} \geq \frac{\gamma_n}{\sum_{j=0}^{\infty} |c_j|} \rightarrow \infty.$$

Hence, $T_{k,n} \rightarrow \infty$, uniformly in $k \geq 1$, as $n \rightarrow \infty$. Furthermore, set

$$\bar{\xi}_{k,n} := \xi_k 1\{|\xi_k| \geq T_{k,n}\} \text{ and } \underline{\xi}_{k,n} := \xi_k 1\{|\xi_k| < T_{k,n}\}.$$

By Lemma 1 in Astrauskas (1983), for $k \geq 1$ and $\alpha \in (0, 2]$ as $n \rightarrow \infty$ (and hence as $T_{k,n} \rightarrow \infty$) we have

$$\mathbf{E} |\bar{\xi}_{k,n}|^{\alpha-\varepsilon} \asymp T_{k,n}^{-\varepsilon} h(T_{k,n}) \text{ and } \mathbf{E} |\underline{\xi}_{k,n}|^{\alpha+\varepsilon} \asymp T_{k,n}^{\varepsilon} h(T_{k,n}).$$

In view of the above, (75), and using similar arguments as those used in the proof of part (a) we get as $n \rightarrow \infty$,

$$\begin{aligned} \mathbf{E} \left| \gamma_n^{-1} \sum_{k=\lfloor nr \rfloor}^{\infty} b_{k,n} \xi_k \right|^{\alpha-\varepsilon} &\leq 2\gamma_n^{-(\alpha-\varepsilon)} \sum_{k=\lfloor nr \rfloor}^{\infty} |b_{k,n}|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{k,n}|^{\alpha-\varepsilon} \\ &\quad + 2 \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=\lfloor nr \rfloor}^{\infty} |b_{k,n}|^{\alpha+\varepsilon} \mathbf{E} |\underline{\xi}_{k,n}|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\ &\leq 2\gamma_n^{-\alpha} \sum_{k=\lfloor nr \rfloor}^{\infty} |b_{k,n}|^{\alpha} H\left(\gamma_n |b_{k,n}|^{-1}\right) [1 + o(1)] \\ &\quad + 2 \left[\gamma_n^{-\alpha} \sum_{k=\lfloor nr \rfloor}^{\infty} |b_{k,n}|^{\alpha} H\left(\gamma_n |b_{k,n}|^{-1}\right) [1 + o(1)] \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}}, \end{aligned}$$

Next, note that

$$\gamma_n^{-\alpha} \sum_{k=\lfloor nr \rfloor}^{\infty} |b_{k,n}|^{\alpha} H\left(\gamma_n |b_{k,n}|^{-1}\right)$$

is the same form as the term $C''_{3,n}$ of (87) which is $o(1)$ -see also (97). This completes the proof of part (b).

(c) Finally, we show part (c) of this Lemma under **BFR2** and **MI-SM/LM**. The result under **MI-NM** holds trivially.

BFR2. The result under **Assumption BFR2** follows directly from eq. (171) cf. proof of **Lemma 9**.

MI-SM. Without loss of generality set $r = 1$. It follows from (130), (131) and (134) that

$$B_n(n)' := \frac{H(\gamma_n)}{\gamma_n^{\alpha}} \sum_{k=n}^{\infty} |a_{k,n}^{-}|^{\alpha} = o(1), \quad (151)$$

is sufficient for the requisite result -the notation used above is the same as that in the aforementioned equation.

We first prove (151) for $\alpha \in (0, 1]$. In view of the fact that, $\gamma_n^{-\alpha} H(\gamma_n) \sim \kappa_n^{-1}$ write

$$\begin{aligned} B_n(n)' &\leq C\kappa_n^{-1} \sum_{k=n}^{\infty} \sum_{j=0}^n \left| \rho_n^j c_{k-j} \right|^\alpha = C\kappa_n^{-1} \sum_{k=n}^{\infty} \left(\sum_{j=1}^{\lfloor n/2 \rfloor} + \sum_{j=\lfloor n/2 \rfloor + 1}^n \right) \left| \rho_n^j c_{k-j} \right|^\alpha \\ &\leq C\kappa_n^{-1} \sum_{j=1}^{\lfloor n/2 \rfloor} \left| \rho_n^{\alpha j} \right| \sum_{k=\lfloor n/2 \rfloor}^{\infty} |c_k|^\alpha + C\kappa_n^{-1} \sum_{j=\lfloor n/2 \rfloor + 1}^n \left| \rho_n^{\alpha j} \right| \sum_{k=0}^{\infty} |c_k|^\alpha = o(1), \end{aligned} \quad (152)$$

where we have used the fact that $\kappa_n^{-1} \sum_{j=1}^{\lfloor n/2 \rfloor} |\rho_n^{\alpha j}| = O(1)$ and $\kappa_n^{-1} \sum_{j=\lfloor n/2 \rfloor + 1}^n |\rho_n^{\alpha j}| = o(1)$ cf. DK2018, Lemma 6.4.

Next, suppose that $\alpha \in (1, 2]$. It follows from (133) that

$$B_n(n)' \leq 2C\kappa_n^{-1} \sum_{k=n}^{\infty} \left[\left(\sum_{j=0}^n \left| \rho_n^j c_{k-j} \right| \right)^2 + \sum_{j=0}^n \left| \rho_n^j c_{k-j} \right| \right] = o(1),$$

where the negligibility of the first term above follows from Lemma LVAR in DK2018, whilst the negligibility of the second term follows from (152) with $\alpha = 1$.

MI-LM. Without loss of generality set $r = 1$. Let $B_{n,n}$ as in (142). In view of From (145)-(148) we have

$$B_{n,n} = \frac{1}{\kappa_n^b} \sum_{k=n}^{\infty} \left(\sum_{j=0}^n \left| \rho_n^j c_{k-j} \right| \right)^\alpha [1 + o(1)] = o(1).$$

The requisite result follows easily from the above. ■

13. Technical Lemmas

Lemma 3. The following hold.

(i) Let $\varsigma(x) > 0$ SV and locally bounded on $[x_0 - 1, \infty)$ for some $x_0 - 1 > 0$. In addition, for some $\alpha > 0$

$$S(x, x_0) := \alpha \int_{x_0}^x \frac{\varsigma(u)}{u} du \rightarrow \infty$$

as $x \rightarrow \infty$. Moreover, suppose that $\tau \geq 0$, m_n is a positive sequence such that $m_n^{-1} + m_n^{\tau/\alpha} n^{-1} \rightarrow 0$, and $S(m_n n^{1/\alpha}, m_n^{1+\tau/\alpha} x_0^{1/\alpha}) \sim S(m_n n^{1/\alpha}, x_0)$ as $n \rightarrow \infty$. Then as $n \rightarrow \infty$,

$$\sum_{k=x_0 \lfloor m_n^\tau \rfloor}^n \varsigma(m_n k^{1/\alpha})/k \sim S(m_n n^{1/\alpha}, m_n^{1+\tau/\alpha} x_0^{1/\alpha}) = \alpha \int_{m_n^{1+\tau/\alpha} x_0^{1/\alpha}}^{m_n n^{1/\alpha}} \frac{\varsigma(x)}{x} dx.$$

(ii) Let $l > -1$, $m(x)$ SV, $q > 0$ and $q_n \rightarrow \infty$. As $n \rightarrow \infty$

$$\frac{1}{n^{1+l} m(q_n n^q)} \sum_{j=1}^n j^l m(q_n j^q) = \int_0^1 x^l dx + o(1).$$

Proof of Lemma 3.

(i) By Theorem 1.3.3 and Theorem 1.5.3 of BGT 1987 $\varsigma(x^{1/\alpha})/x$ can be approximated (in the “ $\sim$ ” sense) by a regularly varying function of index -1 that is continuous on $[x'_0, \infty)$ with $x'_0 > 0$ and decreasing. In view of this, there is no loss of generality if in the subsequent derivations we assume that $\varsigma(x^{1/\alpha})/x$ is continuous and decreasing for $x \geq x'_0$. For n large enough, such that $m_n (x_0 - 1)^{1/\alpha} \geq x'_0$, write

$$\begin{aligned} \int_{x_0 \lfloor m_n^\tau \rfloor}^n \frac{\varsigma(m_n x^{1/\alpha})}{x} dx &= \int_{u=m_n \lfloor m_n^{\tau/\alpha} \rfloor x_0^{1/\alpha}}^{m_n n^{1/\alpha}} \frac{\varsigma(u)}{m_n^{-\alpha} u^\alpha} m_n^{-\alpha} \alpha u^{\alpha-1} du \\ &= \alpha \int_{u=m_n \lfloor m_n^{\tau/\alpha} \rfloor x_0^{1/\alpha}}^{m_n n^{1/\alpha}} \frac{\varsigma(u)}{u} du \sim_{(1)} \alpha \int_{u=m_n^{1+\tau/\alpha} x_0^{1/\alpha}}^{m_n n^{1/\alpha}} \frac{\varsigma(u)}{u} du \\ &= S(m_n n^{1/\alpha}, m_n x_0^{1/\alpha}) =: S_n, \end{aligned}$$

where note that under our assumptions we have $S_n \rightarrow \infty$. In addition, $\sim_{(1)}$ follows from the fact that $S(x, x_0)$ is SV (cf. BGT 1987; Proposition 1.5.9a). To see this, first note that it be easily checked that

$$S(m_n^{1+\tau/\alpha} x_0^{1/\alpha}, x_0) \sim S(m_n \lfloor m_n^{\tau/\alpha} \rfloor x_0^{1/\alpha}, x_0). \quad (153)$$

Next, write

$$S(m_n n^{1/\alpha}, m_n \lfloor m_n^{\tau/\alpha} \rfloor x_0^{1/\alpha}) = S(m_n n^{1/\alpha}, x_0) - S(m_n \lfloor m_n^{\tau/\alpha} \rfloor x_0^{1/\alpha}, x_0).$$

Hence,

$$\frac{S(m_n n^{1/\alpha}, m_n \lfloor m_n^{\tau/\alpha} \rfloor x_0^{1/\alpha})}{S(m_n n^{1/\alpha}, x_0)} = 1 - \frac{S(m_n \lfloor m_n^{\tau/\alpha} \rfloor x_0^{1/\alpha}, x_0)}{S(m_n n^{1/\alpha}, x_0)}$$

$$\begin{aligned}
&= 1 - \frac{S(m_n^{1+\tau/\alpha} x_0^{1/\alpha}, x_0)}{S(m_n n^{1/\alpha}, x_0)} \frac{S(m_n \lfloor m_n^{\tau/\alpha} \rfloor x_0^{1/\alpha}, x_0)}{S(m_n^{1+\tau/\alpha} x_0^{1/\alpha}, x_0)} \\
&=_{(2)} 1 - \frac{S(m_n^{1+\tau/\alpha} x_0^{1/\alpha}, x_0)}{S(m_n n^{1/\alpha}, x_0)} [1 + o(1)] =_{(3)} 1 + o(1),
\end{aligned}$$

where $=_{(2)}$ follows from (153). For $=_{(3)}$ recall that $S(m_n n^{1/\alpha}, x_0) \sim S(m_n n^{1/\alpha}, m_n^{1+\tau/\alpha} x_0^{1/\alpha})$. Therefore,

$$\frac{S(m_n^{1+\tau/\alpha} x_0^{1/\alpha}, x_0)}{S(m_n n^{1/\alpha}, x_0)} = \frac{S(m_n n^{1/\alpha}, x_0) - S(m_n n^{1/\alpha}, m_n^{1+\tau/\alpha} x_0^{1/\alpha})}{S(m_n n^{1/\alpha}, x_0)} = o(1).$$

Next, note that because $\varsigma(x^{1/\alpha})/x$ is continuous we can write

$$\begin{aligned}
S_n^{-1} \sum_{k=x_0}^n \varsigma(m_n k^{1/\alpha})/k &= S_n^{-1} \int_{x_0-1}^n \frac{\varsigma(m_n x^{1/\alpha})}{x} d\lfloor x \rfloor \\
&=_{(4)} S_n^{-1} \int_{x_0}^n \frac{\varsigma(m_n x^{1/\alpha})}{x} d\lfloor x \rfloor + o(1),
\end{aligned}$$

where $=_{(4)}$ follows from the fact that $\varsigma(x) = o(S(x, x_0))$ (cf. BGT 1987; Proposition 1.5.9a) i.e.

$$\begin{aligned}
S_n^{-1} \int_{x_0-1}^{x_0} \frac{\varsigma(m_n x^{1/\alpha})}{x} d\lfloor x \rfloor &= S_n^{-1} \frac{\varsigma(m_n x_0^{1/\alpha})}{x_0} \\
&\sim S(m_n n^{1/\alpha}, x_0)^{-1} \frac{\varsigma(m_n x_0^{1/\alpha})}{x_0} \\
&\leq S(m_n x_0^{1/\alpha}, x_0)^{-1} \frac{\varsigma(m_n x_0^{1/\alpha})}{x_0} = o(1).
\end{aligned} \tag{154}$$

In addition, using the fact that $\varsigma(x^{1/\alpha})/x$ is decreasing the total variation of $\varsigma(m_n x^{1/\alpha})x^{-1}$ on $[x_0, n]$ is

$$\begin{aligned}
\mathbf{V} \left(\frac{\varsigma(m_n x^{1/\alpha})}{x} \right)_{x_0}^n &= m_n^\alpha \mathbf{V} \left(\frac{\varsigma(m_n x^{1/\alpha})}{m_n^\alpha x} \right)_{x_0}^n \\
&\leq m_n^\alpha \frac{\varsigma(m_n x_0^{1/\alpha})}{m_n^\alpha x_0} = \frac{\varsigma(m_n x_0^{1/\alpha})}{x_0} = o(S_n),
\end{aligned}$$

where the last equality above follows from (154). In view of the above,

$$\begin{aligned}
\left| S_n^{-1} \sum_{k=x_0}^n \varsigma(m_n k^{1/\alpha})/k - 1 \right| &= S_n^{-1} \left| \sum_{k=x_0}^n \varsigma(m_n k^{1/\alpha})/k - \int_{x_0}^n \frac{\varsigma(m_n x^{1/\alpha})}{x} dx \right| \\
&= S_n^{-1} \left| \int_{x_0}^n \frac{\varsigma(m_n x^{1/\alpha})}{x} d\lfloor x \rfloor - \int_{x_0}^n \frac{\varsigma(m_n x^{1/\alpha})}{x} dx \right| + o(1) \\
&= S_n^{-1} \left| \int_{x_0}^n \frac{\varsigma(m_n x^{1/\alpha})}{x} d(x - \lfloor x \rfloor) \right| + o(1) \\
&= S_n^{-1} \left| \int_{x_0}^n (x - \lfloor x \rfloor) d \left(\frac{\varsigma(m_n x^{1/\alpha})}{x} \right) \right| + o(1)
\end{aligned}$$

$$\begin{aligned}
&\leq S_n^{-1} \int_{x_0}^n |x - \lfloor x \rfloor| d\mathbf{V} \left(\frac{\varsigma(m_n x^{1/\alpha})}{x} \right) + o(1) \\
&\leq S_n^{-1} \int_{x_0}^n d\mathbf{V} \left(\frac{\varsigma(m_n x^{1/\alpha})}{x} \right) + o(1) \\
&\leq S_n^{-1} \mathbf{V} \left(\frac{\varsigma(m_n x^{1/\alpha})}{x} \right)_{x_0}^n + o(1) \leq S_n^{-1} \frac{\varsigma(m_n x_0^{1/\alpha})}{x_0} = o(1),
\end{aligned}$$

as required.

(ii) Let $l_2 > -1$ with $l_1 := l - l_2 > 0$, and set $R_r(x) := x^r m(x)$, $r \in \mathbb{R}$. Then

$$\begin{aligned}
&\left| \frac{1}{n^{1+l} m(q_n n^q)} \sum_{j=1}^n j^l m(q_n j^q) - \frac{1}{n^{1+l}} \sum_{j=1}^n j^l \right| \leq \frac{1}{n} \sum_{j=1}^n \left| \frac{(q_n j^q)^{l_1/q} m(q_n j^q)}{(q_n n^q)^{l_1/q} m(q_n n^q)} - \left(\frac{j^q}{n^q} \right)^{l_1/q} \right| \\
&= \frac{1}{n} \sum_{j=1}^n \left| \frac{(q_n j^q)^{l_1/q} m(q_n j^q)}{(q_n n^q)^{l_1/q} m(q_n n^q)} - \left(\frac{j^q}{n^q} \right)^{l_1/q} \right| \left(\frac{j^q}{n^q} \right)^{l_2/q} = \frac{1}{n} \sum_{j=1}^n \left| \frac{R_{l_1/q}(q_n j^q)}{R_{l_1/q}(q_n n^q)} - \left(\frac{j^q}{n^q} \right)^{l_1/q} \right| \left(\frac{j^q}{n^q} \right)^{l_2/q} \\
&\leq \sup_{0 < y \leq 1} \left| \frac{R_{l_1/q}(q_n n^q y)}{R_{l_1/q}(q_n n^q)} - y^{l_1/q} \right| \frac{1}{n} \sum_{j=1}^n \left(\frac{j^q}{n^q} \right)^{l_2/q} \\
&= \sup_{0 < y \leq 1} \left| \frac{R_{l_1/q}(q_n n^q y)}{R_{l_1/q}(q_n n^q)} - y^{l_1/q} \right| \left[\int_0^1 x^{l_2} dx + o(1) \right] = o(1),
\end{aligned}$$

where the last equality above is a consequence of the uniform convergence of regularly varying functions -see e.g. Theorem 1.5.2. in BGT 1987. In view of the above, the result follows from the fact that $n^{-(1+l)} \sum_{j=1}^n j^l = \int_0^1 x^l dx + o(1)$. ■

Lemma 4. Let $0 < m_n$ an integer valued sequence such that $m_n^{-1} + m_n \kappa_n^{-1} \rightarrow 0$. Under **Assumption MI-SM**, for all $r > 0$,

$$\begin{aligned}
\kappa_n^{-1} \sum_{k=1}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha &= \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha + o(1) \\
&= \alpha^{-1} \left| \sum_{l=0}^{\infty} c_l \right|^\alpha + o(1).
\end{aligned}$$

Lemma 5. Under **Assumption MI-SM**, the following hold:

(a) As $n \rightarrow \infty$,

$$\kappa_n^{-1} \sum_{k=1}^n |a_k - \rho_n^k c_\infty|^\alpha = o(1).$$

(b) For each $0 < r \leq 1$, as $n \rightarrow \infty$,

$$\kappa_n^{-1} \sum_{k=1}^{\lfloor nr \rfloor} |a_k|^\alpha = \kappa_n^{-1} \sum_{k=1}^{\lfloor nr \rfloor} |\rho_n^k c_\infty|^\alpha + o(1) = \alpha^{-1} \left| \sum_{l=0}^{\infty} c_l \right|^\alpha + o(1).$$

(c) For each $0 < r \leq 1$, as $n \rightarrow \infty$,

$$\kappa_n^{-1} \sum_{k=1}^{\lfloor nr \rfloor} |a_k|^\alpha \operatorname{sgn} \left(\frac{1}{\gamma_n} a_k \right) = \kappa_n^{-1} \sum_{k=1}^{\lfloor nr \rfloor} |a_k|^\alpha \operatorname{sgn} (\rho_n^k c_\infty) + o(1).$$

Proof of Lemma 4. Note that for n large enough, $0 < \rho_n^j \leq 1$, for all $j \geq 0$. Hence,

$$\begin{aligned} \kappa_n^{-1} \sum_{k=1}^{m_n} \left| \sum_{l=0}^k \rho_n^{k-l} c_l \right|^\alpha &\leq \kappa_n^{-1} \sum_{k=1}^{m_n} \left(\sum_{l=0}^k \rho_n^{k-l} |c_l| \right)^\alpha \\ &\leq \kappa_n^{-1} \sum_{k=1}^{m_n} \left(\sum_{l=0}^\infty |c_l| \right)^\alpha = O\left(\frac{m_n}{\kappa_n}\right) \rightarrow 0. \end{aligned} \quad (155)$$

Next, consider

$$\begin{aligned} \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha &= \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} \left| \sum_{l=0}^k \rho_n^{k-l} c_l \right|^\alpha \\ &= \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} \left| \left(\sum_{l=0}^{m_n} + \sum_{l=m_n+1}^k \right) \rho_n^{k-l} c_l \right|^\alpha. \end{aligned} \quad (156)$$

Using the fact $\lim_{n \rightarrow \infty} \rho_n^{-l} = 1$ for each l and $\sum_{l=0}^\infty |c_l| < \infty$ dominated convergence gives $\sum_{l=0}^{m_n} |\rho_n^{-l} c_l - c_l| \rightarrow 0$. It follows from DK2018, Lemma 6.4, that

$$\begin{aligned} \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} \left| \sum_{l=0}^{m_n} \rho_n^{k-l} c_l \right|^\alpha &= \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} |\rho_n^{\alpha k}| \left| \sum_{l=0}^{m_n} \rho_n^{-l} c_l \right|^\alpha \\ &= \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} |\rho_n^{\alpha k}| \left| \sum_{l=0}^{m_n} c_l \right|^\alpha [1 + o(1)] \\ &= \int_{m_n/\kappa_n}^{\lfloor (n-1)r \rfloor/\kappa_n} e^{-\alpha x} dx \left| \sum_{l=0}^\infty c_l \right|^\alpha + o(1) \\ &= \int_0^\infty e^{-\alpha x} dx \left| \sum_{l=0}^\infty c_l \right|^\alpha + o(1) = \alpha^{-1} \left| \sum_{l=0}^\infty c_l \right|^\alpha. \end{aligned} \quad (157)$$

Next, set

$$R_n(\alpha) := \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} \left| \sum_{l=m_n+1}^k \rho_n^{k-l} c_l \right|^\alpha.$$

We shall show that

$$R_n(\alpha) = o(1). \quad (158)$$

First, we consider the $\alpha \in (0, 1]$. As $n \rightarrow \infty$

$$\begin{aligned} R_n(\alpha) &\leq \kappa_n^{-1} \sum_{k=m_n}^n \sum_{l=m_n+1}^k \rho_n^{\alpha(k-l)} |c_l|^\alpha \\ &= \kappa_n^{-1} \sum_{l=m_n+1}^n \sum_{k=m_n \vee l}^n \rho_n^{\alpha(k-l)} |c_l|^\alpha \leq \kappa_n^{-1} \sum_{l=m_n}^n \rho_n^{-\alpha l} |c_l|^\alpha \sum_{k=l}^n \rho_n^{\alpha k} \\ &= \kappa_n^{-1} \sum_{l=m_n}^n \rho_n^{-\alpha l} |c_l|^\alpha \left[\frac{1 - \rho_n^{\alpha(n+1)}}{1 - \rho_n^\alpha} - \frac{1 - \rho_n^{\alpha l}}{1 - \rho_n^\alpha} \right] \\ &= \frac{1}{1 - \rho_n^\alpha} \kappa_n^{-1} \sum_{l=m_n}^n \rho_n^{-\alpha l} |c_l|^\alpha \left[\rho_n^{\alpha l} - \rho_n^{\alpha(n+1)} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1 - \rho_n^\alpha} \kappa_n^{-1} \sum_{l=m_n}^n |c_l|^\alpha - \frac{\rho_n^{\alpha(n+1)}}{1 - \rho_n^\alpha} \kappa_n^{-1} \sum_{l=m_n}^n |c_l|^\alpha \\
&=_{(1)} [1 + o(1)] C \sum_{l=m_n}^\infty |c_l|^\alpha = o(1),
\end{aligned}$$

where $=_{(1)}$ follows from the fact that $\rho_n^{\alpha(n+1)} = o(1)$ and the fact that $(1 - \rho_n^\alpha)^{-1} \kappa_n^{-1} \rightarrow 1/\alpha$. For the latter, see Footnote 11. Moreover, an argument similar to the one shows that (129), and $R_n(\alpha) = o(1)$ for $\alpha \in (1, 2]$. Next, we show that

$$R'_n := \left| \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha - \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} \left| \sum_{l=0}^{m_n} \rho_n^{k-l} c_l \right|^\alpha \right| = o(1)$$

First, suppose that $0 < \alpha \leq 1$. In view of (156), for

$$\begin{aligned}
R'_n &= \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} \left| \left(\sum_{l=0}^{m_n} + \sum_{l=m_n+1}^k \right) \rho_n^{k-l} c_l \right|^\alpha - \left| \sum_{l=0}^{m_n} \rho_n^{k-l} c_l \right|^\alpha \\
&\leq_{(2)} R_n(\alpha) =_{(3)} o(1),
\end{aligned} \tag{159}$$

where $\leq_{(2)}$ follows from Loève's inequality and $=_{(3)}$ from the fact that $R_n(\alpha) = o(1)$. Next, suppose that $1 < \alpha \leq 2$. Using the fact that, see (42),

$$||y + x|^\alpha - |y|^\alpha| \leq C \left(|y|^{\alpha-1} |x| + |x|^\alpha \right),$$

for n large enough,

$$\begin{aligned}
R'_n &\leq \kappa_n^{-1} \sum_{k=m_n}^{n-1} \left| \sum_{l=0}^{m_n} \rho_n^{k-l} c_l \right|^{\alpha-1} \left| \sum_{l=m_n+1}^k \rho_n^{k-l} c_l \right| + \kappa_n^{-1} \sum_{k=m_n}^{n-1} \left| \sum_{l=m_n+1}^k \rho_n^{k-l} c_l \right|^\alpha \\
&\leq_{(4)} R_n(1) \sum_{l=0}^{m_n} |c_l| + R_n(\alpha) \leq R_n(1) \sum_{l=0}^\infty |c_l| + R_n(\alpha) = o(1),
\end{aligned}$$

where $\leq_{(4)}$ follows from the fact that $|\rho_n^{k-l}| \leq 1$ for n large and all $k \geq l$, the last approximation follows from the fact that $R_n(\alpha) = o(1)$, $\alpha \in (0, 2]$. Therefore, for all $\alpha \in (0, 2]$

$$R'_n = o(1). \tag{160}$$

Using (155)-(158) and (160) for all $\alpha \in (0, 2]$ we get

$$\begin{aligned}
&\kappa_n^{-1} \sum_{k=1}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha = \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} |a_k|^\alpha + o(1) \\
&= \kappa_n^{-1} \sum_{k=m_n}^{\lfloor (n-1)r \rfloor} \left| \sum_{l=0}^{m_n} \rho_n^{k-l} c_l \right|^\alpha + o(1) = \alpha^{-1} \left| \sum_{l=0}^\infty c_l \right|^\alpha + o(1),
\end{aligned}$$

as required. ■

Proof of Lemma 5. (a) Write

$$\kappa_n^{-1} \sum_{k=1}^n \left| a_k - \rho_n^k c_\infty \right|^\alpha$$

$$\begin{aligned}
&\leq C\kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=0}^k \rho_n^{k-l} c_l - \sum_{l=0}^k \rho_n^k c_l \right|^\alpha + C\kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=k+1}^\infty \rho_n^k c_l \right|^\alpha \\
&=: C [R_{1n}(\alpha) + R_{2n}(\alpha)].
\end{aligned}$$

We shall show that as $n \rightarrow \infty$

$$R_{1n}(\alpha) + R_{2n}(\alpha) = o(1). \quad (161)$$

First, suppose that $\alpha \in (0, 1]$. Note that for n large enough, $0 < \rho_n^j \leq 1$, for all $j \geq 0$. In addition, for each $j \geq 0$ fixed $\rho_n^j \rightarrow 1$, as $n \rightarrow \infty$. In view of this, as $n \rightarrow \infty$, the first term above is

$$\begin{aligned}
R_{1n}(\alpha) &= \kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=0}^k \rho_n^{k-l} (1 - \rho_n^l) c_l \right|^\alpha \leq \kappa_n^{-1} \sum_{k=1}^n \sum_{l=0}^k \rho_n^{\alpha(k-l)} (1 - \rho_n^l)^\alpha |c_l|^\alpha \\
&= \kappa_n^{-1} \sum_{l=0}^n (1 - \rho_n^l)^\alpha |c_l|^\alpha \sum_{k=l}^n \rho_n^{\alpha(k-l)} = \kappa_n^{-1} \sum_{l=0}^n (1 - \rho_n^l)^\alpha |c_l|^\alpha \sum_{k=l}^n \rho_n^{\alpha(k-l)} \\
&= \kappa_n^{-1} \sum_{l=0}^n (1 - \rho_n^l)^\alpha |c_l|^\alpha \sum_{k=0}^{n-l} \rho_n^{\alpha k} \leq \kappa_n^{-1} \sum_{l=0}^n (1 - \rho_n^l)^\alpha |c_l|^\alpha \sum_{k=0}^n \rho_n^{\alpha k} \\
&\leq_{(1)} C \sum_{l=0}^\infty (1 - \rho_n^l)^\alpha |c_l|^\alpha =_{(2)} o(1),
\end{aligned}$$

where $\leq_{(1)}$ follows from the fact that $\kappa_n^{-1} \sum_{k=0}^n \rho_n^{\alpha k} = O(1)$, and $=_{(2)}$ from dominated convergence. Next, let $m_n > 0$ an integer valued sequence such that $m_n^{-1} + m_n \kappa_n^{-1} \rightarrow 0$. As $n \rightarrow \infty$, the second term

$$\begin{aligned}
R_{2n}(\alpha) &\leq \kappa_n^{-1} \sum_{k=1}^n \sum_{l=k+1}^\infty \rho_n^{\alpha k} |c_l|^\alpha \\
&= \kappa_n^{-1} \left(\sum_{k=m_n+1}^n + \sum_{k=1}^{m_n} \right) \rho_n^{\alpha k} \sum_{l=k+1}^\infty |c_l|^\alpha \\
&\leq \kappa_n^{-1} \sum_{k=m_n+1}^n \rho_n^{\alpha k} \sum_{l=m_n+1}^\infty |c_l|^\alpha + \kappa_n^{-1} \sum_{k=1}^{m_n} \rho_n^{\alpha k} \sum_{l=0}^\infty |c_l|^\alpha \\
&= C \sum_{l=m_n+1}^\infty |c_l|^\alpha + O(m_n/\kappa_n) = o(1).
\end{aligned}$$

Hence, (161) holds.

Next, suppose that $1 < \alpha \leq 2$. First, consider $R_{1n}(\alpha)$. As $n \rightarrow \infty$

$$\begin{aligned}
\kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=0}^k \rho_n^{k-l} c_l - \sum_{l=0}^k \rho_n^k c_l \right|^\alpha &= \kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=0}^k \rho_n^{k-l} (1 - \rho_n^l) c_l \right|^\alpha \\
&= \kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=0}^k \rho_n^{k-l} (1 - \rho_n^l) c_l \right|^{\alpha-1} \left| \sum_{l=0}^k \rho_n^{k-l} (1 - \rho_n^l) c_l \right| \\
&\leq \kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=0}^\infty 2 |c_l| \right|^{\alpha-1} \left| \sum_{l=0}^k \rho_n^{k-l} (1 - \rho_n^l) c_l \right|
\end{aligned}$$

$$= C \kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=0}^k \rho_n^{k-l} (1 - \rho_n^l) c_l \right| = C R_{1n}(1) = o(1).$$

Finally, as $n \rightarrow \infty$

$$\begin{aligned} R_{2n}(\alpha) &= \kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=k+1}^{\infty} \rho_n^k c_l \right|^{\alpha-1} \left| \sum_{l=k+1}^{\infty} \rho_n^k c_l \right| \\ &\leq \left(\sum_{l=k+1}^{\infty} |c_l| \right)^{\alpha-1} \kappa_n^{-1} \sum_{k=1}^n \left| \sum_{l=k+1}^{\infty} \rho_n^k c_l \right| \\ &= C R_{2n}(1) = o(1), \end{aligned}$$

as required. The proof for $0 < \alpha \leq 1$ is analogous for the one provided above and therefore omitted.

(b) We will prove the result for the case where $1 < \alpha \leq 2$ only. Recall that by (42),

$$\begin{aligned} \kappa_n^{-1} \sum_{k=1}^n \left| |a_k|^\alpha - |\rho_n^k c_\infty|^\alpha \right| &\leq C \cdot \left[\kappa_n^{-1} \sum_{k=1}^n \left| \rho_n^k c_\infty \right|^{\alpha-1} |a_k - \rho_n^k c_\infty| + \kappa_n^{-1} \sum_{k=1}^n |a_k - \rho_n^k c_\infty|^\alpha \right] \\ &\leq C \cdot \left[\kappa_n^{-1} \sum_{k=1}^n |a_k - \rho_n^k c_\infty| + \kappa_n^{-1} \sum_{k=1}^n |a_k - \rho_n^k c_\infty|^\alpha \right] = o(1), \end{aligned}$$

where the last approximation follows from part (a) above. This shows the first approximation of part (b).

Finally, we show the second approximation. By Lemma 6.4 in DK2018 we get

$$\begin{aligned} \kappa_n^{-1} \sum_{k=1}^{\lfloor (n-1)r \rfloor} \left| \rho_n^k c_\infty \right|^\alpha &= |c_\infty|^\alpha \kappa_n^{-1} \sum_{k=1}^{\lfloor (n-1)r \rfloor} \rho_n^{\alpha k} \\ &= |c_\infty|^\alpha \kappa_n^{-1} \left[\sum_{k=1}^{\kappa_n \ln(\kappa_n)} \rho_n^{\alpha k} + \sum_{k=\kappa_n \ln(\kappa_n)}^{\lfloor (n-1)r \rfloor} \rho_n^{\alpha k} \right] = |c_\infty|^\alpha \kappa_n^{-1} \sum_{k=1}^{\kappa_n \ln(\kappa_n)} \rho_n^{\alpha k} + o(1) \\ &= |c_\infty|^\alpha \kappa_n^{-1} \sum_{k=1}^{\kappa_n \ln(\kappa_n)} e^{-\alpha \frac{k}{\kappa_n}} + o(1) = |c_\infty|^\alpha \int_{1/\kappa_n}^{\kappa_n \ln(\kappa_n)/\kappa_n} e^{-\alpha x} dx + o(1) \\ &= |c_\infty|^\alpha \int_0^\infty e^{-\alpha x} dx + o(1) = \alpha^{-1} |c_\infty|^\alpha + o(1), \end{aligned}$$

as required.

(c) Set

$$K_{n,k}(r) = \left\{ 1 \leq k \leq \lfloor nr \rfloor : \operatorname{sgn} \left(\frac{1}{\gamma_n} a_k \right) \neq \operatorname{sgn} \left(\rho_n^k c_\infty \right) \right\}.$$

Next, note that as $n \rightarrow \infty$,

$$\begin{aligned} \kappa_n^{-1} \sum_{k=1}^{\lfloor nr \rfloor} |a_k|^\alpha \left| \operatorname{sgn} \left(\frac{1}{\gamma_n} a_k \right) - \operatorname{sgn} \left(\rho_n^k c_\infty \right) \right| \\ = \kappa_n^{-1} \sum_{k \in K_{n,k}(r)} |a_k|^\alpha \left| \operatorname{sgn} \left(\frac{1}{\gamma_n} a_k \right) - \operatorname{sgn} \left(\rho_n^k c_\infty \right) \right| \end{aligned}$$

$$\leq_{(2)} 2\kappa_n^{-1} \sum_{k=1}^n \left| a_k - \rho_n^k c_\infty \right|^\alpha =_{(3)} o(1),$$

where $\leq_{(2)}$ follows from the fact that

$$|X|^\alpha |\operatorname{sgn}(X) - \operatorname{sgn}(Y)| \leq 2|X - Y|^\alpha,$$

while $=_{(3)}$ follows from part (a). ■

Lemma 6. Let $\delta_1 > 0$, $\delta_2 > 1$, and $\delta := \delta_2 - \delta_1 > 0$. Suppose that **Assumption BFR1** holds. Define

$$I_t(k) := \int_{k-\delta_1 t}^k x^{-(1+\frac{1}{\alpha})} dx = \alpha \left[(k - \delta_1 t)^{-1/\alpha} - k^{-1/\alpha} \right],$$

and

$$Q_t(k) := \int_k^{k+\delta_1 t} x^{-(1+\frac{1}{\alpha})} dx = \alpha \left[k^{-1/\alpha} - (k + \delta_1 t)^{-1/\alpha} \right].$$

(a) Fix $\eta \in (0, 1)$. Then for some $t_0 \in \mathbb{N}$ large enough, depending on η , we have

$$(i) \left| \sum_{j=k-\delta_1 t}^k c_j / I_t(k) - 1 \right| < \eta, \text{ for } k \geq \delta_2(t + t_0) + 1;$$

$$(ii) \left| \sum_{j=k}^{k+\delta_1 t} c_j / Q_t(k) - 1 \right| < \eta, \text{ for } \delta t + \delta_2 t_0 \geq k \geq t_0 \text{ and } t \geq t_0;$$

(b) Let $t_0 \in \mathbb{N}$. As $t \rightarrow \infty$,

$$\inf_{k \geq \delta_2(t+t_0)+1} \gamma_t I_t(k)^{-1}, \inf_{\delta t + \delta_2 t_0 \geq k \geq t_0} \gamma_t Q_t(k)^{-1} \rightarrow \infty.$$

Proof of Lemma 6. Let $0 < \eta < 1$ and choose t_0 large enough such that

$$\left| c_j j^{(1+\frac{1}{\alpha})} - 1 \right|, \frac{3}{2} \left\{ \alpha j \left[1 - \left(\frac{2\delta_2 - \delta_1}{2\delta_2} \right)^{1/\alpha} \right] \right\}^{-1}, \frac{3}{2\delta_1} \left(\frac{\delta_2}{\delta_2 - \delta_1} \right)^{(1+\frac{1}{\alpha})} j^{-1} < \eta/2, \quad (162)$$

for $j \geq t_0$. Further, set $t' := t + t_0$. Euler summation yields

$$\left| \sum_{j=k-\delta_1 t}^k j^{-(1+\frac{1}{\alpha})} - I_t(k) \right| \leq (k - \delta_1 t)^{-(1+\frac{1}{\alpha})} \text{ and } \left| \sum_{j=k}^{k+\delta_1 t} j^{-(1+\frac{1}{\alpha})} - Q_t(k) \right| \leq k^{-(1+\frac{1}{\alpha})}. \quad (163)$$

In view of (162), and (163) for $k \geq \delta_2 t' + 1$ we have

$$\begin{aligned} & \left| \sum_{j=k-\delta_1 t}^k c_j - I_t(k) \right| \leq \sum_{j=k-\delta_1 t}^k \left| c_j - j^{-(1+\frac{1}{\alpha})} \right| + \left| \sum_{j=k-\delta_1 t}^k j^{-(1+\frac{1}{\alpha})} - I_t(k) \right| \\ & \leq \sum_{j=k-\delta_1 t}^k \left| c_j j^{(1+\frac{1}{\alpha})} - 1 \right| j^{-(1+\frac{1}{\alpha})} + (k - \delta_1 t)^{-(1+\frac{1}{\alpha})} \leq (\eta/2) \sum_{j=k-\delta_1 t}^k j^{-(1+\frac{1}{\alpha})} + (k - \delta_1 t)^{-(1+\frac{1}{\alpha})} \\ & = (\eta/2) \left(\sum_{j=k-\delta_1 t}^k j^{-(1+\frac{1}{\alpha})} - I_t(k) \right) + (\eta/2) I_t(k) + (k - \delta_1 t)^{-(1+\frac{1}{\alpha})} \end{aligned}$$

$$\leq (1 + \eta/2) (k - \delta_1 t)^{-(1+\frac{1}{\alpha})} + (\eta/2) I_t(k)$$

Therefore, for $k \geq \delta_2 t' + 1$,

$$\begin{aligned} \left| \sum_{j=k-\delta_1 t}^k c_j / I_t(k) - 1 \right| &\leq (1 + \eta/2) (k - \delta_1 t)^{-(1+\frac{1}{\alpha})} / I_t(k) + \eta/2 \\ &= (1 + \eta/2) \frac{(k - \delta_1 t)^{-(1+\frac{1}{\alpha})}}{\alpha [(k - \delta_1 t)^{-1/\alpha} - k^{-1/\alpha}]} + \eta/2. \end{aligned} \quad (164)$$

Consider the term

$$\begin{aligned} \frac{(k - \delta_1 t)^{-(1+\frac{1}{\alpha})}}{\alpha [(k - \delta_1 t)^{-1/\alpha} - k^{-1/\alpha}]} &= \frac{k^{-(1+\frac{1}{\alpha})} (1 - \delta_1 t/k)^{-(1+\frac{1}{\alpha})}}{\alpha k^{-1/\alpha} [(1 - \delta_1 t/k)^{-1/\alpha} - 1]} \\ &= \frac{k^{-1} (1 - \delta_1 t/k)^{-(1+\frac{1}{\alpha})}}{\alpha [(1 - \delta_1 t/k)^{-1/\alpha} - 1]} \leq_{(1)} k^{-1} \frac{\alpha (1 - \delta_1 t/k)^{-(1+\frac{1}{\alpha})}}{\alpha \delta_1 t/k} \\ &= \frac{1}{(1 - \delta_1 t/k)^{(1+\frac{1}{\alpha})} \delta_1 t} \leq \frac{1}{[1 - \delta_1 t/(\delta_2 t' + 1)]^{(1+\frac{1}{\alpha})} \delta_1 t} \\ &\leq \frac{1}{[1 - \delta_1 t/\delta_2 t]^{(1+\frac{1}{\alpha})} \delta_1 t} \leq \left(\frac{\delta_2}{\delta_2 - \delta_1} \right)^{(1+\frac{1}{\alpha})} \frac{1}{\delta_1 t}, \end{aligned} \quad (165)$$

where $\leq_{(1)}$ follows from the mean value theorem²¹. In view of (164)-(165), for $k \geq \delta_2 t' + 1$ we get

$$\begin{aligned} \left| \sum_{j=k-\delta_1 t}^k c_j / I_t(k) - 1 \right| &\leq \frac{1}{\delta_1 t} \left(\frac{\delta_2}{\delta_2 - \delta_1} \right)^{(1+\frac{1}{\alpha})} (1 + \eta/2) + \eta/2 \\ &\leq \frac{1}{\delta_1} \left(\frac{\delta_2}{\delta_2 - \delta_1} \right)^{(1+\frac{1}{\alpha})} (1 + \eta/2) / t_0 + \eta/2 \leq \frac{\eta}{3} (1 + \eta/2) + \eta/2 \leq \eta, \end{aligned}$$

and this shows part (a.i).

Next, we show part (ii). In view of (162) and (163) for $\delta t + \delta_2 t_0 \geq k \geq t_0$ we have

$$\left| \sum_{j=k}^{k+\delta_2 t} c_j - Q_t(k) \right| \leq \sum_{j=k}^{k+\delta_2 t} \left| c_j - j^{-(1+\frac{1}{\alpha})} \right| + \left| \sum_{j=k}^{k+\delta_2 t} j^{-(1+\frac{1}{\alpha})} - Q_t(k) \right|$$

²¹Let $f(y) = (1 - y)^{-1/\alpha}$. Then for $0 \leq \bar{y} \leq y < 1$,

$$f(y) - f(0) = \frac{1}{\alpha} \frac{y}{(1 - \bar{y})^{1+\frac{1}{\alpha}}} \geq \frac{1}{\alpha} y.$$

Setting $y = t/k$, the above yields

$$(1 - t/k)^{-1/\alpha} - 1 \geq \frac{1}{\alpha} t/k.$$

$$\begin{aligned}
&\leq \sum_{j=k}^{k+\delta_2 t} \left| c_j j^{(1+\frac{1}{\alpha})} - 1 \right| j^{-(1+\frac{1}{\alpha})} + k^{-(1+\frac{1}{\alpha})} \leq (\eta/2) \sum_{j=k}^{k+\delta_2 t} j^{-(1+\frac{1}{\alpha})} + k^{-(1+\frac{1}{\alpha})} \\
&= (\eta/2) \left(\sum_{j=k}^{k+\delta_2 t} j^{-(1+\frac{1}{\alpha})} - Q_t(k) \right) + (\eta/2) Q_t(k) + k^{-(1+\frac{1}{\alpha})} \\
&\leq (1 + \eta/2) k^{-(1+\frac{1}{\alpha})} + (\eta/2) Q_t(k).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left| \frac{\sum_{j=k}^{k+\delta_2 t} c_j}{Q_t(k)} - 1 \right| = \left| \frac{\sum_{j=k}^{k+\delta_2 t} c_j - Q_t(k)}{Q_t(k)} \right| \\
&\leq \frac{(1 + \eta/2) k^{-(1+\frac{1}{\alpha})} + (\eta/2) Q_t(k)}{Q_t(k)} \leq \frac{3k^{-(1+\frac{1}{\alpha})}}{2Q_t(k)} + \eta/2.
\end{aligned} \tag{166}$$

Next, for $\delta t + \delta_2 t_0 \geq k \geq t_0$ and $t \geq t_0$

$$\begin{aligned}
&\frac{k^{-(1+\frac{1}{\alpha})}}{Q_t(k)} = \frac{k^{-(1+\frac{1}{\alpha})}}{\alpha [k^{-1/\alpha} - (k + \delta_1 t)^{-1/\alpha}]} = \frac{k^{-(1+\frac{1}{\alpha})}}{\alpha k^{-1/\alpha} [1 - (1 + \delta_1 t/k)^{-1/\alpha}]} \\
&= \frac{1}{\alpha k [1 - \{1 + \delta_1 t/k\}^{-1/\alpha}]} \\
&\stackrel{k=\delta t + \delta_2 t_0}{\leq} \frac{1}{\alpha k [1 - \{1 + \delta_1 t/(\delta t + \delta_2 t_0)\}^{-1/\alpha}]} \stackrel{k, t=t_0}{\leq} \frac{1}{\alpha t_0 [1 - \{1 + \delta_1 t_0/(\delta t_0 + \delta_2 t_0)\}^{-1/\alpha}]} \\
&= \frac{1}{\alpha t_0 [1 - \{1 + \delta_1/(\delta + \delta_2)\}^{-1/\alpha}]} = \frac{1}{\alpha t_0 [1 - \{\frac{\delta + \delta_2}{\delta + \delta_2} + \frac{\delta_1}{\delta + \delta_2}\}^{-1/\alpha}]} \\
&= \frac{1}{\alpha t_0 [1 - (\frac{\delta + \delta_2 + \delta_1}{\delta + \delta_2})^{-1/\alpha}]} = \frac{1}{\alpha t_0 [1 - (\frac{2\delta_2 - \delta_1}{2\delta_2})^{1/\alpha}]} < \eta/3,
\end{aligned} \tag{167}$$

where the last inequality follows from (162).

Using (166)-(167), for $\delta t + \delta_2 t_0 \geq k \geq t_0$ and $t \geq t_0$

$$\left| \frac{\sum_{j=k}^{k+\delta_1 t} c_j}{Q_t(k)} - 1 \right| < \eta,$$

as required.

Finally, we show part (b). First, consider $I_t(k)^{-1}$, $k \geq \delta_2 t' + 1$. We have

$$\begin{aligned}
I_t(k) &= \alpha [(k - \delta_1 t)^{-1/\alpha} - k^{-1/\alpha}] \leq \alpha k^{-1/\alpha} [(1 - t/k)^{-1/\alpha} - 1] \\
&\leq_{(2)} k^{-1/\alpha} (1 - t/k)^{-(1+1/\alpha)} \delta_1 t/k \leq (1 - t/k)^{-(1+1/\alpha)} t/k^{1+1/\alpha} \\
&\leq (1 - t/(\delta_2 t))^{-(1+1/\alpha)} t/(\delta_2 t)^{1+1/\alpha} \\
&= (1 - 1/\delta_2)^{-(1+1/\alpha)} (1/\delta_2)^{(1+1/\alpha)} / t^{1/\alpha} = [\delta_2(1 - 1/\delta_2)]^{-(1+1/\alpha)} / t^{1/\alpha},
\end{aligned}$$

where $\leq_{(2)}$ follows from the mean value theorem²². In view of this, as $t \rightarrow \infty$

$$\inf_{k \geq \delta_2 t' + 1} \gamma_t I_t(k)^{-1} \geq [\delta_2(1 - 1/\delta_2)]^{(1+1/\alpha)} \gamma_t t^{1/\alpha} \rightarrow \infty.$$

Next, consider $Q_t(k)^{-1}$, $\delta t + \delta_2 t_0 \geq k \geq t_0$. Notice that $Q_t(k)^{-1}$ is increasing in k .²³ Hence, as $t \rightarrow \infty$

$$\begin{aligned} Q_t(k)^{-1} &\geq Q_t(t_0)^{-1} = \left[\frac{\alpha}{t_0^{-1/\alpha} - (t_0 + t)^{-1/\alpha}} \right]^{-1} \\ &= \frac{t_0^{-1/\alpha} - (t_0 + t)^{-1/\alpha}}{\alpha} \rightarrow t_0^{-1/\alpha} / \alpha. \end{aligned}$$

Therefore, as $t \rightarrow \infty$,

$$\inf_{\delta t + \delta_2 t_0 \geq k \geq t_0} \gamma_t Q_t(k)^{-1} \geq \gamma_t Q_t(t_0)^{-1} \rightarrow \infty,$$

as required. ■

²²Set $f(u) = (1 - u)^{-1/\alpha}$. Then for $0 < \bar{u} \leq u < 1$,

$$\begin{aligned} 0 < (1 - u)^{-1/\alpha} - 1 &= f(u) - f(0) = f'(\bar{u})u = \frac{1}{\alpha}(1 - \bar{u})^{-(1+1/\alpha)}u \\ &\leq \frac{1}{\alpha}(1 - u)^{-(1+1/\alpha)}u. \end{aligned}$$

²³Note that

$$\frac{\partial Q_t(k)}{\partial k} = \alpha \frac{\partial \left(k^{-1/\alpha} - (k + \delta_1 t)^{-1/\alpha} \right)}{\partial k} = (k + \delta_1 t)^{-\frac{1}{\alpha}(\alpha+1)} - k^{-\frac{1}{\alpha}(\alpha+1)} < 0.$$

Lemma 7. Suppose that **Assumption BFR2** holds and $a_k = \sum_{j=0}^k \phi_j c_{k-j}$. There is some $t_0 > 0$ and constants $0 < \underline{D} \leq \overline{D} < \infty$ such that

$$\underline{D} \leq \inf_{t_0 \leq k} k^{1/\alpha} |a_k| \leq \sup_{0 \leq k} k^{1/\alpha} |a_k| \leq \overline{D}.$$

Proof of Lemma 7. It follows directly from **Lemma 8** and the fact that $\sum_{j=0}^{\infty} c_j \neq 0$. ■

Lemma 8. The following hold.

Suppose that $\phi_j \sim \text{cons.} \cdot j^{-1/\alpha}$, $\alpha \in (0, 2]$, $\sum_{j=0}^{\infty} |c_j| < \infty$, $\sum_{j=0}^{\infty} c_j \neq 0$, and one of the following holds:

(a) $\limsup_{j \rightarrow \infty} \bar{c}_j |c_j| < \infty$, where $\bar{c}_j := \varsigma(j) j^{\frac{1}{\alpha-1}}$ with $\varsigma(x)$ SV such that

$$\begin{aligned} \lim_{x \rightarrow \infty} \varsigma(x)^{-1} &= 0, \text{ if } \alpha < 1; \\ \lim_{x \rightarrow \infty} \varsigma(x)^{-1} \ln(x) &= 0, \text{ if } \alpha = 1; \\ \varsigma(x) &= 1, \text{ if } \alpha \in (1, 2]. \end{aligned}$$

(b) $\sum_{j=1}^{\infty} j^{1/\alpha} |c_j| < \infty$.

Then as $s \rightarrow \infty$

$$\sum_{j=0}^s \phi_j c_{s-j} = \text{cons.} \cdot s^{-1/\alpha} \sum_{j=0}^{\infty} c_j + o(s^{-1/\alpha}).$$

Remark. **Lemma 8** implies that as $s \rightarrow \infty$

$$\sum_{j=0}^s \phi_j c_{s-j} \sim \text{cons.} \cdot s^{-1/\alpha} \sum_{j=0}^{\infty} c_j.$$

Proof of Lemma 8. Without loss of generality, suppose that $\text{cons.} = 1$. Set $\lambda(s) := s^{-1/\alpha}$. Note that for all $\varepsilon > 0$, there some N_ε such that $|\phi_{s-l}/\lambda(s-l) - 1| < \varepsilon$ when $s-l > N_\varepsilon$. We have

$$\begin{aligned} \sum_{j=0}^s \phi_j c_{s-j} &= \sum_{l=0}^s \phi_{s-l} c_l = \sum_{l=0}^{s-1-N_\varepsilon} \lambda(s-l) c_l + S_\varepsilon(s) \\ &\quad [l = s-j] \end{aligned} \tag{168}$$

where

$$S_\varepsilon(s) := \sum_{l=0}^{s-1-N_\varepsilon} [\phi_{s-l}/\lambda(s-l) - 1] \lambda(s-l) c_l + \sum_{l=s-N_\varepsilon}^s \phi_{s-l} c_l$$

We shall show that as $s \rightarrow \infty$

$$\sum_{l=0}^{s-1-N_\varepsilon} \lambda(s-l) c_l = \lambda(s) \sum_{j=0}^{\infty} c_j + o(\lambda(s)), \tag{169}$$

and as $s \rightarrow \infty$ first and then $\varepsilon \rightarrow 0$

$$\lambda^{-1}(s) S_\varepsilon(s) = o(1). \tag{170}$$

The requisite result follows from (168)-(170).

We start with the proof of (170). As $s \rightarrow \infty$ first and then $\varepsilon \rightarrow 0$, the first component of $S_\varepsilon(s)$ is

$$\begin{aligned} \sum_{l=0}^{s-1-N_\varepsilon} |\phi_{s-l}/\lambda(s-l) - 1| |\lambda(s-l) c_l| &\leq \varepsilon \sum_{l=0}^{s-1-N_\varepsilon} |\lambda(s-l) c_l| \\ &=_{(1)} \varepsilon \left[|\lambda(s)| \sum_{l=0}^{\infty} |c_l| + o(\lambda(s)) \right] = o(\lambda(s)), \end{aligned}$$

where $=_{(1)}$ follows from (169) applied $|c_j|$ in place of c_j .

Next, we consider the second component of $S_\varepsilon(s)$. First, suppose that condition (a) holds. Then

$$\begin{aligned} \left| \sum_{l=s-N_\varepsilon}^s \phi_{s-l} c_l \right| &\leq \sup_{0 \leq j \leq N_\varepsilon} |\phi_j| \sum_{l=s-N_\varepsilon}^s |c_l| \leq C \sup_{0 \leq j \leq N_\varepsilon} |\phi_j| \sum_{l=s-N_\varepsilon}^s |\bar{c}_l|^{-1} \\ &= C \sum_{l=s-N_\varepsilon}^s \left| \varsigma(l) l^{\frac{1}{\alpha \wedge 1}} \right|^{-1}. \end{aligned}$$

Set $\bar{R}(l) = \left[\varsigma(l) l^{\frac{1}{\alpha \wedge 1}} \right]^{-1}$. For any $\eta > 0$ and s large enough, the sum term above is

$$\begin{aligned} \sum_{l=s-N_\varepsilon}^s \left| \varsigma(l) l^{\frac{1}{\alpha \wedge 1}} \right|^{-1} &= \bar{R}(s) \sum_{l=s-N_\varepsilon}^s \frac{\bar{R}(l)}{\bar{R}(s)} \\ &= \bar{R}(s) \sum_{l=s-N_\varepsilon}^s \left[\frac{\bar{R}\left(\frac{l}{s}\right)}{\bar{R}(s)} - \left(\frac{l}{s}\right)^{-\frac{1}{\alpha \wedge 1}} \right] + \bar{R}(s) \sum_{l=s-N_\varepsilon}^s \left(\frac{l}{s}\right)^{-\frac{1}{\alpha \wedge 1}} \\ &\leq \bar{R}(s) \sum_{l=s-N_\varepsilon}^s \sup_{1-\frac{N_\varepsilon}{s} \leq y \leq 1} \left[\frac{\bar{R}(ys)}{\bar{R}(s)} - y^{-\frac{1}{\alpha \wedge 1}} \right] + N_\varepsilon \left(\frac{s}{s-N_\varepsilon}\right)^{1/\alpha} \bar{R}(s) \\ &\leq_{(2)} \bar{R}(s) N_\varepsilon \left[\eta + \left(\frac{s}{s-N_\varepsilon}\right)^{\frac{1}{\alpha \wedge 1}} \right] = C \frac{\bar{R}(s)}{\lambda(s)} \lambda(s) \\ &= \frac{C}{\varsigma(s)} \lambda(s) I\{\alpha \in (0, 1]\} + \frac{Cs^{1/\alpha}}{\varsigma(s)s} \lambda(s) I\{\alpha \in (1, 2]\} = o(\lambda(s)), \end{aligned}$$

as required. We note that $\leq_{(2)}$ follows from the uniform convergence of regularly varying functions. Next, suppose that condition (b) holds. In this case, for s large enough, the second component of $S_\varepsilon(s)$ is

$$\begin{aligned} \left| \sum_{l=s-N_\varepsilon}^s \phi_{s-l} c_l \right| &\leq C \sum_{l=s-N_\varepsilon}^s |c_l| \leq C (s-N_\varepsilon)^{-1/\alpha} \sum_{l=s-N_\varepsilon}^{\infty} l^{1/\alpha} |c_l| \\ &= C \left(\frac{s}{s-N_\varepsilon}\right)^{1/\alpha} \lambda(s) \sum_{l=s-N_\varepsilon}^{\infty} l^{1/\alpha} |c_l| = o(\lambda(s)). \end{aligned}$$

Therefore, (170) holds either under condition (a) or (b).

Finally, we show (169). Set $m_s := \lfloor s\delta \rfloor$ where $1/2 \leq \delta < 1$. Consider

$$\sum_{l=0}^{s-1-N_\varepsilon} \lambda(s-l) c_l = \left(\sum_{l=0}^{m_s} + \sum_{l=m_s+1}^{s-1-N_\varepsilon} \right) \lambda(s-l) c_l =: T_1(s) + T_2(s).$$

As $s \rightarrow \infty$, the first term

$$\begin{aligned} \left| T_1(s) - \lambda(s) \sum_{l=0}^{m_s} c_l \right| &\leq \sum_{l=0}^{m_s} |\lambda(s) - \lambda(s-l)| |c_l| \\ &\leq \sum_{l=0}^{m_s} \left| \frac{\lambda(s-l)}{\lambda(s)} - 1 \right| \lambda(s) |c_l| = \lambda(s) \sum_{l=0}^{m_s} q_{l,s} |c_l| \end{aligned}$$

where $q_{l,s} := \left| \left(\frac{s-l}{s} \right)^{-1/\alpha} - 1 \right|$. Next, we show that $\sum_{l=0}^{m_s} q_{l,s} |c_l| = o(1)$ as $s \rightarrow \infty$. Note that for $\alpha \in (0, 1)$ the mean value theorem yields

$$q_{l,s} = \frac{s^{1/\alpha} - (s-l)^{1/\alpha}}{(s-l)^{1/\alpha}} \leq \frac{1}{\alpha} \frac{s^{\frac{1}{\alpha}-1} l}{(s-l)^{1/\alpha}},$$

whilst for $\alpha \in [1, 2]$ by Loève's inequality

$$q_{l,s} = \frac{s^{1/\alpha} - (s-l)^{1/\alpha}}{(s-l)^{1/\alpha}} \leq \frac{l^{1/\alpha}}{(s-l)^{1/\alpha}}.$$

Therefore, as $s \rightarrow \infty$

$$\begin{aligned} \sum_{l=0}^{m_s} q_{l,s} |c_l| &\leq C \sum_{l=0}^{m_s} |c_l| \left[\frac{l^{1/\alpha}}{(s-l)^{1/\alpha}} 1_{\{\alpha \in [1, 2]\}} + \frac{s^{\frac{1}{\alpha}-1} l}{(s-l)^{1/\alpha}} 1_{\{\alpha \in (0, 1)\}} \right] \\ &\leq \frac{1}{(s-m_s)^{1/\alpha}} \sum_{l=0}^{m_s} l^{1/\alpha} |c_l| + \frac{C s^{\frac{1}{\alpha}-1}}{(s-m_s)^{1/\alpha}} \sum_{l=0}^{m_s} l |c_l| =_{(4)} o(1), \end{aligned}$$

where $=_{(4)}$ follows from Kronecker's lemma (e.g. p. 31 in Hall and Heyde, 1980).

Next, we consider $T_2(s)$. First, assume that condition (a) holds. Given that $\bar{c}_l$ is regularly varying of a positive index, there is no loss of generality if we assume it is increasing -see e.g. BGT 1987, Thm 1.5.3. Hence, as $s \rightarrow \infty$, we have

$$\begin{aligned} |T_2(s)| &\leq \sum_{l=m_s}^{s-1} \lambda(s-l) \bar{c}_l^{-1} \bar{c}_l |c_l| \\ &\leq C_\zeta(m_s)^{-1} m_s^{-\frac{1}{\alpha \wedge 1}} \sum_{l=m_s}^{s-1} \lambda(s-l) = C_\zeta(m_s)^{-1} m_s^{-\frac{1}{\alpha \wedge 1}} \sum_{l=1}^{s-m_s+1} \lambda(l) \\ &\leq C_\zeta(s)^{-1} s^{-\frac{1}{\alpha \wedge 1}} \sum_{l=1}^{s-m_s+1} \lambda(l) =: CQ(s). \end{aligned}$$

The result follows if we show that $Q(s) = o(\lambda(s))$. Note that for $\alpha \in (0, 1)$, $\zeta(s)^{-1} s^{-\frac{1}{\alpha \wedge 1}} = o(\lambda(s))$ and $\sum_{l=1}^{\infty} \lambda(l) < \infty$. Hence, $Q(s) = o(\lambda(s))$. For $\alpha = 1$, $\sum_{l=1}^{s-m_s+1} \lambda(l) \sim \ln(s)$. Hence,

$$Q(s) \sim \frac{\lambda(s)}{\zeta(s)} \ln(s) =_{(5)} o(\lambda(s)),$$

where $=_{(5)}$ follows due to the requirement of condition (a) for $\alpha = 1$. For $\alpha \in (1, 2]$, using Euler summation (see e.g. **Lemma 3**) we get as $s \rightarrow \infty$ first, and then as $\delta \rightarrow 1$

$$\begin{aligned} Q(s) &= s^{-1} \sum_{l=1}^{s-m_s+1} l^{-1/\alpha} = \lambda(s) s^{-1} \sum_{l=1}^{s-m_s+1} \left(\frac{l}{s}\right)^{-1/\alpha} \\ &= \lambda(s) \int_0^{1-\delta} x^{-1/\alpha} dx (1 + o(1)) = o(\lambda(s)). \end{aligned}$$

Finally, we consider $T_2(s)$ under condition (b). As $s \rightarrow \infty$,

$$\begin{aligned} |T_2(s)| &\leq \sup_{l > N_\varepsilon} \lambda(l) \sum_{l=m_s}^{s-1-N_\varepsilon} |c_l| \leq C \sum_{l=m_s}^{\infty} l^{1/\alpha} \\ &\leq C \lambda(m_s) \sum_{l=m_s}^{\infty} l^{1/\alpha} |c_l| = o(\lambda(s)), \end{aligned}$$

as required. ■

Lemma 9. (linear process approximation). *Suppose that **Assumptions BFR2** and **INN** hold. Then for all $\alpha' \in (0, \alpha)$, as $n \rightarrow \infty$*

$$\sup_{0 \leq r \leq 1} \left\| \gamma_n^{-1} x_{[nr]} - \gamma_n^{-1} \tilde{x}_{[nr]} \right\|_{L_{\alpha'}} = o(1),$$

where

$$\tilde{x}_n := \left(\sum_{j=0}^{\infty} c_j \right) \sum_{k=1}^{n-1} k^{-\frac{1}{\alpha}} \xi_{n-k}.$$

Proof of Lemma 9. Let $\varepsilon > 0$ such that $\alpha - \varepsilon > 0$. We shall show that for each $0 < r \leq 1$

$$\mathbf{E} \left| \gamma_n^{-1} x_{[nr]}^- \right|^{\alpha - \varepsilon} \rightarrow 0 \quad (171)$$

and

$$\mathbf{E} \left| \gamma_n^{-1} x_{[nr]}^+ - \gamma_n^{-1} \tilde{x}_{[nr]} \right|^{\alpha - \varepsilon} \rightarrow 0. \quad (172)$$

We start with the proof of (171). Without loss of generality set $r = 1$, and assume that $R_{\nu-\alpha}(x) = x^{\nu-\alpha} H(x)$ is decreasing, where $\nu = 1$ for $\alpha \in (1, 2)$ and $\nu \in (0, \alpha)$ for $\alpha \in (0, 1]$. Moreover, recall that under our assumptions $\sum_{l=0}^{\infty} |c_l|^\nu < \infty$. In addition, to simplify the notation assume that $\sum_{l=0}^{\infty} c_l = 1$. Using arguments similar to those used for establishing (117) and (121), we have

$$\begin{aligned} \mathbf{E} \left| \gamma_n^{-1} x_n^- \right|^{\alpha - \varepsilon} &\leq C \gamma_n^{-\nu} \sum_{j=0}^n |\phi_j|^\nu R_{\nu-\alpha}(\gamma_n j^{1/\alpha}) \sum_{l=n-j}^{\infty} |c_l|^\nu \\ &\leq C \gamma_n^{-\alpha} \sum_{j=1}^n j^{-1} H(\gamma_n j^{1/\alpha}) \sum_{l=n-j}^{\infty} |c_l|^\nu \\ &= C \gamma_n^{-\alpha} \left(\sum_{j=[n/2]}^n + \sum_{j=1}^{[n/2]-1} \right) j^{-1} H(\gamma_n j^{1/\alpha}) \sum_{l=n-j}^{\infty} |c_l|^\nu \\ &\leq C \gamma_n^{-\alpha} \sum_{j=[n/2]}^n j^{-1} H(\gamma_n j^{1/\alpha}) \sum_{l=0}^{\infty} |c_l|^\nu \\ &\quad + C \gamma_n^{-\alpha} \sum_{j=1}^{[n/2]-1} j^{-1} H(\gamma_n j^{1/\alpha}) \sum_{l=[n/2]}^{\infty} |c_l|^\nu \\ &=_{(1)} C \gamma_n^{-\alpha} \left[\int_{\gamma_n [n/2]^{1/\alpha}}^{\gamma_n n^{1/\alpha}} \frac{H(x)}{x} dx + \int_1^{\gamma_n n^{1/\alpha}} \frac{H(x)}{x} dx \sum_{l=[n/2]}^{\infty} |c_l|^\nu \right] [1 + o(1)] =_{(2)} o(1), \end{aligned}$$

where $=_{(1)}$ follows from **Lemma 3(i)**, and $=_{(2)}$ from the fact that $\gamma_n^{-\alpha} \int_{\gamma_n [n/2]^{1/\alpha}}^{\gamma_n n^{1/\alpha}} \frac{H(x)}{x} dx = o(1)$.

Next, we show (172). Again, without loss of generality set $r = 1$. It can be easily checked that as $n \rightarrow \infty$

$$\inf_{1 \leq k \leq n} S_{n,k} := \inf_{1 \leq k \leq n} \gamma_n (k^{-1/\alpha})^{-1} \rightarrow \infty.$$

Set

$$\bar{\xi}_{n,k} := \xi_k 1\{|\xi_k| \geq S_{n,k}\} \quad \text{and} \quad \underline{\xi}_{n,k} := \xi_k 1\{|\xi_k| < S_{n,k}\}.$$

Hence, by Lemma 1 in Astrauskas (1983)

$$\mathbf{E} \left| \bar{\xi}_{n,k} \right|^{\alpha-\varepsilon} \asymp S_{n,k}^{-\varepsilon} h(S_{n,k}) \quad \text{and} \quad \mathbf{E} \left| \underline{\xi}_{n,k} \right|^{\alpha+\varepsilon} \asymp S_{n,k}^{\varepsilon} h(S_{n,k}),$$

as $n \rightarrow \infty$. Let $r_k := \left(\frac{a_k}{k^{-1/\alpha}} - 1 \right)$. Due to **Lemma 8**, for $\epsilon > 0$ arbitrarily small we have $|r_k| < \epsilon$ for all k large enough, $k \geq N_\epsilon$ say. Consider

$$\begin{aligned} \mathbf{E} \left| \gamma_n^{-1} x_n^+ - \gamma_n^{-1} \tilde{x}_n \right|^{\alpha-\varepsilon} &= \mathbf{E} \left| \gamma_n^{-1} \sum_{k=0}^{n-1} a_k \xi_{n-k} - \gamma_n^{-1} \sum_{k=1}^{n-1} k^{-\frac{1}{\alpha}} \xi_{n-k} \right|^{\alpha-\varepsilon} \\ &\leq \left| \gamma_n^{-1} \sum_{k=N_\epsilon}^{n-1} \left(\frac{a_k}{k^{-\frac{1}{\alpha}}} - 1 \right) k^{-\frac{1}{\alpha}} \xi_{n-k} \right|^{\alpha-\varepsilon} = \mathbf{E} \left| \gamma_n^{-1} \sum_{k=N_\epsilon}^{n-1} r_k k^{-\frac{1}{\alpha}} \xi_{n-k} \right|^{\alpha-\varepsilon} + o(1). \end{aligned} \quad (173)$$

Next, arguments similar to those used for (106) give

$$\begin{aligned} \mathbf{E} \left| \gamma_n^{-1} \sum_{k=N_\epsilon}^{n-1} r_k k^{-1/\alpha} \xi_{n-k} \right|^{\alpha-\varepsilon} &\leq 2 \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=N_\epsilon}^{n-1} (|r_k| k^{-1/\alpha})^{\alpha-\varepsilon} \mathbf{E} \left| \bar{\xi}_{n,k} \right|^{\alpha-\varepsilon} \\ &\quad + 2 \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=N_\epsilon}^{n-1} (|r_k| k^{-1/\alpha})^{\alpha+\varepsilon} \mathbf{E} \left| \underline{\xi}_{n,k} \right|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\ &\leq 2 \left\{ \epsilon^{\alpha-\varepsilon} \sum_{k=N_\epsilon}^{n-1} S_{n,k}^{-\alpha+\varepsilon} S_{n,k}^{-\varepsilon} h(S_{n,k}) + \left[\epsilon^{\alpha+\varepsilon} \sum_{k=N_\epsilon}^{n-1} S_{n,k}^{-\alpha-\varepsilon} S_{n,k}^{\varepsilon} h(S_{n,k}) \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right\} [1 + o(1)] \\ &\leq 2 \epsilon^{\alpha-\varepsilon} \left\{ \sum_{k=N_\epsilon}^{n-1} S_{n,k}^{-\alpha} h(S_{n,k}) + \left[\sum_{k=N_\epsilon}^{n-1} S_{n,k}^{-\alpha} h(S_{n,k}) \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right\} [1 + o(1)] \\ &\leq 2 \epsilon^{\alpha-\varepsilon} \left\{ \sum_{k=1}^n S_{n,k}^{-\alpha} H(S_{n,k}) + \left[\sum_{k=1}^n S_{n,k}^{-\alpha} H(S_{n,k}) \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right\} [1 + o(1)] \leq_{(3)} C \epsilon^{\alpha-\varepsilon} \end{aligned} \quad (174)$$

where $\leq_{(3)}$ follows from the fact that (cf. **Lemma 3**)

$$\sum_{k=1}^n S_{n,k}^{-\alpha} H(S_{n,k}) = \gamma_n^{-\alpha} \sum_{k=1}^n k^{-1} H(\gamma_n k^{1/\alpha}) = 1 + o(1).$$

The result follows from (173), (174) and the fact that ϵ can be made arbitrarily small. $\blacksquare$

14. Verification of HL4

Lemma 10. Suppose the **Assumption INN** and that x_t satisfies either **Assumption BFR** or **MI** holds. Then, for every bounded and Lipschitz function $g : \mathbb{R} \rightarrow \mathbb{R}$, as $n \rightarrow \infty$

$$\frac{1}{n} \sum_{t=1}^n g(\gamma_n^{-1} x_t) = \frac{1}{n} \sum_{t=1}^n \mathbf{E} [g(\gamma_n^{-1} x_t) \mid \mathcal{F}_0] + o_p(1).$$

Proof of Lemma 10. Let $\mathbf{E}_t[\cdot] := \mathbf{E}[\cdot | \mathcal{F}_t]$ and as in DK2021, write

$$\sum_{t=1}^n \left[g(\gamma_n^{-1} x_t) - \mathbf{E}_0 g(\gamma_n^{-1} x_t) \right] = \sum_{s=0}^{n-1} M_{n,s},$$

where $M_{n,s} = \sum_{t=s+1}^n \left[\mathbf{E}_{t-s} g(\gamma_n^{-1} x_t) - \mathbf{E}_{(t-s)-1} g(\gamma_n^{-1} x_t) \right]$ -see DK2021, Proof of Proposition B.1(iii). Note that the summand in $M_{n,s}$ forms a martingale difference w.r.t. $\mathcal{F}_{t-s}$.

Let

$$\alpha_\varepsilon := \alpha + \varepsilon \text{ with } \begin{cases} \varepsilon = 0, & \text{if } \alpha = 2 \\ \varepsilon > 0, & \text{if } \alpha \in (0, 2) \text{ such that } \alpha_\varepsilon \in (1, 2] \end{cases} \quad (175)$$

Note that in the following calculations all relevant moments exist because g is bounded. Therefore, in view of this and the fact that $\alpha_\varepsilon > 1$ we have

$$\begin{aligned} I_n &:= \frac{1}{n} \mathbf{E} \left| \sum_{t=1}^n \left[g(\gamma_n^{-1} x_t) - \mathbf{E}_0 g(\gamma_n^{-1} x_t) \right] \right| = \frac{1}{n} \mathbf{E} \left| \sum_{s=0}^{n-1} M_{n,s} \right| \\ &\leq \frac{1}{n} \sum_{s=0}^{n-1} \mathbf{E} |M_{n,s}| \leq \frac{1}{n} \sum_{s=0}^{n-1} [\mathbf{E} |M_{n,s}|^{\alpha_\varepsilon}]^{1/\alpha_\varepsilon}. \end{aligned} \quad (176)$$

Next, for $0 \leq s \leq n-1$ set

$$x_{t,s}^* := x_t^* := \sum_{k=0, k \neq s}^{\infty} a_{k,t} \xi_{t-k} + a_{s,t} \xi^*$$

with $\xi^* \stackrel{d}{=} \xi_1$ and $\xi^* \perp \xi_1$. Then $x_t^* \stackrel{d}{=} x_t$. In fact, $\{\mathbf{E}_{t-s} g(\gamma_n^{-1} x_t), \mathbf{E}_{(t-s)-1} g(\gamma_n^{-1} x_t)\} \stackrel{d}{=} \{\mathbf{E}_{t-s} g(\gamma_n^{-1} x_t), \mathbf{E}_{t-s} g(\gamma_n^{-1} x_t^*)\}$. Set $\Delta_{s,n} := \gamma_n^{-1} |a_s| |\xi_1 - \xi^*|$. Recall that the coefficients $a_{k,t}, a_k$ are defined in (7). Due to the fact that g is bounded and Lipschitz, we have $|g(x) - g(z)| \leq C_0 (1 \wedge |x - z|)$ for some $C_0 \in (0, \infty)$. In view of the above,

$$\begin{aligned} \mathbf{E} |M_{n,s}|^{\alpha_\varepsilon} &= \mathbf{E} \left| \sum_{t=s+1}^n \mathbf{E}_{t-s} g(\gamma_n^{-1} x_t) - \mathbf{E}_{(t-s)-1} g(\gamma_n^{-1} x_t) \right|^{\alpha_\varepsilon} \\ &\leq 2 \mathbf{E} \sum_{t=s+1}^n \left| \mathbf{E}_{t-s} g(\gamma_n^{-1} x_t) - \mathbf{E}_{(t-s)-1} g(\gamma_n^{-1} x_t) \right|^{\alpha_\varepsilon} = 2 \mathbf{E} \sum_{t=s+1}^n \left| \mathbf{E}_{t-s} g(\gamma_n^{-1} x_t) - \mathbf{E}_{t-s} g(\gamma_n^{-1} x_t^*) \right|^{\alpha_\varepsilon} \\ &\leq 2 \sum_{t=s+1}^n \mathbf{E} \left[\mathbf{E}_{t-s} |g(\gamma_n^{-1} x_t) - g(\gamma_n^{-1} x_t^*)| \right]^{\alpha_\varepsilon} \leq 2 \sum_{t=s+1}^n \mathbf{E} \left\{ \left[\mathbf{E}_{t-s} |g(\gamma_n^{-1} x_t) - g(\gamma_n^{-1} x_t^*)| \right]^{\alpha_\varepsilon} \right\}^{1/\alpha_\varepsilon} \\ &= 2 \sum_{t=s+1}^n \mathbf{E} \mathbf{E}_{t-s} |g(\gamma_n^{-1} x_t) - g(\gamma_n^{-1} x_t^*)|^{\alpha_\varepsilon} = 2 \sum_{t=s+1}^n \mathbf{E} |g(\gamma_n^{-1} x_t) - g(\gamma_n^{-1} x_t^*)|^{\alpha_\varepsilon} \\ &\leq 2C_0^{\alpha_\varepsilon} \sum_{t=s+1}^n \mathbf{E} \left\{ \left[\gamma_n^{-1} |a_{s,t}| |\xi_{t-s} - \xi^*| \wedge 1 \right] \right\}^{\alpha_\varepsilon} \stackrel{(1)}{=} C \sum_{t=s+1}^n \mathbf{E} \left\{ \left[\gamma_n^{-1} |a_s| |\xi_1 - \xi^*| \wedge 1 \right] \right\}^{\alpha_\varepsilon} \\ &= C \sum_{t=s+1}^n \mathbf{E} [(\Delta_{s,n} \wedge 1)^{\alpha_\varepsilon}] = C \sum_{t=s+1}^n \mathbf{E} \{[(\Delta_{s,n} I\{\Delta_{s,n} \leq 1\} + I\{\Delta_{s,n} > 1\})]^{\alpha_\varepsilon}\} \\ &\leq C \sum_{t=s+1}^n \mathbf{E} \left[\Delta_{s,n}^{\alpha_\varepsilon} I\{\Delta_{s,n} \leq 1\} + I\{\Delta_{s,n} > 1\} \right], \end{aligned} \quad (177)$$

where $=_{(1)}$ holds for $0 \leq s \leq t-1$. Recall that (see the proof of **Lemma FS $_{\alpha}$ M**)

$$\lim_{n \rightarrow \infty} \gamma_n \inf_{0 \leq s \leq n} |a_s(n)|^{-1} = \infty, \quad (178)$$

where $a_s := a_s(n)$.

In view of **Assumption INN**, (178), and the fact that $\xi_1 - \xi^*$ is a (symmetric) random variable in the domain of attraction of an α -stable law (see Remark 1), as $n \rightarrow \infty$

$$\begin{aligned} \mathbf{E}(I \{\Delta_{s,n} > 1\}) &= P(\gamma_n^{-1} |a_s| |\xi_1 - \xi^*| > 1) \\ &= C \left(\frac{|a_s|}{\gamma_n} \right)^{\alpha} h \left(\frac{\gamma_n}{|a_s|} \right) [1 + o(1)], \end{aligned} \quad (179)$$

uniformly in $0 \leq s \leq n$. Next, using (178) once more and Lemma 1 of Astrauskas (1983) as $n \rightarrow \infty$ we have

$$\begin{aligned} \mathbf{E}(\Delta_{s,n}^{\alpha_{\varepsilon}} I \{\Delta_{s,n} \leq 1\}) &= \gamma_n^{-\alpha_{\varepsilon}} |a_s|^{\alpha_{\varepsilon}} \mathbf{E}(|\xi_1 - \xi^*|^{\alpha_{\varepsilon}} I \{|\xi_1 - \xi^*| \leq \gamma_n |a_s|^{-1}\}) \\ &= C \left(\frac{|a_s|}{\gamma_n} \right)^{\alpha} \left\{ H \left(\frac{\gamma_n}{|a_s|} \right) 1_{\{\alpha = 2\}} + h \left(\frac{\gamma_n}{|a_s|} \right) 1_{\{\alpha \in (0, 2)\}} \right\} [1 + o(1)], \end{aligned} \quad (180)$$

uniformly in $0 \leq s \leq n$. Since $h(x) = O(H(x))$ as $x \rightarrow \infty$, the contributions involving h can be bounded by the corresponding expressions involving H , after an adjustment of the multiplicative constant. We shall use these bounds throughout the remainder of the proof. In view of this, and (176)-(180) we get

$$\begin{aligned} I_n &\leq \frac{C}{n} \sum_{s=0}^{n-1} \left(\sum_{t=s+1}^n \left(\frac{|a_s|}{\gamma_n} \right)^{\alpha} H \left(\frac{\gamma_n}{|a_s|} \right) \right)^{1/\alpha_{\varepsilon}} [1 + o(1)] \\ &\leq \frac{C n^{1/\alpha_{\varepsilon}}}{n} \sum_{s=0}^n \left(\frac{|a_s|}{\gamma_n} \right)^{\alpha/\alpha_{\varepsilon}} H \left(\frac{\gamma_n}{|a_s|} \right)^{1/\alpha_{\varepsilon}} [1 + o(1)] =: C I'_n(\varepsilon, \alpha) [1 + o(1)]. \end{aligned} \quad (181)$$

as $n \rightarrow \infty$. Next, we shall show that, under each assumption (i.e. under **BFR** and **MI**), there is some $\varepsilon > 0$ such that

$$I'_n := I'_n(\varepsilon, \alpha) = o(1), \text{ for } \alpha \in (0, 2) \text{ and } \alpha_{\varepsilon} > 1, \quad (182)$$

and

$$I'_n(0, 2) = o(1), \quad (183)$$

as $n \rightarrow \infty$.

BFR ($\alpha \in (0, 2]$). Using the approximation $a_s \asymp s^{-1/\alpha}$ (see e.g. (79)), and similar arguments as those used for the proof of **Lemma FS $_{\alpha}$ M**, (see e.g. (80)) together with **Lemma 3(ii)** we have as $n \rightarrow \infty$

$$\begin{aligned} I'_n &= \frac{n^{1/\alpha_{\varepsilon}}}{\gamma_n^{\alpha/\alpha_{\varepsilon}} n} \sum_{s=0}^n |a_s|^{\alpha/\alpha_{\varepsilon}} H \left(\frac{\gamma_n}{|a_s|} \right)^{1/\alpha_{\varepsilon}} = \frac{n^{1/\alpha_{\varepsilon}}}{\gamma_n^{\alpha/\alpha_{\varepsilon}} n} \sum_{s=1}^n s^{-1/\alpha_{\varepsilon}} H \left(s^{1/\alpha} \gamma_n \right)^{1/\alpha_{\varepsilon}} [1 + o(1)] \\ &= \left[\frac{H \left(n^{1/\alpha} \gamma_n \right)}{\gamma_n^{\alpha}} \right]^{1/\alpha_{\varepsilon}} \int_0^1 x^{-1/\alpha_{\varepsilon}} dx [1 + o(1)] =_{(1)} o(1), \end{aligned}$$

where for $=_{(1)}$ we have used the fact that $\gamma_n^\alpha \sim L\left(n^{1/\alpha}\gamma_n\right)$ with $H(x) = o(L(x))$, as $x \rightarrow \infty$. Hence, (182) and (183) hold.

MI-SM ($\alpha \in (0, 2]$). We start with a proof under **Assumption SM(ii)**. In addition to (175), let

$$0 < \delta < 1 - \nu \text{ and } 0 \leq \varepsilon < \alpha \left(\frac{1}{\delta + \nu} - 1 \right), \quad (184)$$

where recall that $\nu \in (0, \alpha \wedge 1)$ -cf. **Assumption SM**. The former restriction in (184) ensures that $\left[0, \alpha \left(\frac{1}{\delta + \nu} - 1 \right)\right]$ is not empty, while the latter restriction ensures that $\sum_{s=0}^\infty |c_s|^{\frac{\alpha}{\alpha_\varepsilon} - \delta} < \infty$.²⁴ Next, by (178), for all $\delta > 0$ as $n \rightarrow \infty$

$$H\left(\frac{\gamma_n}{|a_s|}\right)^{1/\alpha_\varepsilon} \left(\frac{\gamma_n}{|a_s|}\right)^{-\delta} \rightarrow 0,$$

uniformly in $s \geq 1$. In view of this and (178) we get as $n \rightarrow \infty$

$$\begin{aligned} I'_n &= \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\alpha/\alpha_\varepsilon} n} \sum_{s=0}^n |a_s|^{\alpha/\alpha_\varepsilon} H\left(\frac{\gamma_n}{|a_s|}\right)^{1/\alpha_\varepsilon} \\ &\leq \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon} - \delta} n} \sum_{s=0}^n |a_s|^{\frac{\alpha}{\alpha_\varepsilon} - \delta} = \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon} - \delta} n} \sum_{s=0}^n \left| \sum_{l=0}^s \rho_n^l c_{s-l} \right|^{\frac{\alpha}{\alpha_\varepsilon} - \delta} \\ &\leq \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon} - \delta} n} \sum_{s=0}^n \sum_{l=0}^s \left| \rho_n^l c_{s-l} \right|^{\frac{\alpha}{\alpha_\varepsilon} - \delta} = \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon} - \delta} n} \sum_{l=0}^n \sum_{s=l}^n \left| \rho_n^l c_{s-l} \right|^{\frac{\alpha}{\alpha_\varepsilon} - \delta} \\ &\leq \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon} - \delta} n} \sum_{l=0}^n \rho_n^{\left(\frac{\alpha}{\alpha_\varepsilon} - \delta\right)l} \sum_{s=0}^\infty |c_s|^{\frac{\alpha}{\alpha_\varepsilon} - \delta} \leq_{(1)} C \frac{n^{1/\alpha_\varepsilon} \kappa_n}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon} - \delta} n}, \end{aligned} \quad (185)$$

where $\leq_{(1)}$ follows from Lemma 6.4 of DK2018, and the fact that $\frac{\alpha}{\alpha_\varepsilon} - \delta > \nu > 0$. Recall that under the current assumptions we have $\kappa_n = n^{\alpha_\kappa} \ell_\kappa(n) \rightarrow \infty$ with $\alpha_\kappa \in [0, 1)$, ℓ_κ SV, and $\gamma_n \sim \kappa_n^{1/\alpha} \ell_1(\kappa_n)^{1/\alpha}$. Hence, as $n \rightarrow \infty$ the last term in (185) is

$$\begin{aligned} &\frac{n^{1/\alpha_\varepsilon} n^{\alpha_\kappa} \ell_\kappa(n)}{\left[\kappa_n^{\frac{1}{\alpha}} \ell_1(\kappa_n)^{\frac{1}{\alpha}} \right]^{\frac{\alpha}{\alpha_\varepsilon} - \delta} n} = \frac{n^{1/\alpha_\varepsilon} n^{\alpha_\kappa} \ell_\kappa(n)}{\left[n^{\frac{\alpha_\kappa}{\alpha}} \ell_\kappa(n)^{\frac{1}{\alpha}} \ell_1(\kappa_n)^{\frac{1}{\alpha}} \right]^{\frac{\alpha}{\alpha_\varepsilon} - \delta} n} \\ &= \frac{\ell_\kappa(n) n^{1/\alpha_\varepsilon} n^{\frac{\alpha_\kappa}{\alpha} \left(\alpha - \frac{\alpha}{\alpha_\varepsilon} + \delta \right)}}{\left[\ell_\kappa(n)^{\frac{1}{\alpha}} \ell_1(\kappa_n)^{\frac{1}{\alpha}} \right]^{\frac{\alpha}{\alpha_\varepsilon} - \delta} n} =: \lambda_n n^{\frac{1}{\alpha_\varepsilon} + \frac{\alpha_\kappa}{\alpha} \left(\alpha - \frac{\alpha}{\alpha_\varepsilon} + \delta \right) - 1} =: \lambda_n n^\eta. \end{aligned}$$

Notice that the term

$$\lambda_n = \frac{\ell_\kappa(n)}{\left[\ell_\kappa(n)^{\frac{1}{\alpha}} \ell_1(\kappa_n)^{\frac{1}{\alpha}} \right]^{\frac{\alpha}{\alpha_\varepsilon} - \delta}} = \frac{\ell_\kappa(n)}{\left[\ell_\kappa(n)^{\frac{1}{\alpha}} \ell_1(n^{\alpha_\kappa} \ell_\kappa(n))^{\frac{1}{\alpha}} \right]^{\frac{\alpha}{\alpha_\varepsilon} - \delta}}$$

is SV in n (see e.g. BGT 1987, Proposition 1.5.7). In view of this, the result will follow if we show that $\eta < 0$. Indeed, for $\alpha_\kappa = 0$ it can be easily checked that $\eta < 0$. For $\alpha_\kappa \in (0, 1)$, the following additional restrictions on δ and ε ensure that η is strictly negative.

²⁴By (184) to $\frac{\alpha}{\alpha_\varepsilon} - \delta > \nu$. Therefore, $\sum_{s=0}^\infty |c_s|^\nu < \infty$ (cf. **Assumption SM**) implies $\sum_{s=0}^\infty |c_s|^{\frac{\alpha}{\alpha_\varepsilon} - \delta} < \infty$.

- $1 < \alpha \leq 2$: In addition to the restrictions of (175) and (184), δ satisfies

$$\delta < (\alpha - 1) \left(\frac{1}{\alpha_\kappa} - 1 \right) \wedge (1 - \nu).$$

- $0 < \alpha \leq 1$: In addition to the restrictions of (175) and (184), δ and ε satisfy

$$\delta < (\alpha - \nu)(1 - \alpha_\kappa) \text{ and } \frac{(1 - \alpha)(1 - \alpha_\kappa) + \alpha_\kappa \delta}{1 - \alpha_\kappa \left(1 + \frac{\delta}{\alpha}\right)} < \varepsilon.$$

Finally, we show the result under **Assumption SM(i)**. We only need to consider the case $\alpha \in (1, 2]$. Note that **Assumption SM(i)** and **Assumption SM(ii)** are identical for $\alpha \in (0, 1]$. Let $\Delta_{s,n}$ as above. Using a similar argument to the one of Astrauskas (1983, Lemma) together with (178) we get

$$\mathbf{E} |\xi_1 - \xi^*|^\alpha I \left\{ |\xi_1 - \xi^*| \leq \gamma_n |a_s|^{-1} \right\} \asymp S \left(\frac{\gamma_n}{|a_s|} \right),$$

where S a bounded SV function such that

$$S(x) = \int_1^x \frac{h(u)}{u} du.$$

Hence, as $n \rightarrow \infty$

$$\mathbf{E} \Delta_{s,n}^\alpha I \{ \Delta_{s,n} \leq 1 \} = C \left(\frac{|a_s|}{\gamma_n} \right)^\alpha S \left(\frac{\gamma_n}{|a_s|} \right) [1 + o(1)].$$

In view of this and using similar arguments as those used in (181) as $n \rightarrow \infty$ we get

$$\begin{aligned} I_n &\leq \frac{1}{n} \sum_{s=0}^{n-1} [\mathbf{E} |M_{n,s}|^\alpha]^{1/\alpha} \leq C \frac{n^{1/\alpha}}{\gamma_n n} \sum_{s=0}^n |a_s| S \left(\frac{\gamma_n}{|a_s|} \right)^{1/\alpha} \\ &\leq C \frac{n^{1/\alpha}}{\gamma_n n} \sum_{s=0}^n |a_s| \leq \frac{n^{1/\alpha}}{\gamma_n n} \sum_{l=0}^n \rho_n^l \sum_{s=0}^\infty |c_s| \\ &\leq C \frac{n^{1/\alpha} \kappa_n}{\gamma_n n} = C \frac{n^{1/\alpha} \kappa_n}{\kappa_n^{1/\alpha} \ell_1(\kappa_n)^{\frac{1}{\alpha} n}} \\ &= \frac{C}{\ell_1(\kappa_n)^{\frac{1}{\alpha}}} \left(\frac{n}{\kappa_n} \right)^{\frac{1}{\alpha} - 1} = \frac{C}{\ell_1(n^{\alpha \kappa} \ell_\kappa(n))^{\frac{1}{\alpha}}} \left(\frac{n}{n^{\alpha \kappa} \ell_\kappa(n)} \right)^{\frac{1}{\alpha} - 1} = o(1), \end{aligned}$$

as required.

MI-LM ($\alpha \in (1, 2]$, $m \in (1/\alpha, 1)$). Similarly to the previous part, let

$$0 < \delta < 1 \text{ and } 0 \leq \varepsilon < \alpha \left(\frac{1}{\delta} - 1 \right). \quad (186)$$

Then as $n \rightarrow \infty$

$$I'_n = \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\alpha/\alpha_\varepsilon} n} \sum_{s=0}^n |a_s|^{\alpha/\alpha_\varepsilon} H \left(\frac{\gamma_n}{|a_s|} \right)^{1/\alpha_\varepsilon}$$

$$\begin{aligned}
&\leq \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \sum_{s=0}^n |a_s|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} = \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \sum_{s=0}^n \left| \sum_{l=0}^s \rho_n^l c_{s-l} \right|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} \\
&\leq \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \sum_{s=0}^n \sum_{l=0}^s |\rho_n^l c_{s-l}|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} = \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \sum_{l=0}^n \sum_{s=l}^n |\rho_n^l c_{s-l}|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} \\
&\leq \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \sum_{l=0}^n |\rho_n^l|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} \sum_{s=0}^n |c_s|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} \leq_{(2)} C \frac{n^{1/\alpha_\varepsilon} \kappa_n}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \sum_{s=1}^n s^{-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)} [1 + o(1)] \\
&= C \frac{n^{\frac{1}{\alpha_\varepsilon}-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)} \kappa_n}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \frac{1}{n} \sum_{s=1}^n \left(\frac{s}{n}\right)^{-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)} [1 + o(1)] \\
&= C \frac{n^{\frac{1}{\alpha_\varepsilon}-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)} \kappa_n}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \int_0^1 x^{-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)} dx [1 + o(1)],
\end{aligned}$$

where $\leq_{(2)}$ follows from DK2018 (Lemma 6.4), and the fact that $\frac{\alpha}{\alpha_\varepsilon} - \delta > 0$, under (186). Moreover, notice that the integral above is finite because $m(\frac{\alpha}{\alpha_\varepsilon} - \delta) < 1$. Recall that $\gamma_n \sim \kappa_n^{b/\alpha} \ell_1(\kappa_n)^{1/\alpha}$, $b = 1 + \alpha - \alpha m$ with $m \in (1/\alpha, 1)$, and $\kappa_n = n^{\alpha_\kappa} \ell_\kappa(n)$. Write

$$\begin{aligned}
\frac{n^{\frac{1}{\alpha_\varepsilon}-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)} \kappa_n}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} &= \frac{n^{\frac{1}{\alpha_\varepsilon}-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)} n^{\alpha_\kappa} \ell_\kappa(n)}{[\kappa_n^{b/\alpha} \ell_1(\kappa_n)^{1/\alpha}]^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} = \frac{n^{\frac{1}{\alpha_\varepsilon}-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)} n^{\alpha_\kappa} \ell_\kappa(n)}{[(n^{\alpha_\kappa} \ell_\kappa(n))^{b/\alpha} \ell_1(\kappa_n)^{1/\alpha}]^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \\
&= \frac{\ell_\kappa(n)}{[\ell_\kappa(n)^{b/\alpha} \ell_1(\kappa_n)^{1/\alpha}]^{\frac{\alpha}{\alpha_\varepsilon}-\delta}} \frac{n^{\frac{1}{\alpha_\varepsilon} + \alpha_\kappa - m(\frac{\alpha}{\alpha_\varepsilon}-\delta)}}{n^{\frac{\alpha_\kappa}{\alpha} b(\frac{\alpha}{\alpha_\varepsilon}-\delta)}} =: \lambda_n n^\eta,
\end{aligned}$$

where as in the previous part λ_n is SV in n . Therefore, it suffices to show that $\eta < 0$. Lengthy but straightforward calculations show that the latter is true if δ and ε satisfy the additional requirements

$$\delta < \frac{(\alpha m - 1)(1 - \alpha_\kappa)}{\alpha_\kappa b + \alpha m} \text{ and } \varepsilon < \frac{(\alpha m - 1)(1 - \alpha_\kappa) - [\alpha_\kappa b + \alpha m] \delta}{m\delta + \alpha_\kappa + \alpha_\kappa \frac{b}{\alpha} \delta}.$$

MI-NM ($\alpha \in (1, 2]$, $m \in (1, 1 + 1/\alpha)$). Similarly to the previous part, assume that (175) holds. In addition, suppose that

$$\delta < \frac{(\alpha - 1)[1 - \alpha_\kappa]}{\alpha_\kappa} 1_{\{\alpha_\kappa \neq 0\}} + 1_{\{\alpha_\kappa = 0\}} \wedge \frac{m - 1}{m}, \quad (187)$$

and

$$\varepsilon < \frac{\alpha m (1 - \delta) - \alpha}{1 + m\delta} \wedge \alpha (\delta^{-1} - 1). \quad (188)$$

We remark that the requirement $\delta < \frac{m-1}{m}$ ensures that the upper bound $\frac{\alpha m(1-\delta)-\alpha}{1+m\delta}$ in (188) is strictly positive. Then as $n \rightarrow \infty$

$$I'_n = \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\alpha/\alpha_\varepsilon} n} \sum_{s=0}^n |a_s|^{\alpha/\alpha_\varepsilon} H\left(\frac{\gamma_n}{|a_s|}\right)^{1/\alpha_\varepsilon}$$

$$\begin{aligned}
&\leq_{(1)} \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta} n} \sum_{s=0}^n |a_s|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} = \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta} n} \sum_{s=0}^n \left| \sum_{l=0}^s \rho_n^l c_{s-l} \right|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} \\
&=_{(2)} \frac{n^{1/\alpha_\varepsilon}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta} n} \sum_{s=0}^{\delta_n} \left| \sum_{l=0}^s \rho_n^l c_{s-l} \right|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} [1 + o(1)] \\
&= \frac{n^{1/\alpha_\varepsilon} \kappa_n^{1+\frac{\alpha}{\alpha_\varepsilon}-\delta-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta} n} \frac{1}{\kappa_n} \sum_{s=0}^{\delta_n} \left| \frac{1}{\kappa_n} \sum_{l=1}^s \rho_n^{s-l} \left(\frac{l}{\kappa_n} \right)^{-m} \right|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} [1 + o(1)] \\
&=_{(3)} C \int_{x=0}^{\infty} |p(x)|^{\frac{\alpha}{\alpha_\varepsilon}-\delta} dx \cdot \frac{n^{1/\alpha_\varepsilon} \kappa_n^{1+\frac{\alpha}{\alpha_\varepsilon}-\delta-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta} n} [1 + o(1)] \\
&= C \frac{n^{1/\alpha_\varepsilon} \kappa_n^{1+\frac{\alpha}{\alpha_\varepsilon}-\delta-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)}}{\gamma_n^{\frac{\alpha}{\alpha_\varepsilon}-\delta} n} = C \frac{n^{1/\alpha_\varepsilon} \kappa_n^{1+\frac{\alpha}{\alpha_\varepsilon}-\delta-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)}}{\left[\kappa_n^{b/\alpha} \ell_1(\kappa_n)^{1/\alpha} \right]^{\left(\frac{\alpha}{\alpha_\varepsilon}-\delta\right)} n} \\
&= C \frac{n^{1/\alpha_\varepsilon} \kappa_n^{1+\frac{\alpha}{\alpha_\varepsilon}-\delta-m(\frac{\alpha}{\alpha_\varepsilon}-\delta)}}{\ell_1(\kappa_n)^{\frac{1}{\alpha}(\frac{\alpha}{\alpha_\varepsilon}-\delta)} \kappa_n^{\left(\frac{1}{\alpha}+1-m\right)(\frac{\alpha}{\alpha_\varepsilon}-\delta)} n} = C \frac{n^{1/\alpha_\varepsilon} \kappa_n^{1+\frac{\alpha}{\alpha_\varepsilon}-\delta}}{\ell_1(\kappa_n)^{\frac{1}{\alpha}(\frac{\alpha}{\alpha_\varepsilon}-\delta)} \kappa_n^{\left(\frac{1}{\alpha}+1\right)(\frac{\alpha}{\alpha_\varepsilon}-\delta)} n} \\
&= C \frac{n^{\frac{1}{\alpha_\varepsilon}-1} \kappa_n^{1-\frac{1}{\alpha_\varepsilon}+\frac{\delta}{\alpha}}}{\ell_1(\kappa_n)^{\frac{1}{\alpha}(\frac{\alpha}{\alpha_\varepsilon}-\delta)}} = C =: \lambda_n \cdot n^{\frac{1}{\alpha_\varepsilon}-1} n^{\alpha_\kappa(1-\frac{1}{\alpha_\varepsilon}+\frac{\delta}{\alpha})} \\
&=: C \lambda_n n^\eta,
\end{aligned}$$

where $\leq_{(1)}$ follows from (178). Moreover, $=_{(2)}$ follows from **Lemma 14** and the fact that $m\left(\frac{\alpha}{\alpha_\varepsilon}-\delta\right) > 1$. Moreover, $=_{(3)}$ is due to **Lemma 13**.²⁵ Lengthy but straightforward calculations show that under the current assumptions, i.e. (175), (187) and (188), $\eta < 0$. In view of this, the requisite result follows from the fact that λ_n is SV in n . ■

15. Additional Workings for the MI-LM and MI-NM Cases

We start with some notation. For some $m, \theta > 0$ set

$$b_\theta := 1 + \theta - \theta m$$

We have the following results.

Lemma 11. Let $m \in (1, 2)$ and $\theta > 0$ such that $1/m < \theta < 1/(m-1)$. Moreover, suppose that ρ_n satisfies **Assumption MI**. Set

$$T_{1nk} := -\kappa_n^{-1+m} \sum_{j=k+1}^{\infty} c_j \text{ and } T_{2nk} := \frac{1}{\kappa_n^{2-m}} \sum_{l=1}^k \rho_n^{k-l} \sum_{j=l}^{\infty} c_j. \quad (189)$$

Finally, let $\varsigma(\cdot)$ SV such that $\lim_{x \rightarrow \infty} \varsigma(x) = \infty$, and set $\delta_n := \kappa_n \cdot \varsigma(\kappa_n)$, $\mu_n := \mu\left(\kappa_n^{b_\theta/2}\right)$ with $\mu(j) := \left| \frac{c_j}{j^{-m}} - 1 \right| = O(j^{-\nu})$ for some $\nu > 0$.

²⁵Note that (188) implies that $m\left(\frac{\alpha}{\alpha_\varepsilon}-\delta\right) > 1$.

Then as $n \rightarrow \infty$

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |\Delta_{1nk}|^\theta := \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left| T_{1nk} + \int_{\frac{k}{\kappa_n}}^{\infty} x^{-m} dx \right|^\theta = o(1), \quad (190)$$

and

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |\Delta_{2nk}|^\theta := \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left| T_{2nk} - \frac{1}{1-m} e^{-\frac{k}{\kappa_n}} \int_0^{k/\kappa_n} e^y y^{1-m} dy \right|^\theta = o(1). \quad (191)$$

Corollary 1. Suppose that **Assumption MI-NM** holds. Let T_{1nk} and T_{2nk} as in (189). Then as $n \rightarrow \infty$

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |\Delta_{1nk}|^\alpha := \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left| T_{1nk} + \int_{\frac{k}{\kappa_n}}^{\infty} x^{-m} dx \right|^\alpha = o(1),$$

and

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |\Delta_{2nk}|^\alpha := \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left| T_{2nk} + \frac{1}{1-m} e^{-\frac{k}{\kappa_n}} \int_0^{k/\kappa_n} e^y y^{1-m} dy \right|^\alpha = o(1),$$

where δ_n is given in **Lemma 11**.

Remark 21. Note that under the assumptions of **Lemma 11**, for all $a, b > 0$

$$\mu_n^a \varsigma(\kappa_n)^b \rightarrow 0, \quad (192)$$

as $n \rightarrow \infty$.

For the subsequent results recall that

$$p(x) := x^{1-m} - e^{-x} \int_0^x e^y y^{1-m} dy.$$

Remark 22. (a) It can be shown that -see (244) below- that under **MI-NM**,

$$\kappa_n^{1-m} a_k(n) = \kappa_n^{1-m} \sum_{l=0}^k \rho_n^{k-l} c_l = T_{1nk} + T_{2nk}.$$

Moreover,

$$\begin{aligned} & - \int_{\frac{k}{\kappa_n}}^{\infty} x^{-m} dx - \frac{1}{1-m} e^{-\frac{k}{\kappa_n}} \int_0^{k/\kappa_n} e^y y^{1-m} dy \\ &= \frac{(k/\kappa_n)^{1-m}}{1-m} - \frac{1}{1-m} e^{-\frac{k}{\kappa_n}} \int_0^{k/\kappa_n} e^y y^{1-m} dy = \frac{p(k/\kappa_n)}{1-m}. \end{aligned}$$

(b) The following fact is utilized in the proof of **Lemma 1** and is a direct consequence of **Lemma 1**

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left| \kappa_n^{m-1} a_k(n) - \frac{p(k/\kappa_n)}{1-m} \right|^\alpha = o(1), \quad (193)$$

as $n \rightarrow \infty$.

Lemma 12. Let $m \in (1, 2)$ and $\theta > 0$ such that $1/m < \theta < 1/(m-1)$. Then as $n \rightarrow \infty$

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |p(k/\kappa_n)|^\theta = \int_0^\infty |p(x)|^\theta dx + o(1),$$

with δ_n given in **Lemma 11**.

Corollary 2. Suppose that **Assumption MI-NM** holds. Then as $n \rightarrow \infty$

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |p(k/\kappa_n)|^\alpha = \int_0^\infty |p(x)|^\alpha dx + o(1),$$

and

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} p(k/\kappa_n)^{(\alpha)} = \int_0^\infty p(x)^{(\alpha)} dx + o(1),$$

where δ_n is given in **Lemma 11**.

Lemma 13. Let $m \in (1, 2)$ and $\theta > 0$ such that $1/m < \theta < 1/(m-1)$. Then as $n \rightarrow \infty$

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk} + T_{2nk}|^\theta = \frac{1}{(m-1)^\theta} \int_0^\infty |p(x)|^\theta dx + o(1),$$

with δ_n given in **Lemma 11**.

Corollary 3. Suppose that **Assumption MI-NM** holds. Then as $n \rightarrow \infty$

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk} + T_{2nk}|^\alpha = \frac{1}{(m-1)^\alpha} \int_0^\infty |p(x)|^\alpha dx + o(1),$$

with δ_n given in **Lemma 11**.

Proof of Lemma 11. Notice that under the assumptions of **Lemma 11** we have

$$\theta m > 1 \quad \text{and} \quad b_\theta := 1 + \theta - \theta m > 0. \quad (194)$$

- T_{1nk} : We start with the proof of (190). Set

$$\begin{aligned} T_{1nk} &:= -\kappa_n^{m-1} \sum_{j=k+1}^{\infty} c_j, \quad T_{1nk}^{(1)} := -\kappa_n^{-1+m} \sum_{j=k+1}^{\infty} j^{-m} \\ T_{1nk}^{(2)} &:= -\int_{\frac{k+1}{\kappa_n}}^{\infty} x^{-m} dx, \quad T_{1nk}^{(3)} := -\int_{\frac{k}{\kappa_n}}^{\infty} x^{-m} dx. \end{aligned} \quad (195)$$

In addition, let

$$\begin{aligned} \Delta_{1nk}^{(1)} &:= T_{1nk} - T_{1nk}^{(1)}, \quad \Delta_{1nk}^{(2)} := T_{1nk}^{(1)} - T_{1nk}^{(2)}, \\ \Delta_{1nk}^{(3)} &:= T_{1nk}^{(2)} - T_{1nk}^{(3)}, \quad \text{and} \quad \Delta_{1nk} := T_{1nk} - T_{1nk}^{(3)}. \end{aligned}$$

First, note that

$$\frac{1}{\kappa_n} \sum_{k=1}^{\kappa_n^{b_\theta/2}} |T_{1nk}|^\theta \leq \frac{\kappa_n^{b_\theta/2}}{\kappa_n^{b_\theta}} \left(\sum_{k=1}^{\infty} |c_k| \right)^\theta = O\left(\kappa_n^{-b_\theta/2}\right) = o(1). \quad (196)$$

Write

$$\frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/2}}^{\delta_n} |\Delta_{1nk}|^\theta \leq \frac{C}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/2}}^{\delta_n} \sum_{i=1}^3 |\Delta_{1nk}^{(i)}|^\theta. \quad (197)$$

We shall show that the terms on the r.h.s. above are $o(1)$, as $n \rightarrow \infty$.

As $n \rightarrow \infty$ the first term

$$\begin{aligned} \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/2}}^{\delta_n} |\Delta_{1nk}^{(1)}|^\theta &= \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/2}}^{\delta_n} \left| \kappa_n^{-1+m} \sum_{j=k+1}^{\infty} c_j - \kappa_n^{-1+m} \sum_{j=k+1}^{\infty} j^{-m} \right|^\theta \\ &\leq \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/2}}^{\delta_n} \left| \kappa_n^{-1+m} \sum_{j=k+1}^{\infty} \left| \frac{c_j}{j^{-m}} - 1 \right| j^{-m} \right|^\theta \leq \mu_n^\theta \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left| \kappa_n^{-1+m} \sum_{j=k+1}^{\infty} j^{-m} \right|^\theta \\ &= \mu_n^\theta \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk}^{(1)}|^\theta =_{(1)} o(1), \end{aligned} \quad (198)$$

where $=_{(1)}$ will be proved later.

Next, consider

$$\begin{aligned} \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |\Delta_{1nk}^{(2)}|^\theta &= \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left| \kappa_n^{-1+m} \sum_{j=k+1}^{\infty} j^{-m} - \int_{\frac{k+1}{\kappa_n}}^{\infty} x^{-m} dx \right|^\theta \\ &\leq \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} \mathbf{V} \left(x^{-m} \right)_{\frac{k+1}{\kappa_n}}^{\infty} \right]^\theta \leq \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} \left(\frac{k+1}{\kappa_n} \right)^{-m} \right]^\theta \\ &= \frac{1}{\kappa_n^{b_\theta}} \sum_{k=1}^{\delta_n} (k+1)^{-\theta m} \leq \frac{1}{\kappa_n^{b_\theta}} \sum_{k=1}^{\infty} (k+1)^{-\theta m} = O(\kappa_n^{-b_\theta}) = o(1), \end{aligned} \quad (199)$$

where recall that $b_\theta > 0$.

The third term

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |\Delta_{1nk}^{(3)}|^\theta = \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left| \left(\frac{k+1}{\kappa_n} \right)^{1-m} - \left(\frac{k}{\kappa_n} \right)^{1-m} \right|^\theta. \quad (200)$$

The summand in (200) is

$$\begin{aligned} \left(\frac{\kappa_n}{k} \right)^{m-1} - \left(\frac{\kappa_n}{k+1} \right)^{m-1} &= \left(\frac{\kappa_n}{k} \right)^{m-1} - \left(\frac{\kappa_n}{k} + \frac{\kappa_n}{k+1} - \frac{\kappa_n}{k} \right)^{m-1} \\ &= \left(\frac{\kappa_n}{k} \right)^{m-1} - \left(\frac{\kappa_n}{k} + \frac{k\kappa_n - (k+1)\kappa_n}{(k+1)k} \right)^{m-1} \\ &= \left(\frac{\kappa_n}{k} \right)^{m-1} - \left(\frac{\kappa_n}{k} - \frac{\kappa_n}{(k+1)k} \right)^{m-1} \\ &= \left(\frac{\kappa_n}{(k+1)k} + \frac{\kappa_n}{k} - \frac{\kappa_n}{(k+1)k} \right)^{m-1} - \left(\frac{\kappa_n}{k} - \frac{\kappa_n}{(k+1)k} \right)^{m-1} \\ &\leq \left(\frac{\kappa_n}{(k+1)k} \right)^{m-1} + \left(\frac{\kappa_n}{k} - \frac{\kappa_n}{(k+1)k} \right)^{m-1} - \left(\frac{\kappa_n}{k} - \frac{\kappa_n}{(k+1)k} \right)^{m-1} \end{aligned}$$

$$= \left[\frac{\kappa_n}{(k+1)k} \right]^{m-1}.$$

Hence,

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |\Delta_{1nk}^{(3)}|^\theta \leq \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{\kappa_n}{(k+1)k} \right]^{\theta(m-1)} =_{(2)} o(1). \quad (201)$$

To see why $=_{(2)}$ holds, first assume that $2\theta(m-1) > 1$.²⁶ In this case

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{\kappa_n}{(k+1)k} \right]^{\theta(m-1)} = \frac{1}{\kappa_n^{b_\theta}} \sum_{k=1}^{\delta_n} \left[\frac{1}{(k+1)k} \right]^{\theta(m-1)} \leq \frac{1}{\kappa_n^{b_\theta}} \sum_{k=1}^{\infty} k^{-2\theta(m-1)} = o(1).$$

Next, suppose that $0 < 2\theta(m-1) \leq 1$. Then

$$\begin{aligned} \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{\kappa_n}{(k+1)k} \right]^{(m-1)\theta} &\leq \frac{1}{\kappa_n} \sum_{k=1}^{\kappa_n} \left(\frac{\kappa_n}{k^2} \right)^{(m-1)\theta} + \frac{1}{\kappa_n} \sum_{k=\kappa_n}^{\delta_n} \left(\frac{\kappa_n}{k^2} \right)^{(m-1)\theta} \\ &\leq \frac{1}{\kappa_n^{1+(m-1)\theta}} \sum_{k=1}^{\kappa_n} \left(\frac{\kappa_n^2}{k^2} \right)^{(m-1)\theta} + \frac{1}{\kappa_n} \sum_{k=\kappa_n}^{\delta_n} \left(\frac{\kappa_n}{\kappa_n^2} \right)^{(m-1)\theta} \\ &\leq \frac{1}{\kappa_n^{1+(m-1)\theta}} \sum_{k=1}^{\kappa_n} \left(\frac{\kappa_n^2}{k^2} \right)^{(m-1)\theta} + \frac{\delta_n}{\kappa_n^{1+(m-1)\theta}} \\ &= \frac{1}{\kappa_n^{(m-1)\theta}} \left[\int_{1/\kappa_n}^1 x^{-2(m-1)\theta} dx + o(1) \right] + \frac{\delta_n}{\kappa_n^{1+(m-1)\theta}} = o(1), \end{aligned}$$

where we have used the fact that $\delta_n = \kappa_n \varsigma(\kappa_n)$ with ς slowly varying, and

$$\int_{1/\kappa_n}^1 x^{-2(m-1)\theta} dx = \begin{cases} \text{bounded, for } 0 < 2\theta(m-1) < 1 \\ \ln(\kappa_n), \text{ for } 2\theta(m-1) = 1 \end{cases}.$$

Finally, we show that $=_{(1)}$ in (198) holds true. Write

$$\begin{aligned} \frac{\mu_n^\theta}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk}^{(1)}|^\theta &\leq C \left(\frac{\mu_n^\theta}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk}^{(1)} - T_{1nk}^{(3)}|^\theta + \frac{\mu_n^\theta}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk}^{(3)}|^\theta \right) \\ &\leq C \left[\frac{\mu_n^\theta}{\kappa_n} \sum_{k=1}^{\delta_n} (|\Delta_{1nk}^{(2)}|^\theta + |\Delta_{1nk}^{(3)}|^\theta) + \frac{\mu_n^\theta}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk}^{(3)}|^\theta \right] \\ &=_{(3)} o(\mu_n^\theta) + C \frac{\mu_n^\theta}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk}^{(3)}|^\theta \\ &=_{(4)} o(1) + C \mu_n^\theta \left[\frac{1}{|1-m|^\theta} \int_{1/\kappa_n}^{\delta_n/\kappa_n} x^{-\theta(m-1)} dx + o(1) \right] \\ &= o(1) + C \mu_n^\theta \left[x^{-\theta m + \theta + 1} \right]_{1/\kappa_n}^{\delta_n/\kappa_n} = o(1) + C \mu_n^\theta \left[(\delta_n/\kappa_n)^{b_\theta} - (1/\kappa_n)^{b_\theta} \right] = o(1), \\ &\leq o(1) + C \mu_n^\theta \left[(\kappa_n \varsigma(\kappa_n)/\kappa_n)^{b_\theta} \right] = O(\mu_n^\theta \varsigma(\kappa_n)^{b_\theta}) =_{(5)} o(1), \end{aligned}$$

where $=_{(3)}$ follows from (200) and (201), whilst $=_{(4)}$ follows from Euler summation, see, e.g. **Lemma 12**. Finally, $=_{(5)}$ is due to (192). Hence, (190) is a direct consequence of (197), (198), (199), and (201).

²⁶By definition, $2\theta(m-1) > 0$.

- T_{2nk} : Next, we prove (191). Let

$$p > \frac{b_\theta}{2-m} \vee 1,$$

where note that $0 < 2-m < 1$. Hence,

$$0 < b_\theta/p < 2-m < 1. \quad (202)$$

Set

$$\begin{aligned} T_{2nk} &:= T_{2nk}^{(0)} := \frac{1}{\kappa_n^{2-m}} \sum_{l=1}^k \rho_n^{k-l} \sum_{j=l}^\infty c_j, \quad T_{2nk}^{(1)} := \frac{1}{\kappa_n^{2-m}} \sum_{l=\kappa_n^{b_\theta/p}}^k \rho_n^{k-l} \sum_{j=l}^\infty c_j, \\ T_{2nk}^{(2)} &:= \frac{1}{\kappa_n^{2-m}} \sum_{l=\kappa_n^{b_\theta/p}}^k \rho_n^{k-l} \sum_{j=l}^\infty j^{-m}, \quad T_{2nk}^{(3)} := \frac{1}{\kappa_n} \sum_{l=\kappa_n^{b_\theta/p}}^k \rho_n^{k-l} \int_{l/\kappa_n}^\infty y^{-m} dy, \\ T_{2nk}^{(4)} &:= \frac{1}{\kappa_n} \sum_{l=\kappa_n^{b_\theta/p}}^k e^{-\frac{k-l}{\kappa_n}} \int_{l/\kappa_n}^\infty y^{-m} dy = -\frac{1}{1-m} \kappa_n^{-1} e^{-\frac{k}{\kappa_n}} \sum_{l=\kappa_n^{b_\theta/p}}^k e^{\frac{l}{\kappa_n}} \left(\frac{l}{\kappa_n}\right)^{1-m}, \\ T_{2nk}^{(5)} &:= -\frac{1}{1-m} e^{-\frac{k}{\kappa_n}} \int_{\kappa_n^{b_\theta/p}/\kappa_n}^{k/\kappa_n} e^y y^{1-m} dy, \text{ and} \\ T_{2nk}^{(6)} &:= -\frac{1}{1-m} e^{-\frac{k}{\kappa_n}} \int_0^{k/\kappa_n} e^y y^{1-m} dy. \end{aligned} \quad (203)$$

Moreover, for $i = 1, \dots, 6$ set

$$\Delta_{2nk}^{(i)} := T_{2nk}^{(i)} - T_{2nk}^{(i-1)}, \text{ and } \Delta_{2nk} := T_{2nk} - T_{2nk}^{(6)}.$$

Consider

$$\begin{aligned} \frac{1}{\kappa_n} \sum_{k=1}^{\kappa_n^{b_\theta/p}} |T_{2nk}|^\theta &= \frac{1}{\kappa_n} \sum_{k=1}^{\kappa_n^{b_\theta/p}} \left| \frac{1}{\kappa_n^{2-m}} \sum_{l=1}^k \rho_n^{k-l} \sum_{j=l}^\infty c_j \right|^\theta \\ &\leq \frac{1}{\kappa_n} \sum_{k=1}^{\kappa_n^{b_\theta/p}} \left(\frac{1}{\kappa_n^{2-m}} \sum_{l=1}^{\kappa_n^{b_\theta/p}} \sum_{j=0}^\infty |c_j| \right)^\theta \leq C \frac{\kappa_n^{b_\theta/p}}{\kappa_n} \left(\frac{\kappa_n^{b_\theta/p}}{\kappa_n^{2-m}} \right)^\theta = o(1), \end{aligned} \quad (204)$$

where the last equality follows from (202) above. Next, we shall show that

$$\frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |\Delta_{2nk}|^\theta \leq C \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \sum_{i=1}^6 |\Delta_{2nk}^{(i)}|^\theta = o(1). \quad (205)$$

- (i) $\Delta_{2nk}^{(1)}$: Consider the first summand on the r.h.s. of (205)

$$\begin{aligned} \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |\Delta_{2nk}^{(1)}|^\theta &= \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |T_{2nk} - T_{2nk}^{(1)}|^\theta \\ &= \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| \frac{1}{\kappa_n^{2-m}} \sum_{l=1}^{\kappa_n^{b_\theta/p}-1} \rho_n^{k-l} \sum_{j=l}^\infty c_j \right|^\theta \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left(\frac{1}{\kappa_n^{2-m}} \sum_{l=1}^{\kappa_n^{b_\theta/p}-1} \sum_{j=l}^{\infty} |c_j| \right)^\theta \leq \left(\frac{\kappa_n^{b_\theta/p}}{\kappa_n^{2-m}} \right)^\theta \frac{C\delta_n}{\kappa_n} \left(\sum_{j=0}^{\infty} |c_j| \right)^\theta \\
&= C \left(\frac{\kappa_n^{b_\theta/p}}{\kappa_n^{2-m}} \right)^\theta \frac{\delta_n}{\kappa_n} = C \left(\frac{\kappa_n^{b_\theta/p}}{\kappa_n^{2-m}} \right)^\theta \varsigma(\kappa_n) = o(1),
\end{aligned}$$

where we have used the fact that $b_\theta/p < 2 - m$ and that $\varsigma(\cdot)$ is slowly varying.

(ii) $\Delta_{2nk}^{(2)}$:

$$\begin{aligned}
\frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| \Delta_{2nk}^{(2)} \right|^\theta &= \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| \frac{1}{\kappa_n^{2-m}} \sum_{l=\kappa_n^{b_\theta/p}}^k \rho_n^{k-l} \sum_{j=l}^{\infty} j^{-m} - \frac{1}{\kappa_n^{2-m}} \sum_{l=\kappa_n^{b_\theta/p}}^k \rho_n^{k-l} \sum_{j=l}^{\infty} c_j \right|^\theta \\
&\leq \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| \frac{1}{\kappa_n^{2-m}} \sum_{l=\kappa_n^{b_\theta/p}}^k \rho_n^{k-l} \sum_{j=l}^{\infty} j^{-m} \left| \frac{c_j}{j^{-m}} - 1 \right| \right|^\theta \\
&\leq \frac{\mu_n^\theta}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| \frac{1}{\kappa_n^{2-m}} \sum_{l=\kappa_n^{b_\theta/p}}^k \rho_n^{k-l} \sum_{j=l}^{\infty} j^{-m} \right|^\theta \\
&= \frac{\mu_n^\theta}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| T_{2nk}^{(2)} \right|^\theta =_{(5)} o(1), \tag{206}
\end{aligned}$$

where $=_{(5)}$ will be established in detail later.

(iii) $\Delta_{2nk}^{(3)}$: Next, noting that

$$\begin{aligned}
&\left| \frac{1}{\kappa_n} \sum_{j=l}^{\infty} \left(\frac{j}{\kappa_n} \right)^{-m} - \int_{l/\kappa_n}^{\infty} x^{-m} dx \right| \\
&\leq \kappa_n^{-1} \left[\mathbf{V}(x^{-m}) \right]_{l/\kappa_n}^{\infty} \leq \kappa_n^{-1} (l/\kappa_n)^{-m}, \tag{207}
\end{aligned}$$

we get

$$\begin{aligned}
\frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| \Delta_{2nk}^{(3)} \right|^\theta &= \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| T_{2nk}^{(3)} - T_{2nk}^{(2)} \right|^\theta \\
&\leq \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| \frac{1}{\kappa_n} \sum_{l=\kappa_n^{b_\theta/p}}^k \rho_n^{k-l} \int_{l/\kappa_n}^{\infty} y^{-m} dy - \frac{1}{\kappa_n} \sum_{l=\kappa_n^{b_\theta/p}}^k \rho_n^{k-l} \frac{1}{\kappa_n} \sum_{j=l}^{\infty} \left(\frac{j}{\kappa_n} \right)^{-m} \right|^\theta \\
&\leq_{(6)} \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left| \frac{1}{\kappa_n^2} \sum_{l=1}^k \rho_n^{k-l} (l/\kappa_n)^{-m} \right|^\theta =_{(7)} o(1), \tag{208}
\end{aligned}$$

where the $\leq_{(6)}$ follows from (207) whilst $=_{(7)}$ follows from the fact that

$$\frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left| \frac{1}{\kappa_n^2} \sum_{l=1}^k \rho_n^{k-l} (l/\kappa_n)^{-m} \right|^\theta$$

$$\leq \frac{C}{\kappa_n \kappa_n^{\theta(2-m)}} \sum_{k=1}^{\delta_n} \left(\sum_{l=1}^{\infty} l^{-m} \right)^{\theta} = \frac{C \delta_n}{\kappa_n \kappa_n^{\theta(2-m)}} = o(1),$$

where the last equality is due to the fact that $2 - m > 0$ and $\delta_n = \kappa_n \varsigma(\kappa_n)$, with $\varsigma(\cdot)$ SV.

(iv) $\Delta_{2nk}^{(4)}$:

$$\begin{aligned} & \frac{1}{\kappa_n} \sum_{k=\kappa_n}^{\delta_n} \left| \Delta_{2nk}^{(4)} \right|^{\theta} \\ &= \frac{1}{\kappa_n} \sum_{k=\kappa_n}^{\delta_n} \left| \frac{1}{\kappa_n} \sum_{l=\kappa_n}^k e^{-\frac{k-l}{\kappa_n}} \int_{l/\kappa_n}^{\infty} y^{-m} dy - \frac{1}{\kappa_n} \sum_{l=\kappa_n}^k \rho_n^{k-l} \int_{l/\kappa_n}^{\infty} y^{-m} dy \right|^{\theta} \\ &\leq \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} \sum_{l=\kappa_n}^k \left| e^{-\frac{k-l}{\kappa_n}} - \rho_n^{k-l} \right| \int_{l/\kappa_n}^{\infty} y^{-m} dy \right]^{\theta} \\ &= \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} \sum_{l=\kappa_n}^k \left| e^{-\frac{k-l}{\kappa_n}} - \rho_n^{k-l} \right| (l/\kappa_n)^{1-m} \right]^{\theta} \\ &\leq_{(8)} \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left| \kappa_n^{-1} \sum_{l=\kappa_n}^k e^{-\frac{k-l}{\kappa_n}} \frac{k-l}{\kappa_n^2} \left(\frac{l}{\kappa_n} \right)^{1-m} \right|^{\theta} \\ &\leq C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\theta} \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left| \kappa_n^{-1} \sum_{l=\kappa_n}^k e^{-\frac{k-l}{\kappa_n}} \left(\frac{l}{\kappa_n} \right)^{1-m} \right|^{\theta} \\ &= C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\theta} \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left| T_{2nk}^{(4)} \right|^{\theta} =_{(9)} o(1), \end{aligned} \tag{209}$$

where $\leq_{(8)}$ follows from DK2018, Lemma 6.4 whilst $=_{(9)}$ will be proved in detail later.

(v) $\Delta_{2nk}^{(5)}$: Set $\omega(y) := e^y y^{1-m}$. We have

$$\begin{aligned} & \frac{1}{\kappa_n} \sum_{k=\kappa_n}^{\delta_n} \left| \Delta_{2nk}^{(5)} \right|^{\theta} = \frac{1}{\kappa_n} \sum_{k=\kappa_n}^{\delta_n} \left| T_{2nk}^{(5)} - T_{2nk}^{(4)} \right|^{\theta} \\ &= \left| \frac{1}{1-m} \right|^{\theta} \frac{1}{\kappa_n} \sum_{k=\kappa_n}^{\delta_n} \left| e^{-\frac{k}{\kappa_n}} \int_{\kappa_n}^{k/\kappa_n} e^y y^{1-m} dy - \frac{e^{-\frac{k}{\kappa_n}}}{\kappa_n} \sum_{l=\kappa_n}^k e^{\frac{l}{\kappa_n}} \left(\frac{l}{\kappa_n} \right)^{1-m} \right|^{\theta} \\ &\leq \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{e^{-\frac{k}{\kappa_n}}}{\kappa_n} \mathbf{V}[\omega(y)]_{\kappa_n}^{k/\kappa_n} \right]^{\theta} \leq \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{e^{-\frac{k}{\kappa_n}}}{\kappa_n} \mathbf{V}[\omega(y)]_{1/\kappa_n}^{k/\kappa_n} \right]^{\theta}, \end{aligned} \tag{210}$$

where recall that $\mathbf{V}[\omega(y)]_{1/\kappa_n}^{k/\kappa_n}$ is the total variation of $\omega(y)$ on $[1/\kappa_n, k/\kappa_n]$. It follows from the proof of **Proposition 1** that

$$e^{-\frac{k}{\kappa_n}} \mathbf{V}[\omega(y)]_{1/\kappa_n}^{k/\kappa_n} \leq 2e^{-\frac{k}{\kappa_n}} [\omega(k/\kappa_n) + \omega(m-1) + \omega(1/\kappa_n)].$$

Hence,

$$\begin{aligned}
\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{e^{-\frac{k}{\kappa_n}}}{\kappa_n} \mathbf{V} [\omega(y)]_{1/\kappa_n}^{k/\kappa_n} \right]^\theta &\leq \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{e^{-\frac{k}{\kappa_n}}}{\kappa_n} [\omega(k/\kappa_n) + \omega(m-1) + \omega(1/\kappa_n)] \right]^\theta \\
&= \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} \left[(k/\kappa_n)^{1-m} + e^{-\frac{k}{\kappa_n}} \omega(m-1) + e^{-\frac{k}{\kappa_n}} \omega(1/\kappa_n)^{1-m} \right] \right]^\theta \\
&\leq \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} (k/\kappa_n)^{1-m} \right]^\theta + \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} e^{-\frac{k}{\kappa_n}} \omega(m-1) \right]^\theta \\
&\quad + \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} e^{-\frac{k-1}{\kappa_n}} (1/\kappa_n)^{1-m} \right]^\theta =: R_{1n} + R_{2n} + R_{3n}.
\end{aligned}$$

We shall show that

$$R_{in} = o(1), \quad i = 1, 2, 3,$$

which in view of (210) implies that

$$\frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |\Delta_{2nk}^{(5)}|^\theta = o(1). \quad (211)$$

The first term

$$\begin{aligned}
R_{1n} &= \frac{C}{\kappa_n^{\theta+b_\theta}} \sum_{k=1}^{\delta_n} k^{\theta(1-m)} = \frac{C}{\kappa_n^{\theta+b_\theta}} \sum_{k=1}^{\delta_n} k^{-\theta(m-1)} \\
&= \frac{C}{\kappa_n^{\theta+b_\theta}} \delta_n^{1+\theta(1-m)} \frac{1}{\delta_n} \sum_{k=1}^{\delta_n} \left(\frac{k}{\delta_n} \right)^{\theta(1-m)} = \frac{C}{\kappa_n^{\theta+b_\theta}} \delta_n^{b_\theta} \left[\int_0^1 x^{-\theta(m-1)} dx + o(1) \right] \\
&\leq \frac{C \varsigma(\kappa_n)^{b_\theta}}{\kappa_n^\theta} \left[\int_0^1 x^{-\theta(m-1)} dx + o(1) \right] = o(1),
\end{aligned}$$

where we have used the fact that $\theta(m-1) < 1$. The second term

$$\begin{aligned}
R_{2n} &= \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} e^{-\frac{k}{\kappa_n}} \right]^\theta \\
&= \frac{C}{\kappa_n^\theta} \left[\int_0^\infty e^{-\theta x} dx + o(1) \right] = O(\kappa_n^{-\theta}) = o(1).
\end{aligned}$$

Finally,

$$\begin{aligned}
R_{3n} &= \frac{C}{\kappa_n} \sum_{k=1}^{\delta_n} \left[\frac{1}{\kappa_n} e^{-\frac{k-1}{\kappa_n}} (1/\kappa_n)^{1-m} \right]^\theta \\
&= \frac{C}{\kappa_n^{\theta(2-m)}} \left[\int_0^\infty e^{-\theta x} dx + o(1) \right] = O(\kappa_n^{\theta(2-m)}) = o(1).
\end{aligned}$$

Hence, (211) holds.

(vi) Consider

$$\frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |\Delta_{2nk}^{(6)}|^\theta = \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |T_{2nk}^{(6)} - T_{2nk}^{(5)}|^\theta$$

$$\begin{aligned}
&= C \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left| e^{-\frac{k}{\kappa_n}} \int_0^{\kappa_n^{b_\theta/p}/\kappa_n} e^y y^{1-m} dy \right|^\theta \leq C e^{\theta \frac{b_\theta/p}{\kappa_n}} \left(\int_0^{\kappa_n^{b_\theta/p}/\kappa_n} y^{1-m} dy \right) \cdot \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} e^{-\theta \frac{k}{\kappa_n}} \\
&= C \left(\frac{\kappa_n^{b_\theta/p}}{\kappa_n} \right)^{2-m} \left[\int_{\kappa_n^{b_\theta/p}/\kappa_n}^{\delta_n/\kappa_n} e^{-\theta x} dx + o(1) \right] = o(1) \cdot \left[\int_0^\infty e^{-\theta x} dx + o(1) \right] = o(1),
\end{aligned}$$

where we have used the facts that $\beta_\theta/p \in (0, 1)$ and $2 - m > 0$.

Finally, we establish (206) and (209). We start with the latter. Write

$$\begin{aligned}
&\left(\frac{\delta_n}{\kappa_n^2} \right)^\theta \frac{1}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |T_{2nk}^{(4)}|^\theta \leq \left(\frac{\delta_n}{\kappa_n^2} \right)^\theta \frac{C}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \left[|T_{2nk}^{(5)} - T_{2nk}^{(4)}|^\theta + |T_{2nk}^{(6)} - T_{2nk}^{(5)}|^\theta \right] \\
&+ \left(\frac{\delta_n}{\kappa_n^2} \right)^\theta \frac{C}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |T_{2nk}^{(6)}|^\theta =_{(10)} o \left(\left(\frac{\delta_n}{\kappa_n^2} \right)^\theta \right) + C \left(\frac{\delta_n}{\kappa_n^2} \right)^\theta \left[\int_{\kappa_n^{b_\theta/p}/\kappa_n}^{\delta_n/\kappa_n} \left[e^{-x} \int_0^x y^{1-m} dy \right]^\theta dx + o(1) \right] \\
&\leq C \left(\frac{\delta_n}{\kappa_n^2} \right)^\theta \int_{1/\kappa_n}^{\delta_n/\kappa_n} \left[e^{-x} \int_0^x e^y y^{1-m} dy \right]^\theta dx + o(1) \\
&\leq_{(11)} C \left(\frac{\delta_n}{\kappa_n^2} \right)^\theta \int_1^{\delta_n/\kappa_n} x^{-\theta(m-1)} dx + o(1) \\
&= C \left(\frac{\delta_n}{\kappa_n^2} \right)^\theta (\delta_n/\kappa_n)^{-\theta(m-1)+1} + o(1) = C \left(\frac{\delta_n}{\kappa_n^2} \right)^\theta (\delta_n/\kappa_n)^{b_\theta} + o(1) \\
&= C \frac{\zeta(\kappa_n)^{\theta+b_\theta}}{\kappa_n^\theta} + o(1) = o(1),
\end{aligned}$$

where $=_{(10)}$ follows from parts (v)-(vi) above and an Euler summation argument (see e.g. **Lemma 12**), whilst $\leq_{(11)}$ from the fact that $e^{-x} \int_0^x e^y y^{1-m} dy \sim x^{1-m}$ (cf. **Proposition 2**).

Next, we conclude with the proof of $=_{(5)}$ in (206). Using similar arguments as those used above for the proof of (209) we get

$$\begin{aligned}
&\frac{\mu_n^\theta}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |T_{2nk}^{(2)}|^\theta \leq C \left\{ \frac{\mu_n^\theta}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \sum_{i=3}^6 |T_{2nk}^{(i)} - T_{2nk}^{(i-1)}|^\theta + \frac{\mu_n^\theta}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |T_{2nk}^{(6)}|^\theta \right\} \\
&= C \left\{ \frac{\mu_n^\theta}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} \sum_{i=3}^6 |\Delta_{2nk}^{(i)}|^\theta + \frac{\mu_n^\theta}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |T_{2nk}^{(6)}|^\theta \right\} \\
&= o(\mu_n^\theta) + \frac{\mu_n^\theta}{\kappa_n} \sum_{k=\kappa_n^{b_\theta/p}}^{\delta_n} |T_{2nk}^{(6)}|^\theta \\
&= o(1) + \mu_n^\theta \left[\int_{\kappa_n^{b_\theta/p}/\kappa_n}^{\delta_n/\kappa_n} \left[e^{-x} \int_0^x y^{1-m} dy \right]^\theta dx + o(1) \right] \\
&\leq_{(11)} C \mu_n^\theta \int_1^{\delta_n/\kappa_n} x^{-\theta(m-1)} dx + o(1) = C \mu_n^\theta (\delta_n/\kappa_n)^{b_\theta} + o(1) \\
&\leq C \mu_n^\theta \zeta(\kappa_n)^{b_\theta} + o(1) = o(1),
\end{aligned}$$

where $\leq_{(11)}$ follows from **Proposition 2**. This completes the proof. $\blacksquare$

Proof of Corollary 1. The result follows directly from **Lemma 11** if we set $\theta = \alpha$ with α as in **Assumption MI-NM**. Notice in this case the conditions $m \in (1, 2)$ and $1/m < \theta < 1/(m-1)$ of **Lemma 11** are satisfied. Recall that under **Assumption MI-NM** $\alpha \in (1, 2]$ and $m \in (1, 1 + 1/\alpha)$. Hence, we do have $1/m < \alpha < 1/(m-1)$ and $m \in (1, 2)$. ■

Proof of Lemma 12. Set

$$\mathcal{P}(x) = |p(x)|^\theta.$$

We shall show that the total variation of $\mathcal{P}(x)$ on $[1/\kappa_n, \delta_n/\kappa_n]$ satisfies

$$\frac{1}{\kappa_n} \mathbf{V} [\mathcal{P}(x)]_{1/\kappa_n}^{\delta_n/\kappa_n} \rightarrow 0, \quad (212)$$

as $n \rightarrow \infty$. Noting that

$$\left| \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |p(k/\kappa_n)|^\theta - \int_{1/\kappa_n}^{\delta_n/\kappa_n} |p(x)|^\theta dx \right| \leq \frac{1}{\kappa_n} \mathbf{V} [\mathcal{P}(x)]_{1/\kappa_n}^{\delta_n/\kappa_n}, \quad (213)$$

the requisite result follows from (212), (213) and dominated convergence.

We set out to prove (212). By **Proposition 3**, $p(x)$ has a single root, x_* , such that $x_* > 2 - m$, and

$$|p(x)| = \begin{cases} p(x), & 0 < x \leq x_* \\ -p(x), & x > x_* \end{cases}.$$

Hence,

$$\mathcal{P}(x) = \begin{cases} p(x)^\theta, & 0 < x \leq x_* \\ (-p(x))^\theta, & x > x_* \end{cases}.$$

Therefore, for n large enough (such that $1/\kappa_n < x_* < \delta_n/\kappa_n$) we can write

$$\begin{aligned} \mathbf{V} [\mathcal{P}(x)]_{1/\kappa_n}^{\delta_n/\kappa_n} &= \mathbf{V} [\mathcal{P}(x)]_{1/\kappa_n}^{x_*} + \mathbf{V} [\mathcal{P}(x)]_{x_*}^{\delta_n/\kappa_n} \\ &= \int_{1/\kappa_n}^{x_*} \left| \frac{\partial \mathcal{P}(x)}{\partial x} \right| dx + \int_{x_*}^{\delta_n/\kappa_n} \left| \frac{\partial \mathcal{P}(x)}{\partial x} \right| dx. \end{aligned} \quad (214)$$

Denote the derivative of $p(x)$ as $p(x)'$. In view of the above,

$$\begin{aligned} \mathbf{V} [\mathcal{P}(x)]_{1/\kappa_n}^{\delta_n/\kappa_n} &= \int_{1/\kappa_n}^{x_*} |\theta p(x)^{\theta-1} p(x)'| dx + \int_{x_*}^{\delta_n/\kappa_n} |[-\theta p(x)]^{\theta-1} p(x)'| dx \\ &= \int_{1/\kappa_n}^{x_*} |p(x)|^{\theta-1} |\theta p(x)'| dx + \int_{x_*}^{\delta_n/\kappa_n} |p(x)|^{\theta-1} |\theta p(x)'| dx. \end{aligned}$$

Next, consider

$$p(x)' = (1-m)x^{-m} + e^{-x} \int_0^x e^y y^{1-m} dy - x^{1-m}.$$

First, suppose that $x \geq x_*$. As $x \rightarrow \infty$ we have

$$p(x)' = (1-m)x^{-m} + e^{-x} \int_0^{x_*} e^y y^{1-m} dy + e^{-x} \int_{x_*}^x e^y y^{1-m} dy - x^{1-m}$$

$$\begin{aligned}
&= (1-m)x^{-m} + O(e^{-x}) + e^{-x} \left[\left[e^y y^{1-m} \right]_{x_*}^x - (1-m) \int_{x_*}^x e^y y^{-m} dy \right] - x^{1-m} \\
&= (1-m)x^{-m} + O(e^{-x}) + e^{-x} \left[e^x x^{1-m} - e^{x_*} x_*^{1-m} - (1-m) \int_{x_*}^x e^y y^{-m} dy \right] - x^{1-m} \\
&= (1-m)x^{-m} + O(e^{-x}) - (1-m) e^{-x} \int_{x_*}^x e^y y^{-m} dy = o(x^{-m}),
\end{aligned}$$

where we have used the fact that $e^{-x} \int_{x_*}^x e^y y^{-m} dy \sim x^{-m}$ (cf. **Proposition 2**). Furthermore, it follows from the proof of **Lemma 16** that

$$p(x) \asymp x^{-m}.$$

Hence, as $n \rightarrow \infty$

$$\begin{aligned}
\frac{1}{\kappa_n} \int_{x_*}^{\delta_n/\kappa_n} |\theta p(x)^{\theta-1} p(x)'| dx &\leq \frac{C}{\kappa_n} \int_{x_*}^{\delta_n/\kappa_n} (x^{-m})^{\theta-1} x^{-m} dx \\
&= \frac{C}{\kappa_n} \int_{x_*}^{\delta_n/\kappa_n} x^{-\theta m} dx \leq \frac{C}{\kappa_n} \int_{x_*}^{\infty} x^{-\theta m} dx = o(1).
\end{aligned} \tag{215}$$

Next, note that for $x > 0$

$$\begin{aligned}
|p(x)|^{\theta-1} &\leq C (x^{1-m} + x^{2-m})^{\theta-1} = C [x^{1-m} (1+x)]^{\theta-1} \\
&\leq_{(1)} C \cdot \begin{cases} x^{(1-m)(\theta-1)} + x^{(2-m)(\theta-1)}, & \theta \geq 1 \\ x^{(m-2)(1-\theta)}, & 0 < \theta < 1 \end{cases} \leq C \cdot [x^{(1-m)(\theta-1)} + x^{(2-m)(\theta-1)}].
\end{aligned}$$

where $\leq_{(1)}$ follows from standard arguments²⁷. In addition, for $x > 0$

$$|p(x)'| \leq C (x^{-m} + x^{1-m} + x^{2-m}).$$

Hence,

$$\begin{aligned}
&\frac{1}{\kappa_n} \int_{1/\kappa_n}^{x_*} |p(x)|^{\theta-1} |\theta p(x)'| dx \\
&\leq \frac{C}{\kappa_n} \int_{1/\kappa_n}^{x_*} [x^{(\theta-1)(1-m)} + x^{(\theta-1)(2-m)}] (x^{-m} + x^{1-m} + x^{2-m}) dx.
\end{aligned} \tag{216}$$

The leading term above is

$$\begin{aligned}
&\frac{C}{\kappa_n} \int_{1/\kappa_n}^{x_*} [x^{(\theta-1)(1-m)-m} + x^{(\theta-1)(2-m)-m}] dx \\
&= \frac{C}{\kappa_n} \int_{1/\kappa_n}^{x_*} [x^{\theta-\theta m-1+m-m} + x^{2\theta-\theta m-2+m-m}] dx \\
&= \frac{C}{\kappa_n} \int_{1/\kappa_n}^{x_*} [x^{\theta-\theta m-1} + x^{2\theta-\theta m-2}] dx = \frac{C}{\kappa_n} \left[\left(\frac{1}{\kappa_n} \right)^{\theta-\theta m} + \left(\frac{1}{\kappa_n} \right)^{2\theta-\theta m-1} \right] + O(\kappa_n^{-1})
\end{aligned}$$

²⁷In particular, for $\theta < 1$ note that

$$\begin{aligned}
(x^{1-m} + x^{2-m})^{\theta-1} &= [x^{1-m} (1+x)]^{\theta-1} = [x^{-(m-1)} (1+x)]^{\theta-1} \\
&= \left[\frac{1+x}{x^{m-1}} \right]^{\theta-1} = \left[\frac{x^{m-1}}{1+x} \right]^{1-\theta} \leq \left[\frac{x^{m-1}}{x} \right]^{1-\theta} = [x^{m-2}]^{1-\theta}.
\end{aligned}$$

$$\begin{aligned}
&= C \left[\left(\frac{1}{\kappa_n} \right)^{1+\theta-\theta m} + \left(\frac{1}{\kappa_n} \right)^{\theta(2-m)} \right] + o(1) \\
&= C \left[\left(\frac{1}{\kappa_n} \right)^{b_\theta} + \left(\frac{1}{\kappa_n} \right)^{\theta(2-m)} \right] + o(1) = o(1),
\end{aligned} \tag{217}$$

where we have used the fact that $b_\theta := 1 + \theta - \theta m > 0$, cf. (194), and $\theta(2-m) > 0$. Equation (212) follows directly from (214)-(217). ■

Proof of Lemma 13. We start with some preliminary facts. Recall that (cf. (42)) for all $\theta > 0$ and some $C_\theta \in (0, \infty)$

$$|Y|^\theta - |X|^\theta \leq C_\theta \cdot \begin{cases} (|X|^{\theta-1}|Y-X| + |Y-X|^\theta), & \theta > 1 \\ |Y-X|^\theta, & 0 < \theta \leq 1 \end{cases}.$$

In addition, for any two sequences X_k and Y_k , such that $\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |Y_k - X_k|^\theta = o(1)$ and $\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |X_k|^\theta = O(1)$, Hölder's inequality gives

$$\begin{aligned}
\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |X_k|^{\theta-1} |Y_k - X_k| &\leq \left[\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} (|X_k|^{\theta-1})^{\frac{\theta}{\theta-1}} \right]^{\frac{\theta-1}{\theta}} \left[\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |Y_k - X_k|^\theta \right]^{1/\theta} \\
&= \left[\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |X_k|^\theta \right]^{\frac{\theta-1}{\theta}} \left[\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |Y_k - X_k|^\theta \right]^{1/\theta} = o(1).
\end{aligned} \tag{218}$$

In view of the above, set [check the sign]

$$Y_k = T_{1nk} + T_{2nk} \text{ and } X_k = T_{1nk}^{(3)} + T_{2nk}^{(6)} = \frac{1}{1-m} p(k/\kappa_n),$$

where T_{1nk} , T_{2nk} are given **Lemma 11**, $T_{1nk}^{(3)}$ and $T_{2nk}^{(6)}$ are defined in (195) and (203), respectively, and $p(x)$ is defined in **Lemma 12**. Hence, by **Lemma 12**,

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |X_k|^\theta = \frac{1}{(m-1)^\theta} \int_0^\infty |p(x)|^\theta dx + o(1) = O(1). \tag{219}$$

Moreover, by **Lemma 11** as $n \rightarrow \infty$ we have

$$\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |Y_k - X_k|^\theta \leq \frac{2}{\kappa_n} \sum_{k=1}^{\delta_n} [|\Delta_{1nk}|^\theta + |\Delta_{2nk}|^\theta] = o(1), \tag{220}$$

Hence,

$$\begin{aligned}
&\frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk} - T_{2nk}|^\theta = \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk}^{(3)} - T_{2nk}^{(6)}|^\theta \\
&\quad + \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk} - T_{2nk}|^\theta - \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk}^{(3)} - T_{2nk}^{(6)}|^\theta \\
&=_{(1)} \frac{1}{\kappa_n} \sum_{k=1}^{\delta_n} |T_{1nk}^{(3)} - T_{2nk}^{(6)}|^\theta + o(1) =_{(2)} \frac{1}{(m-1)^\theta} \int_0^\infty |p(x)|^\theta dx + o(1),
\end{aligned}$$

where $=_{(1)}$ follows from (218)-(220) and $=_{(2)}$ from **Lemma 12**. ■

Proposition 1. Suppose that $m \in (1, 1 + 1/\alpha)$, $\alpha \in (1, 2]$ (i.e. **NM** holds). Set $\omega(y) := e^y y^{1-m}$. The total variation of $\omega(y)$ on $[1/\kappa_n, k/\kappa_n]$, $k \in \mathbb{N}$ satisfies

$$\mathbf{V} [\omega(x)]_{1/\kappa_n}^{k/\kappa_n} \leq 2 [\omega(k/\kappa_n) + \omega(m-1) + \omega(1/\kappa_n)].$$

Proof of Proposition 1. First, note that

$$\begin{aligned} |\omega(y)'| &= |e^y y^{1-m} + (1-m) e^y y^{-m}| \\ &= e^y y^{-m} |y + 1 - m| = \begin{cases} e^y y^{-m} (y + 1 - m), & y \geq m-1 \\ -e^y y^{-m} (y + 1 - m), & y < m-1 \end{cases} = \begin{cases} \omega(y)', & y \geq m-1 \\ -\omega(y)', & y < m-1 \end{cases}. \end{aligned}$$

In view of the above,

$$\begin{aligned} \mathbf{V} [\omega(x)]_{1/\kappa_n}^{k/\kappa_n} &= \int_{1/\kappa_n}^{k/\kappa_n} |\omega(y)'| dy \\ &= \left[\int_{m-1}^{k/\kappa_n} \omega(y)' dy - \int_{1/\kappa_n}^{m-1} \omega(y)' dy \right] I_{\{k/\kappa_n \geq m-1\}} - \int_{1/\kappa_n}^{k/\kappa_n} \omega(y)' dy I_{\{k/\kappa_n < m-1\}} \\ &= \left[[\omega(y)]_{m-1}^{k/\kappa_n} - [\omega(y)]_{1/\kappa_n}^{m-1} \right] I_{\{k/\kappa_n \geq m-1\}} - [\omega(y)]_{1/\kappa_n}^{k/\kappa_n} I_{\{k/\kappa_n < m-1\}} \\ &= \{[\omega(k/\kappa_n) - \omega(m-1)] - [\omega(m-1) - \omega(1/\kappa_n)]\} I_{\{k/\kappa_n \geq m-1\}} - [\omega(k/\kappa_n) - \omega(1/\kappa_n)] I_{\{k/\kappa_n < m-1\}} \\ &= \{\omega(k/\kappa_n) - 2\omega(m-1) + \omega(1/\kappa_n)\} I_{\{k/\kappa_n \geq m-1\}} - [\omega(k/\kappa_n) - \omega(1/\kappa_n)] I_{\{k/\kappa_n < m-1\}} \\ &= \{\omega(k/\kappa_n) - 2\omega(m-1)\} I_{\{k/\kappa_n \geq m-1\}} - \omega(k/\kappa_n) I_{\{k/\kappa_n < m-1\}} + \omega(1/\kappa_n) \\ &\leq \{\omega(k/\kappa_n) + 2\omega(m-1)\} I_{\{k/\kappa_n \geq m-1\}} + \omega(k/\kappa_n) I_{\{k/\kappa_n < m-1\}} + \omega(1/\kappa_n) \\ &\leq 2 [\omega(k/\kappa_n) + \omega(m-1) + \omega(1/\kappa_n)], \end{aligned}$$

as required. ■

15.1. Technical Lemmas

Set $b := 1 + \alpha - \alpha m$.

LM ($m \in (1/\alpha, 1)$). For a given $\alpha \in (1, 2]$ set

$$\eta_1 := \frac{1}{2}(\alpha - 1) \text{ and } \eta_2 := \frac{1}{2} \frac{\alpha m - 1}{m}$$

It can be easily checked that $\eta_1, \eta_2 > 0$. Moreover, let

$$\epsilon_- \in [0, \eta_1 \wedge \eta_2]. \quad (221)$$

Define $\alpha_{\epsilon_-} := \alpha - \epsilon_-$ and $b_{\epsilon_-} := 1 + \alpha_{\epsilon_-} - \alpha_{\epsilon_-} m$. Notice that by definition

$$b_{\epsilon_-} > 1, \quad \alpha_{\epsilon_-} > 1 \quad \text{and} \quad \alpha_{\epsilon_-} m > 1. \quad (222)$$

To see this first note that

$$b_{\epsilon_-} > 1 \Leftrightarrow 1 + (\alpha - \epsilon_-) m > 1 \Leftrightarrow (\alpha - \epsilon_-) m > 0$$

$$\Leftrightarrow \alpha > \epsilon_-,$$

which is true because $\alpha > \eta_1 \geq \epsilon_-$. For the latter two results in (222) note that

$$\alpha_{\epsilon_-} \geq \alpha - \eta_1 = \alpha - \frac{1}{2}(\alpha - 1) = \frac{\alpha}{2} + \frac{1}{2} > 1.$$

In addition,

$$\alpha_{\epsilon_-} m \geq (\alpha - \eta_2)m$$

and

$$\begin{aligned} (\alpha - \eta_2)m > 1 &\Leftrightarrow \alpha m - \eta_2 m > 1 \\ &\Leftrightarrow \alpha m - 1 > \eta_2 m \Leftrightarrow \frac{\alpha m - 1}{m} > \eta_2, \end{aligned}$$

which is true.

NM ($m \in (1, 1 + 1/\alpha)$). Set

$$\eta_3 := b/2,$$

and

$$\eta_4 := \frac{1}{2}(2 - \alpha),$$

and note that $\eta_3 > 0$, $\eta_4 \geq 0$. Further, let

$$\epsilon_+ \in \begin{cases} [0, \eta_3 \wedge \eta_4], & \alpha \in (1, 2) \\ [0, \eta_3], & \alpha = 2 \end{cases}, \quad (223)$$

and $\alpha_{\epsilon_+} = \alpha + \epsilon_+$, $b_{\epsilon_+} := 1 + \alpha_{\epsilon_+} - \alpha_{\epsilon_+} m$. Notice that

$$\alpha_{\epsilon_+} m > 1, \quad b_{\epsilon_+} > 0 \quad \text{and} \quad \alpha_{\epsilon_+} m - \epsilon_+ > 1. \quad (224)$$

The first fact is obvious. For the second, note that

$$\begin{aligned} b_{\epsilon_+} &= 1 + \alpha + \epsilon_+ - \alpha m - \epsilon_+ m > 0 \\ &\Leftrightarrow 1 + \alpha + \epsilon_+ (1 - m) > \alpha m \Leftrightarrow \epsilon_+ (1 - m) > \alpha m - \alpha - 1 \\ &\Leftrightarrow \epsilon_+ < \frac{\alpha m - \alpha - 1}{1 - m} = \frac{b}{m - 1}, \end{aligned}$$

which conforms with (223), because

$$\epsilon_+ \leq \eta_3 = \frac{b}{2} = \frac{m - 1}{2} \frac{b}{m - 1} < \frac{1}{2\alpha} \frac{b}{m - 1} < \frac{b}{m - 1}.$$

In addition,

$$\begin{aligned} \alpha_{\epsilon_+} m - \epsilon_+ > 1 &\Leftrightarrow (\alpha + \epsilon_+)m - \epsilon_+ > 1 \\ &\Leftrightarrow \alpha m + (m - 1)\epsilon_+ > 1, \end{aligned}$$

which is true.

Notice that the intervals of (221) and (223) have strictly positive lengths. Next, we introduce some further notation. Set

$$[\epsilon, \alpha_\epsilon, b_\epsilon] := \begin{cases} [\epsilon_-, \alpha_{\epsilon_-}, b_{\epsilon_-}], & \text{under } \mathbf{LM} \\ [\epsilon_+, \alpha_{\epsilon_+}, b_{\epsilon_+}], & \text{under } \mathbf{NM} \end{cases} \quad (225)$$

For any integer valued sequences $a_n \leq b_n$, set

$$V_\epsilon(a_n, b_n) := \kappa_n^{-b_\epsilon} \sum_{k=a_n}^{b_n} \left| \sum_{l=0}^k \rho_n^l c_{k-l} \right|^{\alpha_\epsilon},$$

and

$$Q_\epsilon(a_n, b_n) := \kappa_n^{-b_\epsilon} \sum_{k=b_n}^{\infty} \left| \sum_{l=0}^{a_n-1} \rho_n^l c_{k-l} \right|^{\alpha_\epsilon}.$$

Moreover, for some $\theta > 0$ define

$$b_\theta := 1 + \theta - \theta m,$$

$$\tilde{V}_\theta(a_n, b_n) := \kappa_n^{-b_\theta} \sum_{k=a_n}^{b_n} \left| \sum_{l=0}^k \rho_n^l c_{k-l} \right|^\theta,$$

and

$$\tilde{Q}_\theta(a_n, b_n) := \kappa_n^{-b_\theta} \sum_{k=b_n}^{\infty} \left| \sum_{l=0}^{a_n-1} \rho_n^l c_{k-l} \right|^\theta.$$

Lemma 14. Set $\delta_n := \lfloor \varsigma_n \kappa_n \rfloor$, with $\varsigma_n := [\ln(1 + \kappa_n)]^2$, and $t_n = n \vee \delta_n$. Suppose that $c_k \sim m_0 \cdot k^{-m}$, $m_0 \neq 0$ such that

- (i) $\theta \in (0, 3]$;
- (ii) $\theta m > 1$.

Then as $n \rightarrow \infty$

- (a) $\tilde{V}_\theta(\delta_n, t_n) = o(1)$;
- (b) $\tilde{Q}_\theta(t_n, t_n) = \tilde{Q}_\theta(t_n, \delta_n) + o(1)$;
- (c) $\tilde{Q}_\theta(t_n, \delta_n) = o(1)$.

Remark. The proof provided for **Lemma 14** in fact shows that

$$\kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left(\sum_{l=0}^{t_n-1} \left| \rho_n^l c_{k-l} \right| \right)^\theta = \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left(\sum_{l=0}^{\delta_n-1} \left| \rho_n^l c_{k-l} \right| \right)^\theta = o(1),$$

which implies that $\tilde{Q}_\theta(t_n, t_n) = o(1)$.

The following result is a direct consequence of **Lemma 14**.

Lemma 15. Set $\delta_n := \lfloor \varsigma_n \kappa_n \rfloor$, with $\varsigma_n := [\ln(1 + \kappa_n)]^2$, and $t_n = n \vee \delta_n$. Suppose that

$$c_k \sim m_0 \cdot k^{-m}, \quad m_0 \neq 0,$$

and either condition **MI-NM** or **MI-LM** holds.

Then as $n \rightarrow \infty$

- (a) $V_\epsilon(\delta_n, t_n) = o(1)$;

(b) $Q_\epsilon(t_n, t_n) = Q_\epsilon(t_n, \delta_n) + o(1)$;

(c) $Q_\epsilon(t_n, \delta_n) = o(1)$.

Consider the “reversed” Gamma function

$$\gamma(m; a, x) := \int_{y=a}^x e^y y^{1-m} dy, \quad m \in \mathbb{R}, \quad 0 \leq a < x.$$

Moreover, set

$$\gamma(m; x) := \gamma(m; 0, x).$$

Remark 23. Notice that for $m > 2$, we need $a > 0$ to ensure that $\gamma(m; a, x) < \infty$.

Lemma 16. The following hold.

(i) Let $\theta_1 > 0$, and $\theta_2 \in (0, 1)$ such that $\theta_1 \theta_2 > 1$. Then

$$\int_{x=0}^{\infty} e^{-\theta_1 x} \left(\int_{y=0}^x e^y y^{-\theta_2} dy \right)^{\theta_1} dx < \infty. \quad (226)$$

(ii) Let $\theta_1 > 0$ and $\theta_2 \in (1, 2)$ such that $1/\theta_2 < \theta_1 < 1/(\theta_2 - 1)$. Then

$$\int_{x=0}^{\infty} e^{-\theta_1 x} \left(\int_{y=1}^x e^y y^{-\theta_2} dy \right)^{\theta_1} dx < \infty, \quad (227)$$

and

$$\int_{x=0}^{\infty} \left| x^{1-\theta_2} - e^{-x} \gamma(\theta_2; x) \right|^{\theta_1} dx < \infty. \quad (228)$$

Corollary 4. Let α_ϵ as in (225). The following hold.

(i) Under condition **MI-LM**

$$\mathbb{V}_1(\alpha_\epsilon, m) := \int_{x=0}^{\infty} e^{-\alpha_\epsilon x} \left(\int_{y=0}^x e^y y^{-m} dy \right)^{\alpha_\epsilon} dx < \infty.$$

(ii) Under condition **MI-NM**

$$\mathbb{V}'_1(\alpha_\epsilon, m) := \int_{x=0}^{\infty} e^{-\alpha_\epsilon x} \left(\int_{y=1}^x e^y y^{-m} dy \right)^{\alpha_\epsilon} dx < \infty,$$

and

$$\mathbb{V}_2(\alpha_\epsilon, m) := |1 - m|^{-\alpha_\epsilon} \int_{x=0}^{\infty} \left| x^{1-m} - e^{-x} \gamma(m; x) \right|^{\alpha_\epsilon} dx < \infty.$$

Proof of Corollary 4. It follows directly from **Lemma 16** with $\alpha_\epsilon = \theta_1$, and $m = \theta_2$. ■

Remark 24. **(a)** Notice that under condition **MI-LM** $\sum_{l=0}^{\infty} |c_l|^{\alpha_\epsilon} < \infty$, whilst under condition **MI-NM** $\sum_{l=0}^{\infty} (|c_l| + |c_l|^{\alpha_\epsilon}) < \infty$.

(b) Due to the approximation $c_j \sim m_0 \cdot j^{-m}$, there is no loss of generality if we assume that $|c_j|$ is decreasing.

(c) Recall that $b := 1 + \alpha - \alpha m$. By definition, $b > 0$. In particular,

$$b \in \begin{cases} (0, 1), & \text{under condition MI-NM} \\ (1, \alpha), & \text{under condition MI-LM} \end{cases}$$

(d) Under condition **MI-NM** or **MI-LM** we have $\alpha m > 1$.

(e) For $\alpha = 2$, it can be shown (cf. DK2018) that

$$\mathbb{V}_1(\alpha) = B(1 - m, 2m - 1) \Gamma(2 - 2m),$$

with B and Γ being beta and gamma functions respectively.

Lemma 17. Suppose that the assumptions of **Lemma 15** hold with $m \in (1/\alpha, 1)$, $\alpha > 1$.

(a) As $n \rightarrow \infty$,

$$V_\epsilon(0, \delta_n) = |m_0|^{\alpha_\epsilon} \mathbb{V}_1(\alpha_\epsilon, m) + o(1).$$

(b) Let $(\delta'_n)^{-1} + \delta'_n \kappa_n^{-1} \rightarrow 0$. As $n \rightarrow \infty$,

$$V_\epsilon(0, \delta'_n) = o(1).$$

Lemma 18. Suppose that the assumptions of **Lemma 15** hold with $m \in (1, 1 + 1/\alpha)$, $\alpha > 1$. Then as $n \rightarrow \infty$

$$V_\epsilon(0, \delta_n) = |m_0|^{\alpha_\epsilon} \mathbb{V}_2(\alpha_\epsilon, m) + o(1).$$

Proposition 2. The following hold

(a) Let

$$q(x) := e^{-x} \int_1^x e^y y^{-q_0} dy, \quad q_0 > 0.$$

For all integer $N \geq 1$, as $x \rightarrow \infty$

$$q(x) = x^{-q_0} \left[\sum_{j=0}^{N-1} \frac{(q_0)_j}{x^j} + O(x^{-N}) \right],$$

where $(\cdot)_j$ Pochhammer's symbol for the forward factorial function, i.e. $(q_0)_j = q_0(q_0 + 1) \dots (q_0 + j - 1)$.

(b) Let

$$q(x) := e^{-x} \int_0^x e^y y^{-q_0} dy, \quad 0 < q_0 < 1.$$

For all integer $N \geq 1$, as $x \rightarrow 0$

$$q(x) = \frac{x^{1-q_0}}{1-q_0} \left[\sum_{j=0}^{N-1} \frac{(-x)^j}{(2-q_0)_j} + O(x^{-N}) \right],$$

Remark. (a) It follows from **Proposition 2(a)** that as $x \rightarrow \infty$

$$q(x) \sim x^{-q_0}.$$

In addition,

$$\frac{q(x)}{x^{-q_0}} - 1 \sim \frac{q_0}{x}.$$

(b) It follows from **Proposition 2(b)** that as that as $x \rightarrow 0$

$$q(x) \sim \frac{x^{1-q_0}}{1-q_0}.$$

In addition,

$$\frac{q(x)}{x^{1-q_0}/(1-q_0)} - 1 \sim -\frac{x^{2-q_0}}{2-q_0}.$$

For the following proof recall that

$$V_\theta(\delta_n, t_n) = \kappa_n^{-b_\theta} \sum_{k=\delta_n}^{t_n} \left| \sum_{l=0}^k \rho_n^l c_{k-l} \right|^\theta$$

with $\theta \in (0, 3]$, $b_\theta := 1 + \theta - \theta m$ and $c_k \sim m_0 k^{-m}$, $m_0 \neq 0$ such that $\theta m > 1$.

Proof of Lemma 14. Without loss of generality, we assume that $m_0 = 1$.

(a) We provide a proof of the case $\theta \in (2, 3]$. The proof for $\theta \in (0, 2]$ is similar but simpler. Set $\bar{\delta}_n = \lfloor \delta_n/2 \rfloor$ and recall that $\delta_n = \kappa_n [\ln(1 + \kappa_n)]^2$. Hence, in view of this and the fact that $\kappa_n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \ln(\kappa_n^q) \kappa_n \bar{\delta}_n^{-1} = 0, \text{ for all } q \in \mathbb{R}. \quad (229)$$

Note that (229) together with Lemma 6.4 in DK2018 imply that

$$\lim_{n \rightarrow \infty} \kappa_n^q \rho_n^{\vartheta \bar{\delta}_n} = 0, \text{ for all } \vartheta > 0. \quad (230)$$

In addition, under our assumptions $\sum_{k=0}^\infty |c_k|^\theta < \infty$. In view of these facts write

$$\begin{aligned} V_\theta(\delta_n, t_n) &= \kappa_n^{-b_\theta} \sum_{k=\delta_n}^{t_n} \left| \sum_{l=0}^k \rho_n^l c_{k-l} \right|^\theta \leq \kappa_n^{-b_\theta} \sum_{k=\delta_n}^{t_n} \left(\sum_{l=0}^k |\rho_n^l c_{k-l}| \right)^\theta \\ &= \kappa_n^{-b_\theta} \sum_{k=\delta_n}^{t_n} \left(\sum_{l_1=0}^k |\rho_n^{l_1} c_{k-l_1}| \right)^{\theta-2} \left(\sum_{l_2=0}^k |\rho_n^{l_2} c_{k-l_2}| \right) \left(\sum_{l_3=0}^k |\rho_n^{l_3} c_{k-l_3}| \right) \\ &\leq \kappa_n^{-b_\theta} \sum_{k=\delta_n}^{t_n} \left[\left(\sum_{l_1=0}^{\bar{\delta}_n} |\rho_n^{l_1} c_{k-l_1}| \right)^{\theta-2} + \left(\sum_{l_1=\bar{\delta}_n}^k |\rho_n^{l_1} c_{k-l_1}| \right)^{\theta-2} \right] \left(\sum_{l_2=0}^k |\rho_n^{l_2} c_{k-l_2}| \right) \left(\sum_{l_3=0}^k |\rho_n^{l_3} c_{k-l_3}| \right) \\ &=: A_n + B_n. \end{aligned} \quad (231)$$

Noting that $c_k \sim m_0 k^{-m}$ where the exponent $-m$ is strictly negative, there is no loss of generality if we assume that c_k is decreasing. In view of this convention, the first term is

$$A_n \leq \kappa_n^{-b_\theta} \sum_{k=\delta_n}^{t_n} \left(\sum_{l_1=0}^{\bar{\delta}_n} |\rho_n^{l_1} c_{k-l_1}| \right)^{\theta-2} \left(\sum_{l_2=0}^k |\rho_n^{l_2} c_{k-l_2}| \right) \left(\sum_{l_3=0}^k |\rho_n^{l_3} c_{k-l_3}| \right)$$

$$\begin{aligned}
&= \kappa_n^{-b_\theta} \left(\sum_{l_1=0}^{\bar{\delta}_n} \rho_n^{l_1} \right)^{\theta-2} \sum_{k=\bar{\delta}_n}^{t_n} |c_{k-\bar{\delta}_n}|^{\theta-2} \left(\sum_{l_2=0}^k |\rho_n^{l_2} c_{k-l_2}| \right) \left(\sum_{l_3=0}^k |\rho_n^{l_3} c_{k-l_3}| \right) \\
&= \kappa_n^{-b_\theta} \left(\sum_{l_1=0}^{\bar{\delta}_n} \rho_n^{l_1} \right)^{\theta-2} \sum_{l_2=0}^{t_n} |\rho_n^{l_2}| \sum_{l_3=0}^{t_n} |\rho_n^{l_3}| \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-\bar{\delta}_n}|^{\theta-2} |c_{k-l_2}| |c_{k-l_3}| \\
&= \kappa_n^{-b_\theta} \left(\sum_{l_1=0}^{\bar{\delta}_n} \rho_n^{l_1} \right)^{\theta-2} \prod_{i=2}^3 \left(\sum_{l_i=0}^{\bar{\delta}_n} |\rho_n^{l_i}| + \sum_{l_i=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_i}| \right) \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-\bar{\delta}_n}|^{\theta-2} |c_{k-l_2}| |c_{k-l_3}| \\
&\leq_{(1)} C \kappa_n^{\theta-2-b_\theta} \cdot \prod_{i=2}^3 \left(\sum_{l_i=0}^{\bar{\delta}_n} |\rho_n^{l_i}| + \sum_{l_i=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_i}| \right) \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-\bar{\delta}_n}|^{\theta-2} |c_{k-l_2}| |c_{k-l_3}| \\
&=: C \kappa_n^{\theta-2-b_\theta} \cdot A'_n, \tag{232}
\end{aligned}$$

where $\leq_{(1)}$ follows from Lemma 6.4 of DK2018. Consider the term

$$\begin{aligned}
A'_n &= \prod_{i=2}^3 \left(\sum_{l_i=0}^{\bar{\delta}_n} |\rho_n^{l_i}| + \sum_{l_i=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_i}| \right) \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-\bar{\delta}_n}|^{\theta-2} |c_{k-l_2}| |c_{k-l_3}| \\
&= \sum_{l_2=0}^{\bar{\delta}_n} |\rho_n^{l_2}| \sum_{l_3=0}^{\bar{\delta}_n} |\rho_n^{l_3}| \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-\bar{\delta}_n}|^{\theta-2} |c_{k-l_2}| |c_{k-l_3}| \\
&\quad + \sum_{l_2=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_2}| \sum_{l_3=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_3}| \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-\bar{\delta}_n}|^{\theta-2} |c_{k-l_2}| |c_{k-l_3}| \\
&\quad + 2 \sum_{l_2=0}^{\bar{\delta}_n} |\rho_n^{l_2}| \sum_{l_3=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_3}| \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-\bar{\delta}_n}|^{\theta-2} |c_{k-l_2}| |c_{k-l_3}| \\
&\leq \sum_{l_2=0}^{\bar{\delta}_n} |\rho_n^{l_2}| \sum_{l_3=0}^{\bar{\delta}_n} |\rho_n^{l_3}| \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-\bar{\delta}_n}|^{\theta-2} |c_{k-\bar{\delta}_n}| |c_{k-\bar{\delta}_n}| \\
&\quad + \sum_{l_2=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_2}| \sum_{l_3=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_3}| \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-(\bar{\delta}_n \vee l_2 \vee l_3)}|^{\theta-2} |c_{k-(\bar{\delta}_n \vee l_2 \vee l_3)}| |c_{k-(\bar{\delta}_n \vee l_2 \vee l_3)}| \\
&\quad + 2 \sum_{l_2=0}^{\bar{\delta}_n} |\rho_n^{l_2}| \sum_{l_3=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_3}| \sum_{k=\bar{\delta}_n \vee l_2 \vee l_3}^{t_n} |c_{k-(\bar{\delta}_n \vee l_2 \vee l_3)}|^{\theta-2} |c_{k-(\bar{\delta}_n \vee l_2 \vee l_3)}| |c_{k-(\bar{\delta}_n \vee l_2 \vee l_3)}| \\
&\leq \sum_{l_2=0}^{\bar{\delta}_n} |\rho_n^{l_2}| \sum_{l_3=0}^{\bar{\delta}_n} |\rho_n^{l_3}| \sum_{k=\bar{\delta}_n}^{\infty} |c_k|^\theta + \sum_{l_2=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_2}| \sum_{l_3=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_3}| \sum_{k=0}^{\infty} |c_k|^\theta \\
&\quad + 2 \sum_{l_2=0}^{\bar{\delta}_n} |\rho_n^{l_2}| \sum_{l_3=\bar{\delta}_n+1}^{t_n} |\rho_n^{l_3}| \sum_{k=0}^{\infty} |c_k|^\theta \\
&\leq_{(2)} C \left[\kappa_n^2 \bar{\delta}_n^{1-m\theta} \frac{1}{\bar{\delta}_n} \sum_{k=\bar{\delta}_n}^{\infty} \left(\frac{k}{\bar{\delta}_n} \right)^{-m\theta} + \kappa_n^2 \rho_n^{2\bar{\delta}_n} \sum_{k=0}^{\infty} |c_k|^\theta + \kappa_n^2 \rho_n^{2\bar{\delta}_n} \sum_{k=0}^{\infty} |c_k|^\theta \right] \\
&=_{(3)} C \left[\kappa_n^2 \bar{\delta}_n^{1-m\theta} \int_1^\infty x^{-m\theta} dx + \kappa_n^2 \rho_n^{2\bar{\delta}_n} + \kappa_n^2 \rho_n^{2\bar{\delta}_n} \right],
\end{aligned}$$

where $\leq_{(2)}$ follows from DK2018 (Lemma 6.4), and $=_{(3)}$ from Euler summation (see e.g. 3). Combining (232)-(233) we get

$$\begin{aligned} A_n &\leq C \kappa_n^{\theta-2-b_\theta} \left[\kappa_n^2 \bar{\delta}_n^{1-m\theta} + \kappa_n^2 \rho_n^{2\bar{\delta}_n} + \kappa_n^2 \rho_n^{\bar{\delta}_n} \right] \\ &= C \left[\kappa_n^{\theta-b_\theta} \bar{\delta}_n^{1-m\theta} + \kappa_n^{\theta-b_\theta} \rho_n^{2\bar{\delta}_n} + \kappa_n^{\theta-b_\theta} \rho_n^{\bar{\delta}_n} \right] =_{(4)} C \frac{\kappa_n^\theta \bar{\delta}_n^{1-m\theta}}{\kappa_n^\theta \kappa_n^{1-m\theta}} + o(1) =_{(5)} o(1), \end{aligned} \quad (233)$$

where $=_{(4)}$ is due to (230), and $=_{(5)}$ due to the fact that $\kappa_n = o(\bar{\delta}_n)$.

Next, using (230) once more and DK2018 (Lemma 6.4) the term B_n

$$\begin{aligned} B_n &\leq \kappa_n^{-b_\theta} \sum_{k=\delta_n}^{t_n} \sum_{l_1=\bar{\delta}_n}^k \left| \rho_n^{l_1} c_{k-l_1} \right|^{\theta-2} \sum_{l_2=0}^k \left| \rho_n^{l_2} c_{k-l_2} \right| \sum_{l_3=0}^k \left| \rho_n^{l_3} c_{k-l_3} \right| \\ &= \kappa_n^{-b_\theta} \sum_{l_1=\bar{\delta}_n}^{t_n} \rho_n^{(\theta-2)l_1} \sum_{l_2=0}^{t_n} \rho_n^{l_2} \sum_{l_3=0}^{t_n} \rho_n^{l_3} \sum_{k=\delta_n \vee l_1 \vee l_2 \vee l_3}^{t_n} |c_{k-l_1}|^{\theta-2} |c_{k-l_2}| |c_{k-l_3}| \\ &\leq \kappa_n^{-b_\theta} \sum_{l_1=\bar{\delta}_n}^{t_n} \rho_n^{(\theta-2)l_1} \sum_{l_2=0}^{t_n} \rho_n^{l_2} \sum_{l_3=0}^{t_n} \rho_n^{l_3} \sum_{k=\delta_n \vee l_1 \vee l_2 \vee l_3}^{t_n} \left| c_{k-(\delta_n \vee l_1 \vee l_2 \vee l_3)} \right|^\theta \\ &\leq \kappa_n^{-b_\theta} \sum_{l_1=\bar{\delta}_n}^{t_n} \rho_n^{(\theta-2)l_1} \sum_{l_2=0}^{t_n} \rho_n^{l_2} \sum_{l_3=0}^{t_n} \rho_n^{l_3} \sum_{k=0}^{\infty} |c_k|^\theta \leq C \kappa_n^{3-b_\theta} \rho_n^{(\theta-2)\bar{\delta}_n} \sum_{k=0}^{\infty} |c_k|^\theta = o(1). \end{aligned} \quad (234)$$

The requisite result follows from (231), (233) and (234).

(b) Write

$$\begin{aligned} &|Q_\theta(t_n, t_n) - Q_\theta(t_n, \delta_n)| \\ &\leq \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left| \left| \sum_{l=0}^{t_n-1} \rho_n^l c_{k-l} \right|^\theta - \left| \sum_{l=0}^{\delta_n-1} \rho_n^l c_{k-l} \right|^\theta \right| \\ &\leq C \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left[\left| \sum_{l=\delta_n}^{t_n-1} \rho_n^l c_{k-l} \right|^\theta + \left| \sum_{l=0}^{\delta_n-1} \rho_n^l c_{k-l} \right|^{\theta-1} \left| \sum_{l=\delta_n}^{t_n-1} \rho_n^l c_{k-l} \right| \right] =: C(\Gamma_{1n} + \Gamma_{2n}), \end{aligned}$$

where the last inequality follows from (42). As before, without loss of generality we assume that $|c_k|$ is decreasing. In view of (230), consider

$$\begin{aligned} \Gamma_{1n} &= \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left| \sum_{l=\delta_n}^{t_n-1} \rho_n^l c_{k-l} \right|^\theta \leq \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left(\sum_{l=\delta_n}^{t_n-1} |\rho_n^l c_{k-l}| \right)^\theta \\ &= \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} |c_{k-t_n}|^\theta \left(\sum_{l=\delta_n}^{t_n-1} |\rho_n^l| \right)^\theta \leq C \kappa_n^\theta \rho_n^{\theta\delta_n} \kappa_n^{-b_\theta} \sum_{k=0}^{\infty} |c_k|^\theta \\ &= C \rho_n^{\theta\delta_n} \kappa_n^{-\theta m+1} = o(\kappa_n^{-\theta m+1}) = o(1). \end{aligned}$$

Next,

$$\begin{aligned} \Gamma_{2n} &\leq \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left(\sum_{l=0}^{\delta_n-1} |\rho_n^l c_{k-l}| \right)^{\theta-1} \sum_{l=\delta_n}^{t_n-1} |\rho_n^l c_{k-l}| \\ &\leq \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left(\sum_{l=0}^{\delta_n-1} |\rho_n^l c_{k-t_n}| \right)^{\theta-1} \sum_{l=\delta_n}^{t_n-1} |\rho_n^l c_{k-t_n}| \leq \kappa_n^{-b_\theta} \left(\sum_{l=0}^{\delta_n-1} \rho_n^l \right)^{\theta-1} \sum_{l=\delta_n}^{t_n-1} \rho_n^l \sum_{k=0}^{\infty} |c_k|^\theta \end{aligned}$$

$$\leq C \kappa_n^{-b_\theta} \left(\sum_{l=0}^{\delta_n-1} \rho_n^l \right)^{\theta-1} \sum_{l=\delta_n}^{t_n-1} \rho_n^l = C \kappa_n^{\theta-b_\theta} \rho_n^{\delta_n} = o(1).$$

as required.

(c) As $n \rightarrow \infty$

$$\begin{aligned} Q_\theta(t_n, \delta_n) &= \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left| \sum_{l=0}^{\delta_n-1} \rho_n^l c_{k-l} \right|^\theta = \kappa_n^{-b_\theta} \sum_{k=t_n}^{\infty} \left| \sum_{l=k-(\delta_n-1)}^k \rho_n^{k-l} c_l \right|^\theta \\ &=_{(1)} \frac{1}{\kappa_n} \sum_{k=t_n}^{\infty} \left| \frac{1}{\kappa_n} \sum_{l=k-(\delta_n-1)}^k e^{-\frac{k-l}{\kappa_n}} \left(\frac{l}{\kappa_n} \right)^{-m} \right|^\theta [1 + o(1)] \\ &=_{(2)} \int_{x=t_n/\kappa_n}^{\infty} \left| \int_{y=x-(\delta_n-1)/\kappa_n}^x e^{-x+y} y^{-m} dy \right|^\theta dx [1 + o(1)] \\ &\leq \int_{x=t_n/\kappa_n}^{\infty} \left(\int_{y=(t_n-\delta_n)/\kappa_n}^x e^{-x+y} y^{-m} dy \right)^\theta dx [1 + o(1)] \\ &\leq_{(3)} \int_{x=t_n/\kappa_n}^{\infty} \left(\int_{y=1}^x e^{-x+y} y^{-m} dy \right)^\theta dx [1 + o(1)] =_{(4)} o(1), \end{aligned}$$

where $=_{(1)}$ follows from Lemma 6.4 in DK2018 and the fact that $c_l \sim l^{-m}$, $=_{(2)}$ from Euler summation, and $\leq_{(3)}$ from the fact that $(t_n - \delta_n)/\kappa_n \rightarrow \infty$. Finally, $=_{(4)}$ follows from Lemma 16. ■

Proof of Lemma 15. Lemma 15 follows directly from Lemma 14 if we set $\theta = \alpha_\epsilon$, $b_\theta = b_\epsilon$ with $\epsilon, \alpha_\epsilon$, and b_ϵ defined in (225) -see also (222) and (224). ■

Proof of Lemma 16. (i) Recall that $\theta_1 > 0$ and $\theta_2 \in (1, 2)$ such that $\theta_1 \theta_2 > 1$. Let $N > 0$ and write

$$\begin{aligned} \int_{x=0}^{\infty} e^{-\theta_1 x} \left(\int_{y=0}^x e^y y^{-\theta_2} dy \right)^{\theta_1} dx &\leq \int_{x=0}^N e^{-\theta_1 x} \left(\int_{y=0}^x e^y y^{-\theta_2} dy \right)^{\theta_1} dx \\ + C \int_{x=N}^{\infty} e^{-\theta_1 x} \left(\int_{y=0}^N e^y y^{-\theta_2} dy \right)^{\theta_1} dx &+ C \int_{x=N}^{\infty} e^{-\theta_1 x} \left(\int_{y=N}^x e^y y^{-\theta_2} dy \right)^{\theta_1} dx \\ &=: I_1 + C I_2 + C I_3. \end{aligned} \tag{235}$$

The first term

$$\begin{aligned} I_1 &\leq \int_{x=0}^N e^{-\theta_1 x} \left(\int_{y=0}^x e^y y^{-\theta_2} dy \right)^{\theta_1} dx \\ &\leq C \left(\int_{y=0}^N y^{-\theta_2} dy \right)^{\theta_1} < \infty. \end{aligned}$$

Similarly, the second term

$$I_2 \leq \int_{x=N}^{\infty} e^{-\theta_1 x} dx \left(\int_{y=0}^N e^y y^{-\theta_2} dy \right)^{\theta_1} < \infty.$$

Finally, let N large enough such that

$$\left| \frac{e^{-x} \int_{y=N}^x e^y y^{-\theta_2} dy}{x^{-\theta_2}} - 1 \right| < 1,$$

which is possible due to **Proposition 2**. The third term

$$\begin{aligned}
I_3 &= \int_{x=N}^{\infty} \left(e^{-x} \int_{y=N}^x e^y y^{-\theta_2} dy \right)^{\theta_1} dx \\
&= \int_{x=N}^{\infty} \left(e^{-x} \int_{y=N}^x e^y y^{-\theta_2} dy - x^{-\theta_2} + x^{-\theta_2} \right)^{\theta_1} dx \\
&\leq \int_{x=N}^{\infty} \left(2x^{-\theta_2} \right)^{\theta_1} dx = C \int_{x=N}^{\infty} x^{-\theta_1 \theta_2} dx < \infty,
\end{aligned} \tag{236}$$

as required.

(ii) It can be easily checked that (227) follows from identical arguments to those used for (236). We conclude the proof by establishing the validity of (228). Recall that

$$\gamma(\theta_2; 1, x) := \int_{y=1}^x e^y y^{1-\theta_2} dy.$$

By **Proposition 2** as $x \rightarrow \infty$

$$e^{-x} \gamma(\theta_2; 1, x) - x^{1-\theta_2} = O(x^{-\theta_2}).$$

Therefore, for x large enough, $x \geq x_0$ say,

$$\int_{x=x_0 \vee 1}^{\infty} \left| x^{1-\theta_2} - e^{-x} \gamma(\theta_2; 1, x) \right|^{\theta_1} dx < C \int_{x=x_0 \vee 1}^{\infty} x^{-\theta_1 \theta_2} dx < \infty.$$

Finally, noting that under our assumptions $(2 - \theta_2) \theta_1 > 0$ and $(1 - \theta_2) \theta_1 + 1 > 0$

$$\begin{aligned}
&\int_{x=0}^{x_0 \vee 1} \left| x^{1-\theta_2} - e^{-x} \gamma(\theta_2; 1, x) \right|^{\theta_1} dx \\
&\leq C \left[\int_{x=0}^{x_0 \vee 1} x^{(1-\theta_2)\theta_1} dx + \int_{x=0}^{x_0 \vee 1} \left([y^{2-\theta_2}]_0^x \right)^{\theta_1} dx \right] \\
&= C \left[\int_{x=0}^{x_0 \vee 1} x^{(1-\theta_2)\theta_1} dx + \int_{x=0}^{x_0 \vee 1} x^{(2-\theta_2)\theta_1} dx \right] < \infty,
\end{aligned}$$

as required. ■

Proof of Lemma 17. Without loss of generality assume that $m_0 = 1$. In view of the fact that $\alpha_\epsilon > 1$ and (42), the difference between $V_\epsilon(0, \delta_n)$ and $\kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left(\sum_{l=0}^k e^{-\frac{l}{\kappa_n}} c_{k-l} \right)^{\alpha_\epsilon}$ is

$$\begin{aligned}
&\left| \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left| \sum_{l=0}^k \rho_n^l c_{k-l} \right|^{\alpha_\epsilon} - \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left| \sum_{l=0}^k e^{-\frac{l}{\kappa_n}} c_{k-l} \right|^{\alpha_\epsilon} \right| \\
&\leq C \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left| \sum_{l=0}^k \left(\rho_n^l - e^{-\frac{l}{\kappa_n}} \right) c_{k-l} \right|^{\alpha_\epsilon} + C \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left| \sum_{l=0}^k e^{-\frac{l}{\kappa_n}} c_{k-l} \right|^{\alpha_\epsilon - 1} \left| \sum_{l=0}^k \left(\rho_n^l - e^{-\frac{l}{\kappa_n}} \right) c_{k-l} \right| \\
&\leq C \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left[\sum_{l=0}^k \left| \left(\rho_n^l - e^{-\frac{l}{\kappa_n}} \right) c_{k-l} \right| \right]^{\alpha_\epsilon} \\
&\quad + C \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left[\sum_{l=0}^k \left| e^{-\frac{l}{\kappa_n}} c_{k-l} \right| \right]^{\alpha_\epsilon - 1} \sum_{l=0}^k \left| \left(\rho_n^l - e^{-\frac{l}{\kappa_n}} \right) c_{k-l} \right| := C (B_{1n} + B_{2n}). \tag{237}
\end{aligned}$$

Using the approximation $c_l \sim l^{-m}$, the first term

$$B_{1n} = \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left[\sum_{l=0}^k \left| \left(\rho_n^l - e^{-\frac{l}{\kappa_n}} \right) c_{k-l} \right| \right]^{\alpha_\epsilon} \leq_{(1)} C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left[\sum_{l=0}^k |c_{k-l}| \right]^{\alpha_\epsilon}$$

$$\begin{aligned}
&= C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left[\sum_{l=0}^k |c_l| \right]^{\alpha_\epsilon} = \alpha_\epsilon \left(\frac{\delta_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left[\sum_{l=1}^k l^{-m} \right]^{\alpha_\epsilon} [1 + o(1)] \\
&= C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \kappa_n^{-b_\epsilon} \delta_n^{b_\epsilon} \frac{1}{\delta_n} \sum_{k=0}^{\delta_n} \left[\frac{1}{\delta_n} \sum_{l=0}^k \left(\frac{l}{\delta_n} \right)^{-m} \right]^{\alpha_\epsilon} [1 + o(1)] \\
&= C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \frac{\delta_n^{1+\alpha_\epsilon-\alpha_\epsilon m}}{\kappa_n^{1+\alpha_\epsilon-\alpha_\epsilon m}} \int_0^1 \left(\int_0^x y^{-m} dy \right)^{\alpha_\epsilon} dx [1 + o(1)] \\
&= C \left(\frac{\kappa_n \varsigma_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \varsigma_n^{b_\epsilon} \int_0^1 \left(\int_0^x y^{-m} dy \right)^{\alpha_\epsilon} dx [1 + o(1)] =_{(2)} o(1), \tag{238}
\end{aligned}$$

where $\leq_{(1)}$ follows from DK2018 (Lemma 6.4), and $=_{(2)}$ from the fact that ς_n is SV in κ_n . Similarly, the second term

$$\begin{aligned}
B_{2n} &\leq C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left[\sum_{l=0}^k \left| e^{-\frac{l}{\kappa_n}} c_{k-l} \right| \right]^{\alpha_\epsilon-1} \sum_{l=0}^k |c_{k-l}| \\
&\leq C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \kappa_n^{-b_\epsilon} \left[\sum_{k=0}^{\delta_n} \left\{ \left[\sum_{l=0}^k \left| e^{-\frac{l}{\kappa_n}} c_{k-l} \right| \right]^{\alpha_\epsilon-1} \right\}^{\frac{\alpha_\epsilon-1}{\alpha_\epsilon}} \right]^{\frac{\alpha_\epsilon-1}{\alpha_\epsilon}} \left[\sum_{k=0}^{\delta_n} \left\{ \sum_{l=0}^k |c_{k-l}| \right\}^{\alpha_\epsilon} \right]^{\frac{1}{\alpha_\epsilon}} \\
&= C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \delta_n^{b_\epsilon/\alpha_\epsilon} \kappa_n^{-b_\epsilon/\alpha_\epsilon} \\
&\quad \cdot \left[\kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left\{ \sum_{l=0}^k \left| e^{-\frac{l}{\kappa_n}} c_{k-l} \right| \right\}^{\alpha_\epsilon} \right]^{\frac{\alpha_\epsilon-1}{\alpha_\epsilon}} \left[\delta_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left\{ \sum_{l=0}^k |c_l| \right\}^{\alpha_\epsilon} \right]^{\frac{1}{\alpha_\epsilon}}. \tag{239}
\end{aligned}$$

Euler summation together with the approximation $c_k \sim k^{-m}$ gives

$$\kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left\{ \sum_{l=0}^k \left| e^{-\frac{l}{\kappa_n}} c_{k-l} \right| \right\}^{\alpha_\epsilon} = \int_0^\infty e^{-\alpha_\epsilon x} \left(\int_0^x e^y y^{-m} dy \right)^{\alpha_\epsilon} dx + o(1) \tag{240}$$

and

$$\delta_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left\{ \sum_{l=0}^k |c_l| \right\}^{\alpha_\epsilon} = \int_0^1 \left(\int_0^x y^{-m} dy \right)^{\alpha_\epsilon} dx + o(1), \tag{241}$$

where the integral term of (240) is finite by **Corollary 4**. Hence, in view of (239)-(241) we have

$$B_{2n} \leq C \left(\frac{\delta_n}{\kappa_n^2} \right)^{\alpha_\epsilon} \delta_n^{b_\epsilon/\alpha_\epsilon} \kappa_n^{-b_\epsilon/\alpha_\epsilon} = \frac{\varsigma_n^{1+b_\epsilon/\alpha_\epsilon}}{\kappa_n} = o(1),$$

where we have used the fact that ς_n is SV in κ_n . Once more, using Euler summation together with (238) and (241) we have

$$\begin{aligned}
V_\epsilon(0, \delta_n) &= \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta_n} \left| \sum_{l=0}^k e^{-\frac{l}{\kappa_n}} c_{k-l} \right|^{\alpha_\epsilon} + o(1) \\
&= \int_0^\infty e^{-\alpha_\epsilon x} \left(\int_0^x e^y y^{-m} dy \right)^{\alpha_\epsilon} dx + o(1),
\end{aligned}$$

as required.

(b) Analogous arguments to those used for the proof of part (a) together with Euler summation give

$$\begin{aligned} V_\epsilon(0, \delta'_n) &= \kappa_n^{-b_\epsilon} \sum_{k=0}^{\delta'_n} \left| \sum_{l=0}^k e^{-\frac{l}{\kappa_n}} c_{k-l} \right|^{\alpha_\epsilon} + o(1) \\ &= \int_0^{\delta'_n/\kappa_n} e^{-\alpha_\epsilon x} \left(\int_0^x e^y y^{-m} dy \right)^{\alpha_\epsilon} dx + o(1) = o(1), \end{aligned}$$

as required. ■

Proof of Lemma 18. Without loss of generality we assume that $m_0 = 1$. In addition, note that for n large enough, $0 < \rho_n^j \leq 1$, for all $j \geq 0$. Set

$$\tilde{c}_s := \sum_{j=s}^{\infty} c_j.$$

Write

$$a_k := a_k(n) = \sum_{l=0}^k \rho_n^{k-l} c_l = \sum_{l=0}^k \rho_n^{k-l} (\tilde{c}_l - \tilde{c}_{l+1}). \quad (242)$$

Summation by parts gives

$$\begin{aligned} & \sum_{l=0}^k \rho_n^{-l} (\tilde{c}_l - \tilde{c}_{l+1}) \\ &= \rho_n^{-0} (\tilde{c}_0 - \tilde{c}_{0+1}) + \rho_n^{-1} (\tilde{c}_1 - \tilde{c}_2) + \rho_n^{-2} (\tilde{c}_2 - \tilde{c}_3) + \dots \\ & \quad \dots + \rho_n^{-k-1} (\tilde{c}_{k-1} - \tilde{c}_k) + \rho_n^{-k} (\tilde{c}_k - \tilde{c}_{k+1}) \\ &= \tilde{c}_0 - \rho_n^{-k} \tilde{c}_{k+1} - \tilde{c}_1 + \rho_n^{-1} \tilde{c}_1 - \rho_n^{-1} \tilde{c}_2 + \rho_n^{-2} \tilde{c}_2 - \rho_n^{-2} \tilde{c}_3 + \dots \\ & \quad \dots + \rho_n^{-k-1} \tilde{c}_{k-1} - \rho_n^{-k-1} \tilde{c}_k + \rho_n^{-k} \tilde{c}_k \\ &= \tilde{c}_0 - \rho_n^{-k} \tilde{c}_{k+1} - \sum_{l=1}^k \tilde{c}_l (\rho_n^{-l+1} - \rho_n^{-l}) \\ &= \tilde{c}_0 - \rho_n^{-k} \tilde{c}_{k+1} - \sum_{l=1}^k \rho_n^{-l} \tilde{c}_l (\rho_n - 1) \\ &= \tilde{c}_0 - \rho_n^{-k} \tilde{c}_{k+1} + \kappa_n^{-1} \sum_{l=1}^k \tilde{c}_l \rho_n^{-l}. \end{aligned} \quad (243)$$

In view of (242) and (243),

$$\begin{aligned} \kappa_n^{-(1-m)} a_k &= \kappa_n^{-(1-m)} \sum_{l=0}^k \rho_n^{k-l} c_l \\ &= T_{1nk} + T_{2nk}. \end{aligned} \quad (244)$$

Finally, by (244)

$$\begin{aligned} V_\epsilon(0, \delta_n) &= \kappa_n^{-b_\epsilon} \sum_{k=1}^{\delta_n} |a_k|^{\alpha_\epsilon} \\ &= \kappa_n^{-(1+\alpha_\epsilon-\alpha_\epsilon m)} \sum_{k=1}^{\delta_n} \left| \sum_{l=0}^k \rho_n^{k-l} c_l \right|^{\alpha_\epsilon} = \kappa_n^{-1} \sum_{k=1}^{\delta_n} \left| \kappa_n^{-(1-m)} \sum_{l=0}^k \rho_n^{k-l} c_l \right|^{\alpha_\epsilon} \end{aligned}$$

$$= \kappa_n^{-1} \sum_{k=1}^{\delta_n} |T_{1nk} + T_{2nk}|^{\alpha_\epsilon} = |1 - m|^{-\alpha_\epsilon} \int_{x=0}^{\infty} |p(x)|^{\alpha_\epsilon} dx + o(1),$$

where the last approximation above is a direct consequence of **Corollary 1**. ■

Proof of Proposition 2. (a) Integration by parts gives

$$\begin{aligned} q(x) &= e^{-x} \int_1^x e^y y^{-q_0} dy \\ &= e^{-x} \left\{ \left[e^y y^{-q_0} \right]_1^x + q_0 \int_1^x e^y y^{-q_0-1} dy \right\} \\ &= x^{-q_0} + e^{-x} + q_0 e^{-x} \int_1^x e^y y^{-q_0-1} dy. \end{aligned} \quad (245)$$

Consider the term

$$\begin{aligned} e^{-x} \int_1^x e^y y^{-q_0-1} dy &= e^{-x} \int_1^{x/2} e^y y^{-q_0-1} dy + e^{-x} \int_{x/2}^x e^y y^{-q_0-1} dy \\ &\leq e^{-x/2} \int_1^{x/2} y^{-q_0-1} dy + \left(\frac{x}{2} \right)^{-q_0-1} e^{-x} \int_{x/2}^x e^y dy \\ &\leq C \left[e^{-x/2} + \left(\frac{x}{2} \right)^{-q_0-1} \right] = O(x^{-q_0-1}). \end{aligned} \quad (246)$$

The result for $N = 1$ follows directly from (245) and (246). The general case follows from arguments similar to those used above and induction.

(b) Recall that $0 < q_0 < 1$. Integration by parts gives

$$\begin{aligned} q(x) &= e^{-x} \int_0^x e^y y^{-q_0} dy = e^{-x} \left[e^y \int y^{-q_0} dy \right]_0^x - e^{-x} \int_0^x \int y^{-q_0} dy e^y dy \\ &= e^{-x} \left[e^y \int y^{-q_0} dy \right]_0^x - e^{-x} \int_0^x y^{1-q_0} e^y dy = e^{-x} \left[e^x \frac{x^{1-q_0}}{1-q_0} \right] - e^{-x} \int_0^x y^{1-q_0} e^y dy \\ &= \frac{x^{1-q_0}}{1-q_0} - e^{-x} \int_0^x y^{1-q_0} e^y dy. \end{aligned}$$

Next, note that as $x \rightarrow 0$

$$\frac{e^{-x} \int_0^x y^{1-q_0} e^y dy}{x^{1-q_0}} \leq \frac{\int_0^x y^{1-q_0} dy}{x^{1-q_0}} = \frac{1}{1-q_0} \frac{x^{2-q_0}}{x^{1-q_0}} \rightarrow 0.$$

In view of the above

$$q(x) \sim \frac{x^{1-q_0}}{1-q_0}.$$

Therefore, the requisite result holds for $N = 1$. The general case follows from arguments analogous to those used above and induction. ■

Reversed Gamma Function

For $a < x$ set.

$$\gamma(m; a, x) := \int_{y=a}^x e^y y^{1-m} dy.$$

Then integration by parts gives

$$\begin{aligned}\gamma(m; a, x) &= \left[e^x x^{1-m} - a^{1-m} e^a \right] - (1-m) \int_{y=a}^x e^y y^{-m} dy \\ &= \left[e^x x^{1-m} - a^{1-m} e^a \right] + (m-1) \gamma(m+1; a, x).\end{aligned}$$

Hence,

$$\begin{aligned}x^{1-m} - e^{-x} \gamma(m, 2-m, x) &= x^{1-m} - e^{-x} \int_{y=2-m}^x e^y y^{1-m} dy \\ &= x^{1-m} - e^{-x} \left\{ \left[e^y y^{1-m} \right]_{2-m}^x + (m-1) \int_{y=2-m}^x e^y y^{-m} dy \right\} \\ &= x^{1-m} - e^{-x} \left\{ \left[e^x x^{1-m} - (2-m)^{1-m} e^{2-m} \right] + (m-1) \int_{y=2-m}^x e^y y^{-m} dy \right\} \\ &= e^{-x} \left[(2-m)^{1-m} e^{2-m} - (m-1) \int_{y=2-m}^x e^y y^{-m} dy \right].\end{aligned}$$

For $x > 2-m$ the above imply

$$\begin{aligned}p(x) &= x^{1-m} - e^{-x} \gamma(m, 0, x) = x^{1-m} - e^{-x} \gamma(m, 2-m, x) - e^{-x} \gamma(m, 0, 2-m) \\ &= e^{-x} \left[(2-m)^{1-m} e^{2-m} - (m-1) \int_{y=2-m}^x e^y y^{-m} dy \right] - e^{-x} \gamma(m, 0, 2-m) \\ &= e^{-x} \left[(2-m)^{1-m} e^{2-m} - (m-1) \int_{y=2-m}^x e^y y^{-m} dy - \int_{y=0}^{2-m} e^y y^{1-m} dy \right].\end{aligned}$$

Roots of $p(x)$

Recall that

$$p(x) = x^{1-m} - e^{-x} \int_0^x e^y y^{1-m} dy.$$

Proposition 3. $p(x)$ has a single root x_* on $(0, \infty)$ such that $x_* > 2-m$.

Case $x \in (0, 2-m)$:

$$\begin{aligned}p(x) &= x^{1-m} - e^{-x} \int_0^x e^y y^{1-m} dy \geq x^{1-m} - \int_0^x y^{1-m} dy \\ &\geq x^{1-m} - \frac{x^{2-m}}{2-m} := p_1(x)\end{aligned}$$

Now

$$\begin{aligned}p_1(x) > 0 &\iff x^{1-m} \geq \frac{x^{2-m}}{2-m} \iff (1-m) \ln x > (2-m) \ln(x) - \ln(2-m) \\ &\iff \ln(2-m) > (2-m) \ln(x) - (1-m) \ln(x) \\ &\iff \ln(2-m) > \ln(x) \iff (2-m) > x.\end{aligned}$$

Hence,

$$p(x) > 0 \text{ for } 0 < x < 2-m$$

Case $x \geq 2-m$:

$$p(x) = e^{-x} \left[[2-m]^{1-m} e^{2-m} - (m-1) \int_{y=2-m}^x e^y y^{-m} dy - \int_0^{2-m} e^y y^{1-m} dy \right].$$

The last term above

$$\int_0^{2-m} e^y y^{1-m} dy < e^{2-m} \int_0^{2-m} y^{1-m} dy = e^{2-m} \frac{(2-m)^{2-m}}{2-m} = e^{2-m} (2-m)^{1-m}.$$

Hence,

$$p(x) = e^{-x} \left[\underbrace{[2-m]^{1-m} e^{2-m} - \int_0^{2-m} e^y y^{1-m} dy}_{=: p_2} - (m-1) \int_{y=2-m}^x e^y y^{-m} dy \right],$$

which implies that

$$p(2-m) = e^{-(2-m)} p_2 > 0.$$

Moreover, in view of this we can write

$$p(x) = e^{-x} \left[p_2 - (m-1) \int_{y=2-m}^x e^y y^{-m} dy \right],$$

where $p_2 = [2-m]^{1-m} e^{2-m} - \int_0^{2-m} e^y y^{1-m} dy > 0$. Noting that $\int_{y=2-m}^x e^y y^{-m} dy$ is continuous in x , strictly increasing and diverging, by the intermediate value Theorem there is a some $x^* > 2-m$ such that $p(x^*) = 0$. Further, x^* is a unique root due to the monotonicity of the aforementioned term.

16. Uniformly Bounded Density Results

We start with some notation and preliminary results. First, we will expand the definition of the sequence m_n introduced in (135). Recall that $m \in (1/\alpha, 1)$, $m \in (1, 1 + 1/\alpha)$ under **LM** and **NM** respectively. For **SM**, we shall use the convention $m = 1$. Define

$$m_n := \gamma_n \kappa_n^{-(1-m)}, \text{ with } m \in \begin{cases} (1/\alpha, 1), \text{ under } \mathbf{LM}, \alpha \in (1, 2]; \\ \{1\}, \text{ under } \mathbf{SM}, \alpha \in (0, 2]; \\ (1, 1 + 1/\alpha), \text{ under } \mathbf{NM}, \alpha \in (1, 2]. \end{cases} \quad (247)$$

In addition, recall that

$$\gamma_n \sim \begin{cases} L (\gamma_n n^{1/\alpha})^{1/\alpha}, & \text{under } \mathbf{BFR} \\ \ell_1 (\kappa_n)^{1/\alpha} \kappa_n^{b/\alpha}, & \text{under } \mathbf{MI} \end{cases},$$

where $b = 1 + \alpha - \alpha m$ and $\ell_1(x) \sim H(x^{1/\alpha} \ell_1(x)^{1/\alpha})$. By construction we have

$$m_n \rightarrow \infty.$$

In particular, (see Remark 20)

$$m_n \sim \ell_1(\kappa_n)^{1/\alpha} \kappa_n^{1/\alpha} \rightarrow \infty. \quad (248)$$

Finally, set

$$[\eta_n, \bar{q}_n, \underline{q}_n] := \begin{cases} [n^{1/\alpha}, n-1, \lfloor \gamma_n \rfloor], & \text{under } \mathbf{BFR} \\ [1, \lfloor \delta \kappa_n \rfloor, \lfloor \delta \kappa_n / 2 \rfloor], & \delta \in (0, 1] \text{ under } \mathbf{MI} \end{cases}, \quad (249)$$

and

$$\delta_n := \begin{cases} 1, & \text{under } \mathbf{BFR} \\ \kappa_n^{1-m}, & \text{under } \mathbf{MI} \end{cases}$$

Since the modulus of a characteristic function does not exceed one, if Assumption INN(iii) holds for some $\theta \geq 1$, it also holds for every $\theta' \geq \theta$. Hence, without loss of generality, throughout this section we take $\theta \in \mathbb{N}$.

We have the following technical result that will be utilized in this section.

Lemma 19. Suppose that **Assumption INN**, and **Assumption BFR** or **MI** hold.

(i) There are $n_0, t_0 \geq 1$, sequences $[\eta_n, \bar{q}_n, \underline{q}_n]$ that conform with (249), and constants $0 < \mathcal{D}_0 \leq \mathcal{D}_1 < \infty$ such that

$$\mathcal{D}_0 < \inf_{\underline{q}_t \leq k \leq \bar{q}_t} \delta_t^{-1} |\eta_k a_k(n)| \leq \sup_{\underline{q}_t \leq k \leq \bar{q}_t} \delta_t^{-1} |\eta_k a_k(n)| \leq \mathcal{D}_1, \quad (250)$$

for all $n \geq n_0$ and $t_0 \leq t \leq n$.

(ii) There is an $n_0 \geq 1$ and some $\zeta \in (0, 1)$ such that as $t \rightarrow \infty$

$$\sup_{0 \leq k \leq \bar{q}_t, n \geq n_0} \left| \frac{a_k(n)}{\gamma_t^\zeta} \right| = o(1). \quad (251)$$

Remark 25. Notice that as $n \rightarrow \infty$, $n \geq \bar{q}_n > \underline{q}_n > 0$, and $q_n := \bar{q}_n - \underline{q}_n \rightarrow \infty$.

For all $\lambda \in \mathbb{R}$ define

$$G^*(x) := x^{-\alpha} H(x), \quad x > 0, \quad G(\lambda) := \begin{cases} |\lambda|^\alpha H(|\lambda|^{-1}), & \lambda \neq 0 \\ 0, & \lambda = 0 \end{cases}. \quad (252)$$

Lemma 20. (i) Suppose that **Assumption INN**, and **Assumption BFR** or **MI** hold, and $C_0 \in (0, \infty)$. There are $n_0, t_0 \geq 1$

$$\inf_{n \geq n_0, t_0 \leq t \leq n} \sum_{k=\underline{q}_t+\theta}^{\bar{q}_t} G\left(C_0 \frac{a_k(n)}{\gamma_t}\right) > 0.$$

(ii) Let $r_1, r_2 > 0$. As $n \rightarrow \infty$,

$$\exp(-r_1 \gamma_n^{r_2}) \gamma_n^2 \delta_n^{-2} \eta_{\underline{q}_n+\theta}^2 \rightarrow 0.$$

Under **Assumption INN**, for every $\eta \in (0, 1)$ there is an $N'_\eta > 0$ such that

$$|\psi_\xi(\lambda)| \leq e^{-\eta |\lambda|^\alpha H(|\lambda|^{-1})} \quad \text{for all } 0 < |\lambda| \leq N'_\eta. \quad (253)$$

In addition, there is some $N > 0$ such that $|\lambda|^\alpha H(|\lambda|^{-1}) < 1/2$ for all $0 < |\lambda| \leq N'$ (see e.g. BGT 1987; Proposition 1.3.6(v)). Set $N_\eta := N'_\eta \wedge N'$. Moreover, because ξ has an absolutely continuous distribution, for each $\delta > 0$ there some ρ_δ , $0 < \rho_\delta < 1$ such that (see e.g. Feller, 1971; p. 516)

$$|\psi_\xi(\lambda)| \leq \rho_\delta, \quad |\lambda| > \delta. \quad (254)$$

Combining (253)-(254) we have

$$\begin{aligned}
|\psi_\xi(\lambda)| &\leq e^{-\eta G(\lambda)} 1_{\{|\lambda| \leq N_\eta\}} + e^{\ln \rho_{N_\eta}} 1_{\{|\lambda| > N_\eta\}} \\
&=: e^{-\eta G(\lambda)} 1_{\{|\lambda| \leq N_\eta\}} + e^{-\vartheta_\eta} 1_{\{|\lambda| > N_\eta\}} \\
&\leq e^{-(\eta \wedge \vartheta_\eta) G(\lambda)} 1_{\{|\lambda| \leq N_\eta\}} + e^{-(\eta \wedge \vartheta_\eta)} 1_{\{|\lambda| > N_\eta\}} \\
&=: e^{-\eta_* G(\lambda)} 1_{\{|\lambda| \leq N_\eta\}} + e^{-\eta_*} 1_{\{|\lambda| > N_\eta\}} \tag{255}
\end{aligned}$$

$$\leq e^{-\eta_* [G(\lambda) \wedge 1]} 1_{\{|\lambda| \leq N_\eta\}} + e^{-\eta_* [G(\lambda) \wedge 1]} 1_{\{|\lambda| > N_\eta\}} = e^{-\eta_* [G(\lambda) \wedge 1]}. \tag{256}$$

Noting that $G^*(.)$ (cf. eq. (252)) is regularly varying of index $-\alpha$, there no loss of generality (see e.g. Theorem 1.5.3 in BGT 1987) in the subsequent derivations, if we assume that $G^*(.)$ is decreasing ($G(.)$ increasing). To see this note that for large $|\lambda|^{-1}$, there is some decreasing regularly varying function of index $-\alpha$, $\underline{G}^*(.)$ say, such that $G^*(\mu) \sim \underline{G}^*(\mu)$, $\mu \rightarrow \infty$. Hence, for μ large enough, $\mu \geq 1/N_\eta$ say

$$|G^*(\mu)/\underline{G}^*(\mu) - 1| \leq 1/2.$$

Hence,

$$G^*(\mu) = [G^*(\mu)/\underline{G}^*(\mu) - 1] \underline{G}^*(\mu) + \underline{G}^*(\mu) \geq -\frac{1}{2} \underline{G}^*(\mu) + \underline{G}^*(\mu) = \frac{1}{2} \underline{G}^*(\mu).$$

Setting $\mu = |\lambda|^{-1}$, $|\lambda| \leq N_\eta$ we get

$$|\psi_\xi(\lambda)| \leq e^{-\eta G(\lambda)} = e^{-\eta G^*(|\lambda|^{-1})} \leq e^{-\eta \frac{1}{2} \underline{G}^*(|\lambda|^{-1})}.$$

Therefore, there is no loss of generality if we use the convention $G(\lambda) = \frac{1}{2} \underline{G}^*(|\lambda|^{-1})$.

Remark 26. In view of the the monotonicity convention for $G(.)$, we have

$$\sup_{|\lambda| \geq N_\eta} e^{-\eta_* [G(\lambda) \wedge 1]} \leq e^{-\eta_* [G(N_\eta) \wedge 1]} = e^{-\eta_* G(N_\eta)}.$$

Moreover, by construction, $G(\lambda) < 1/2$ for all $0 < |\lambda| \leq N_\eta$. Hence,

$$\sup_{0 < |\lambda| \leq N_\eta} e^{-\eta_* [G(\lambda) \wedge 1]} \leq \sup_{0 < |\lambda| \leq N_\eta} e^{-\eta_* G(\lambda)}.$$

In addition, we shall use the following facts:

Remark 27.

- (i) Fix $\epsilon > 0$ and note that $|\lambda|^{-\epsilon} H(|\lambda|^{-1}) \rightarrow \infty$ as $\lambda \rightarrow 0$. Hence, for all $|\lambda|$ ($\lambda \neq 0$) small enough

$$|\lambda|^\alpha H(|\lambda|^{-1}) = |\lambda|^{\alpha+\epsilon} |\lambda|^{-\epsilon} H(|\lambda|^{-1}) \geq |\lambda|^{\alpha+\epsilon}.$$

- (ii) We remark that (251) implies that there is $0 < \mathcal{D}_2 < \infty$ such that

$$\sup_{t \geq 1} \sup_{0 \leq k \leq \bar{q}_t, n \geq n_0} \left| \frac{a_k(n)}{\gamma_t^\zeta} \right| \leq \mathcal{D}_2. \tag{257}$$

In addition, as $t \rightarrow \infty$

$$\inf_{0 \leq k \leq \bar{q}_t, n \geq n_0} \left| \frac{a_k(n)}{\gamma_t^\zeta} \right|^{-1} \rightarrow \infty. \tag{258}$$

(iii) Fix $A \in (1, \infty)$ and $\epsilon \in (0, \alpha)$. In view of (258) and Potter's Theorem for all $\lambda \in [1, \mathcal{D}_2^{-1} \gamma_t^{1-\zeta}]$, $\underline{q}_t \leq k \leq \bar{q}_t$, and t large enough we have

$$\begin{aligned} \frac{H\left(\left|\frac{\gamma_t}{a_k(n)}\right|\right)}{H\left(\left|\frac{\gamma_t}{a_k(n)\lambda}\right|\right)} &\leq A \left[\left(\left|\frac{\gamma_t}{a_k(n)}\right|\right)^\epsilon \vee \left(\left|\frac{\gamma_t}{a_k(n)\lambda}\right|\right)^{-\epsilon} \right] \\ &= A \left[|\lambda|^\epsilon \vee |\lambda|^{-\epsilon} \right] = A |\lambda|^\epsilon. \end{aligned}$$

Hence, for all $\lambda \in [1, \mathcal{D}_2^{-1} \gamma_t^{1-\zeta}]$, $\underline{q}_t \leq k \leq \bar{q}_t$, and t large enough

$$\begin{aligned} G\left(\frac{a_k(n)}{\gamma_t} \lambda\right) &= \left|\frac{a_k(n)}{\gamma_t} \lambda\right|^\alpha H\left(\left|\frac{\gamma_t}{a_k(n)\lambda}\right|\right) \\ &\geq A^{-1} |\lambda|^{-\epsilon} \left|\frac{a_k(n)}{\gamma_t} \lambda\right|^\alpha H\left(\left|\frac{\gamma_t}{a_k(n)}\right|\right) = A^{-1} |\lambda|^{\alpha-\epsilon} G\left(\frac{a_k(n)}{\gamma_t}\right). \end{aligned} \quad (259)$$

Let $\zeta \in (0, 1)$. Fix $\epsilon \in (0, \alpha)$ and $A > 1$. Then there is some $t_1 > 0$ such that for all $t \geq t_1$ the following hold

$$\begin{aligned} \frac{H\left(\left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t}\right|^{-1}\right)}{H\left(\left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t^\zeta}\right|^{-1}\right)} &\leq A \left[\left(\left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t}\right|^{-1}\right)^\epsilon \vee \left(\left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t^\zeta}\right|^{-1}\right)^{-\epsilon} \right] \\ &= C \left[\left(\frac{\gamma_t}{\gamma_t^\zeta}\right)^\epsilon \vee \left(\frac{\gamma_t}{\gamma_t^\zeta}\right)^{-\epsilon} \right] = A \left[\gamma_t^{\epsilon(1-\zeta)} \vee \gamma_t^{-\epsilon(1-\zeta)} \right] = A \gamma_t^{\epsilon(1-\zeta)}, \end{aligned}$$

where the inequality above is due to Potter's Theorem. A simple rearrangement of the terms above gives

$$\begin{aligned} G\left(\frac{a_k(n)}{\mathcal{D}_2 \gamma_t^\zeta}\right) &= \left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t^\zeta}\right|^\alpha H\left(\left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t^\zeta}\right|^{-1}\right) \\ &\geq \left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t^\zeta}\right|^\alpha H\left(\left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t}\right|^{-1}\right) \cdot \frac{1}{A \gamma_t^{\epsilon(1-\zeta)}} \\ &= \frac{\gamma_t^\alpha}{A \gamma_t^{\epsilon(1-\zeta)} \gamma_t^{\alpha\zeta}} \left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t}\right|^\alpha H\left(\left|\frac{a_k(n)}{\mathcal{D}_2 \gamma_t}\right|^{-1}\right) \\ &= A^{-1} \gamma_t^{(\alpha-\epsilon)(1-\zeta)} G\left(\frac{a_k(n)}{\mathcal{D}_2 \gamma_t}\right). \end{aligned} \quad (260)$$

Lemma 21. Suppose that the conditions of **Lemma 19** hold. There are some $n_0 \geq t_0 \geq 1$ such that

$$\sup_{n \geq n_0, t_0 \leq t \leq n} \int_{\mathbb{R}} |\lambda| \left| \mathbf{E} \left(e^{i\lambda x_t^+(n)/\gamma_t} \right) \right| d\lambda < \infty.$$

and

$$\sup_{n \geq n_0, t_0 \leq t \leq n} \int_{\mathbb{R}} |\lambda| \left| \mathbf{E} \left(e^{i\lambda \frac{x_t(n)}{\gamma_t}} \right) \right| d\lambda < \infty.$$

Proof of Lemma 21. Fix $\eta \in (0, 1)$ and let N_η, η^* as in (255). Set $\mathcal{N} = N_\eta \mathcal{D}_2^{-1}$, with $\mathcal{D}_2$ given in (257). For some $\zeta \in (0, 1)$, since $x_t^+(n)$ and $x_t^-(n)$ are independent and the modulus of the characteristic function of $x_t^-(n)$ is at most one, it suffices to bound the characteristic function of $x_t^+(n)$. Write

$$\begin{aligned} \int_{\mathbb{R}} |\lambda| \left| \mathbf{E} \left(e^{i\lambda \frac{x_t(n)}{\gamma_t}} \right) \right| d\lambda &\leq \int_{\mathbb{R}} |\lambda| \left| \mathbf{E} \left(e^{i\lambda \gamma_t^{-1} \sum_{k=0}^{t-1} a_k(n) \xi_{t-k}} \right) \right| d\lambda \\ &= \int_{\mathbb{R}} |\lambda| \left| \prod_{k=0}^{t-1} \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| d\lambda \\ &= \left(\int_{|\lambda| < 1} + \int_{1 \leq |\lambda| \leq \gamma_t^{1-\zeta} \mathcal{N}} + \int_{|\lambda| > \gamma_t^{1-\zeta} \mathcal{N}} \right) |\lambda| \left| \prod_{k=0}^{t-1} \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| d\lambda =: I_{0n}(t) + I_{1n}(t) + I_{2n}(t). \end{aligned}$$

Clearly, $I_{0n}(t) \leq 2$.

Next, let ϵ, A as in (259). For t large enough the second term

$$\begin{aligned} I_{1n}(t) &\leq \int_{1 \leq |\lambda| \leq \gamma_t^{1-\zeta} \mathcal{N}} |\lambda| \left| \prod_{k=0}^{t-1} \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| d\lambda \\ &\leq \int_{1 \leq |\lambda| \leq \gamma_t^{1-\zeta} \mathcal{N}} |\lambda| \left| \prod_{k=\underline{q}_t}^{\bar{q}_t} \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| d\lambda \leq_{(1)} \int_{|\lambda| \leq \gamma_t^{1-\zeta} \mathcal{N}} |\lambda| \exp \left[-\eta_* \sum_{k=\underline{q}_t}^{\bar{q}_t} G \left(\left| \frac{\lambda}{\gamma_t} a_k(n) \right| \right) \right] d\lambda \\ &\leq \int_{|\lambda| \leq \gamma_t^{1-\zeta} \mathcal{N}} |\lambda| \exp \left[-A^{-1} \eta_* |\lambda|^{\alpha-\epsilon} \sum_{k=\underline{q}_t}^{\bar{q}_t} G \left(\frac{a_k(n)}{\gamma_t} \right) \right] d\lambda \\ &\quad \left[\mu = \left[\sum_{k=\underline{q}_t}^{\bar{q}_t} G \left(\frac{a_k(n)}{\gamma_t} \right) \right]^{\frac{1}{\alpha-\epsilon}} \lambda \right] \\ &\leq \left[\inf_{n \geq n_0, t_0 \leq t \leq n} \sum_{k=\underline{q}_t}^{\bar{q}_t} G \left(\frac{a_k(n)}{\gamma_t} \right) \right]^{-\frac{2}{\alpha-\epsilon}} \int_{\mathbb{R}} |\mu| \exp \left[-A^{-1} \eta_* |\mu|^{\alpha-\epsilon} \right] d\mu <_{(3)} \infty, \end{aligned}$$

where $\leq_{(1)}$ and $\leq_{(2)}$ follows from (255) and (259) respectively. Moreover, $<_{(3)}$ is a direct consequence of **Lemma 20**.

Next, in view of **Remark 25**,

$$\begin{aligned} I_{2n}(t) &\leq \int_{|\lambda| > \gamma_t^{1-\zeta} \mathcal{N}} |\lambda| \left| \prod_{k=\underline{q}_t}^{\bar{q}_t} \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| d\lambda \\ &= \int_{|\lambda| > \gamma_t^{1-\zeta} \mathcal{N}} |\lambda| \left| \prod_{k=\underline{q}_t}^{\underline{q}_t + \theta - 1} \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| \left| \prod_{k=\underline{q}_t + \theta}^{\bar{q}_t} \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| d\lambda. \end{aligned}$$

For $|\lambda| > \gamma_t^{1-\zeta} \mathcal{N}$ and $\underline{q}_t + \theta \leq k \leq \bar{q}_t$ we have

$$\begin{aligned} \left| \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| &\leq_{(4)} e^{-\eta_* \left[G \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \wedge 1 \right]} \\ &\leq_{(5)} \exp \left\{ -\eta_* \left[G \left(\frac{\gamma_t^{1-\zeta} \mathcal{N}}{\gamma_t} a_k(n) \right) \wedge 1 \right] \right\} \leq_{(6)} \exp \left\{ -\eta_* G \left(\gamma_t^{-\zeta} \mathcal{N} a_k(n) \right) \right\}, \end{aligned}$$

where $\leq_{(4)}$ follows from (256), $\leq_{(5)}$ from the monotonicity of G , and $\leq_{(6)}$ from the fact that $G(|\lambda|) \leq 1/2$, for $|\lambda| \leq N_\eta$. In view of the above, (260), and **Lemma 20** we have

$$\begin{aligned} \sup_{|\lambda| > \gamma_t^{1-\zeta} \mathcal{N}} \prod_{k=\underline{q}_t+\theta}^{\bar{q}_t} \left| \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| &\leq \prod_{k=\underline{q}_t+\theta}^{\bar{q}_t} \exp \left\{ -\eta_* G \left(\gamma_t^{-\zeta} \mathcal{N} a_k(n) \right) \right\} \\ &\leq \exp \left\{ -\eta_* A^{-1} \gamma_t^{(\alpha-\epsilon)(1-\zeta)} \sum_{k=\underline{q}_t+\theta}^{\bar{q}_t} G \left(\mathcal{N} \frac{a_k(n)}{\gamma_t} \right) \right\} \\ &\leq \exp \left\{ -\eta_* C_0 \gamma_t^{(\alpha-\epsilon)(1-\zeta)} \right\} =: Q_t, \end{aligned}$$

for some $C_0 \in (0, \infty)$. Hence,

$$\begin{aligned} I_{2n}(t) &\leq Q_t \int_{|\lambda| > \gamma_t^{1-\zeta} \mathcal{N}} |\lambda| \prod_{k=\underline{q}_t}^{q_t+\theta-1} \left| \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right| d\lambda \\ &\leq_{(7)} Q_t \prod_{k=\underline{q}_t}^{q_t+\theta-1} \left(\int_{|\lambda| > \gamma_t^{1-\zeta} \mathcal{N}} |\lambda| \left| \psi \left(\frac{\lambda}{\gamma_t} a_k(n) \right) \right|^\theta d\lambda \right)^{1/\theta} \\ &\quad \left[\mu = \frac{\lambda}{\gamma_t} a_k(n) \right] \\ &\leq Q_t \gamma_t^2 \prod_{k=\underline{q}_t}^{q_t+\theta-1} \left(|a_k(n)|^{-2} \int_{\mathbb{R}} |\mu| |\psi(\mu)|^\theta d\mu \right)^{1/\theta} \\ &\leq Q_t \gamma_t^2 \sup_{\underline{q}_t \leq k \leq q_t+\theta-1} |a_k(n)|^{-2} \int_{\mathbb{R}} |\mu| |\psi(\mu)|^\theta d\mu \\ &= Q_t \left(\inf_{\underline{q}_t \leq k \leq q_t+\theta-1} \frac{1}{\gamma_t} |a_k(n)| \right)^{-2} \int_{\mathbb{R}} |\mu| |\psi(\mu)|^\theta d\mu \\ &= Q_t \gamma_t^2 \delta_t^{-2} \left(\delta_t^{-1} \inf_{\underline{q}_t \leq k \leq q_t+\theta-1} \eta_k^{-1} \eta_k |a_k(n)| \right)^{-2} \int_{\mathbb{R}} |\mu| |\psi(\mu)|^\theta d\mu \\ &\leq Q_t \gamma_t^2 \delta_t^{-2} \left(\left[\inf_{\underline{q}_t \leq k \leq q_t+\theta-1} \eta_k^{-1} \delta_t^{-1} \inf_{\underline{q}_t \leq k \leq q_t+\theta-1} \eta_k |a_k(n)| \right] \right)^{-2} \int_{\mathbb{R}} |\mu| |\psi(\mu)|^\theta d\mu \\ &\leq_{(5)} \mathcal{D}_0^{-2} Q_t \gamma_t^2 \delta_t^{-2} \left(\left[\inf_{\underline{q}_t \leq k \leq q_t+\theta-1} \eta_k^{-1} \right] \right)^{-2} \int_{\mathbb{R}} |\mu| |\psi(\mu)|^\theta d\mu \\ &\leq \mathcal{D}_0^{-2} Q_t \gamma_t^2 \delta_t^{-2} \eta_{\underline{q}_t+\theta}^2 \int_{\mathbb{R}} |\mu| |\psi(\mu)|^\theta d\mu < C < \infty, \end{aligned}$$

where $\leq_{(7)}$ is a consequence of Hölder's inequality (see e.g. Jeganathan, 2008; eq. (35)), $\leq_{(5)}$ follows from **Lemma 19**. Finally, the last inequality follows from the fact that $Q_t \gamma_t^2 \delta_t^{-2} \eta_{\underline{q}_t+\theta}^2$ is real valued sequence such that $Q_t \gamma_t^2 \delta_t^{-2} \eta_{\underline{q}_t+\theta}^2 \rightarrow 0$ as $t \rightarrow \infty$ -cf. **Lemma 20**. ■

Proof of Lemma 19.

(i) We start with the proof of (250). We shall show the result under **MI-NM** (i.e. $1 < m < 1 + 1/\alpha$). Without loss of generality set $m_0 = 1$. For the other cases see DK2018 (Lemma 6.2). Recall that for $0 \leq k \leq t-1$, we have $a_{k,t} = a_k$. Set $0 \leq k \leq t-1$. For some $\delta > 0$, that will be specified in detail later, set $\underline{q}_t = \lfloor \delta \kappa_t / 2 \rfloor$ and $\bar{q}_t = \lfloor \delta \kappa_t \rfloor$. Using the fact that $m_t/\gamma_t \sim \kappa_t^{m-1}$ -cf. (247)- and the mean value theorem (see e.g. DK2018; Lemma 6.4(i)), as $t(\leq n) \rightarrow \infty$ we have uniformly in $\underline{q}_t \leq k \leq \bar{q}_t$,

$$\begin{aligned}
& \frac{m_t}{\gamma_t} \left| a_k - \sum_{j=0}^k e^{-\frac{k-j}{\kappa_n}} c_j \right| = \frac{m_t}{\gamma_t} \left| \sum_{j=0}^k \rho_n^{k-j} c_j - \sum_{j=0}^k e^{-\frac{k-j}{\kappa_n}} c_j \right| \\
& \leq C \kappa_t^{m-1} \sum_{j=0}^k \left| \rho_n^{k-j} - e^{-\frac{k-j}{\kappa_n}} \right| |c_j| \leq \frac{2C}{\kappa_n^2} \kappa_t^{m-1} \sum_{j=0}^k e^{-\frac{k-j}{\kappa_n}} (k-j) |c_j| \\
& \leq \frac{2C}{\kappa_n^2} \kappa_t^{m-1} \sup_{1 \leq l \leq n} \left(e^{-\frac{l}{\kappa_n}} l \right) \sum_{j=0}^{\infty} |c_j| \leq \frac{2C e^{-1}}{\kappa_n} \kappa_t^{m-1} \sum_{j=0}^{\infty} |c_j| \\
& \leq \frac{2C}{\kappa_n} \kappa_n^{m-1} \sum_{j=0}^{\infty} |c_j| = o(1). \tag{261}
\end{aligned}$$

Next, in view of (261) as $t \rightarrow \infty$

$$\begin{aligned}
& \frac{m_t a_k(n)}{\gamma_t} = \frac{m_t \sum_{j=0}^k \rho_n^{k-j} c_j}{\gamma_t} \\
& \sim \kappa_t^{m-1} \sum_{j=0}^k \rho_n^{k-j} c_j = \kappa_t^{m-1} \sum_{j=0}^k e^{-\frac{k-j}{\kappa_n}} c_j + o(1), \tag{262}
\end{aligned}$$

uniformly in $\underline{q}_t \leq k \leq \bar{q}_t$. Fix $\epsilon > 0$ arbitrarily small and let $|c_j/j^{-m} - 1| < \epsilon$ for all $j \geq N_\epsilon$. Note also that $e^x - 1 \leq x e^x$. Therefore, for $\underline{q}_t \leq k \leq \bar{q}_t$, as $t(\leq n) \rightarrow \infty$, the difference

$$\begin{aligned}
& |R_{nk}| := \left| \sum_{j=0}^k e^{-\frac{k-j}{\kappa_n}} c_j - \sum_{j=0}^k e^{-\frac{k}{\kappa_n}} c_j \right| \leq e^{-\frac{k}{\kappa_n}} \left| \sum_{j=0}^k e^{\frac{j}{\kappa_n}} c_j - \sum_{j=0}^k c_j \right| \\
& \leq C e^{-\frac{k}{\kappa_n}} \sum_{j=0}^k \frac{j}{\kappa_n} |c_j| \leq C e^{-\frac{k}{\kappa_n}} \sum_{j=1}^k \frac{j}{\kappa_n} j^{-m} + C e^{-\frac{k}{\kappa_n}} \sum_{j=1}^k \left| \frac{|c_j|}{j^{-m}} - 1 \right| \frac{j}{\kappa_n} j^{-m} \\
& \leq C \kappa_n^{-1} e^{-\frac{k}{\kappa_n}} \left[\sum_{j=1}^k j^{1-m} + \epsilon \sum_{j=N_\epsilon}^k j^{1-m} + N_\epsilon \sup_{0 \leq j \leq N_\epsilon} |j c_j| \right] \\
& \leq C_0 \kappa_n^{-1} e^{-\frac{k}{\kappa_n}} \sum_{j=1}^k j^{1-m} \\
& = C_0 \kappa_n^{-1} e^{-\frac{k}{\kappa_n}} k^{2-m} \int_0^1 x^{1-m} dx [1 + o(1)] + o(1) \\
& \leq C_1 \kappa_n^{-1} e^{-\frac{k}{\kappa_n}} k^{2-m} [1 + o(1)], \tag{263}
\end{aligned}$$

where $C_0, C_1 \in [1, \infty)$. Set $C_\infty := \sup_{n \geq 1, 1 \leq t \leq n} \frac{\kappa_t}{\kappa_n} \geq 1$ and note that by assumption $C_\infty < \infty$. Hence, $1 \leq C_\infty < \infty$. Choose $\delta := 1 \wedge \frac{1}{8(m-1)C_1 C_\infty}$. In view of (261)-(263), for $\underline{q}_t \leq k \leq \bar{q}_t$ and $t(\leq n) \rightarrow \infty$

$$\left| \frac{m_t a_k(n)}{\gamma_t} \right| \geq \kappa_t^{m-1} e^{-\frac{k}{\kappa_n}} \left| \sum_{j=0}^k c_j \right| - \kappa_t^{m-1} |R_{nk}| + o(1)$$

$$\begin{aligned}
&= \kappa_t^{m-1} e^{-\frac{k}{\kappa_n}} k^{1-m} \int_1^\infty x^{-m} dx [1 + o(1)] - \kappa_t^{m-1} |R_{nk}| + o(1) \\
&\geq \kappa_t^{m-1} e^{-\frac{k}{\kappa_n}} k^{1-m} \int_1^\infty x^{-m} dx [1 + o(1)] - C_1 \kappa_t^{m-1} \kappa_n^{-1} e^{-\frac{k}{\kappa_n}} k^{2-m} [1 + o(1)] \\
&= \kappa_t^{m-1} e^{-\frac{k}{\kappa_n}} k^{1-m} \left[\frac{1}{m-1} [1 + o(1)] - C_1 k \kappa_n^{-1} [1 + o(1)] \right] + o(1) \\
&\geq \kappa_t^{m-1} e^{-\frac{k}{\kappa_n}} k^{1-m} \left[\frac{1}{2(m-1)} - 2C_1 [\delta \kappa_t] \kappa_n^{-1} \right] + o(1) \\
&\geq \kappa_t^{m-1} e^{-\frac{k}{\kappa_n}} k^{1-m} \left[\frac{1}{2(m-1)} - 2C_1 C_\infty \delta \right] + o(1) \\
&\geq \left(\frac{\kappa_t}{k} \right)^{m-1} e^{-\frac{k}{\kappa_n}} \frac{1}{4(m-1)} + o(1) \\
&\geq \left(\frac{\kappa_t}{\delta \kappa_t} \right)^{m-1} e^{-C_\infty} \frac{1}{4(m-1)} + o(1) \geq \delta^{1-m} e^{-C_\infty} \frac{1}{8(m-1)} > 0.
\end{aligned}$$

Similarly, using (261)-(263) for $\underline{q}_t \leq k \leq \bar{q}_t$ as $t \rightarrow \infty$

$$\begin{aligned}
&\left| \frac{m_t a_k(n)}{\gamma_t} \right| = \left| \kappa_t^{m-1} \sum_{j=0}^k e^{-\frac{k-j}{\kappa_n}} c_j \right| + o(1) \\
&\leq C \left[\kappa_t^{m-1} e^{-\frac{k}{\kappa_n}} \left| \sum_{j=0}^k c_j \right| + \kappa_t^{m-1} |R_{nk}| \right] + o(1) \\
&= C \left[\kappa_t^{m-1} e^{-\frac{k}{\kappa_n}} k^{1-m} \int_1^\infty x^{-m} dx (1 + o(1)) + \kappa_t^{m-1} |R_{nk}| \right] + o(1) \\
&\leq C \kappa_t^{m-1} e^{-\frac{k}{\kappa_n}} k^{1-m} \left[\frac{2}{m-1} + C_1 k \kappa_n^{-1} \right] + o(1) \\
&\leq C \left(\frac{\kappa_t}{\delta \kappa_t / 2} \right)^{m-1} \left[\frac{2}{m-1} + C_1 C_\infty \delta \right] + o(1) \\
&\leq 2C \left(\frac{2}{\delta} \right)^{m-1} \left[\frac{2}{m-1} + C_1 C_\infty \delta \right] < \infty,
\end{aligned}$$

as required.

(ii) Next, we show that (251) holds under **BFR1**, **BFR2** and **MI**. We start with the **BFR1** case.

BFR1 As $t \rightarrow \infty$

$$\begin{aligned}
&\gamma_t^{-\zeta} \sup_{0 \leq k \leq \bar{q}_t} |a_k(n)| = \gamma_t^{-\zeta} \sup_{0 \leq k \leq \bar{q}_t} |a_k| \\
&= \gamma_t^{-\zeta} \sup_{0 \leq k \leq \bar{q}_t} \left| \sum_{j=0}^k \phi_j c_{k-j} \right| \leq \gamma_t^{-\zeta} \sum_{j=0}^\infty |c_j| \rightarrow 0,
\end{aligned}$$

for all $\zeta \in (0, 1)$. Note that we have used the fact that $\phi_j = 1$.

BFR2 By **Lemma 8**, as $k \rightarrow \infty$ we have $\sum_{j=0}^k \phi_j c_{k-j} \sim k^{-1/\alpha} \sum_{j=0}^{\infty} c_j$. Hence,

$$\begin{aligned} \gamma_t^{-\zeta} \sup_{0 \leq k \leq \bar{q}_t} |a_k(n)| &= \gamma_t^{-\zeta} \sup_{0 \leq k \leq \bar{q}_t} |a_k| \\ &\leq \gamma_t^{-\zeta} \sup_{k \geq 0} \left| \sum_{j=0}^k \phi_j c_{k-j} \right| \leq C \gamma_t^{-\zeta} \rightarrow 0, \end{aligned}$$

MI Let n large enough, $n \geq n_0$ say, such that $|\rho_n| < 1$. First, suppose that either **Assumptions MI-SM** or **MI-NM** hold. Noting that $\phi_j = \rho_n^j$, as $t \rightarrow \infty$ we have

$$\gamma_t^{-\zeta} \sup_{0 \leq k \leq \bar{q}_t, n \geq n_0} |a_k(n)| = \gamma_t^{-\zeta} \sup_{0 \leq k \leq \bar{q}_t, n \geq n_0} \left| \sum_{j=0}^k \phi_j c_{k-j} \right| \leq \gamma_t^{-\zeta} \sup_{j \geq 0, n \geq n_0} |\rho_n^j| \sum_{j=0}^{\infty} |c_j| \leq C \gamma_t^{-\zeta} \rightarrow 0,$$

for all $\zeta \in (0, 1)$.

Finally, suppose that **Assumption MI-LM** holds. Recall that in this case $|c_j| \asymp j^{-m}$ with $m \in (1/\alpha, 1)$. Moreover, note that as $k \rightarrow \infty$, $\frac{1}{k} \sum_{j=1}^k \left(\frac{j}{k}\right)^{-m} \rightarrow \frac{1}{1-m}$.²⁸ Therefore, the aforementioned term is bounded. Write

$$\begin{aligned} \gamma_t^{-\zeta} \sup_{0 \leq k \leq \bar{q}_t, n \geq n_0} |a_k(n)| &= \gamma_t^{-\zeta} \sup_{0 \leq k \leq \bar{q}_t, n \geq n_0} \left| \sum_{j=0}^k \rho_n^{k-j} c_j \right| \leq C \gamma_t^{-\zeta} \left(|c_0| + \sup_{1 \leq k \leq \bar{q}_t} \sum_{j=1}^k |c_j| \right) \\ &\leq C \gamma_t^{-\zeta} \left(1 + \sup_{1 \leq k \leq \bar{q}_t} \sum_{j=1}^k j^{-m} \right) \leq C \gamma_t^{-\zeta} \sup_{1 \leq k \leq \bar{q}_t} k^{1-m} \frac{1}{k} \sum_{j=1}^k \left(\frac{j}{k}\right)^{-m} + o(1) \\ &\leq C \gamma_t^{-\zeta} \bar{q}_t^{1-m} + o(1) = \frac{C \kappa_t^{1-m}}{\ell_1(\kappa_t)^{\zeta/\alpha} \kappa_t^{\zeta b/\alpha}} + o(1) \\ &= \frac{C \kappa_t^{1-m}}{\ell_1(\kappa_t)^{\zeta/\alpha} \kappa_t^{\zeta \frac{(1+\alpha-\alpha m)}{\alpha}}} + o(1) = \frac{C \kappa_t^{1-m}}{\ell_1(\kappa_t)^{\zeta/\alpha} \kappa_t^{\zeta(\frac{1}{\alpha}+1-m)}} \rightarrow 0, \end{aligned}$$

for all $\zeta \in \left(\frac{1-m}{\frac{1}{\alpha}+1-m}, 1\right)$. Notice that for such ζ the exponent $1-m-\zeta\left(\frac{1}{\alpha}+1-m\right)$ is negative; since $\ell_1(\kappa_t)^{-\zeta/\alpha}$ is slowly varying, the negative power of κ_t dominates and the preceding term converges to zero. ■

Proof of Lemma 20.

(i) Without loss of generality assume that $G(\cdot)$ is increasing. In addition, set $C_0 = 1$. Let n_0 and t_0 as in **Lemma 19**. Using the aforementioned lemma, for all $n \geq n_0$ and $t_0 \leq t \leq n$

$$\sum_{k=\underline{q}_t+\theta}^{\bar{q}_t} G\left(\frac{a_k(n)}{\gamma_t}\right) = \sum_{k=\underline{q}_t+\theta}^{\bar{q}_t} G\left(\frac{\delta_t^{-1} \eta_k a_k(n)}{\delta_t^{-1} \eta_k \gamma_t}\right) \geq \sum_{k=\underline{q}_t+\theta}^{\bar{q}_t} G\left(\frac{\mathcal{D}_0}{\delta_t^{-1} \eta_k \gamma_t}\right).$$

²⁸Notice that

$$\frac{1}{k} \sum_{j=1}^k \left(\frac{j}{k}\right)^{-m} = \int_{1/k}^1 x^{-m} dx + o(1) = \frac{1}{1-m} + o(1).$$

Next, we show that the last term above is bounded away from zero.

BFR. Suppose that **Assumption BFR** holds (i.e. $\delta_t = 1$, $\eta_t = t^{1/\alpha}$, $\bar{q}_t = t - 1$, $q_t = \lfloor \gamma_t \rfloor$). As $t \rightarrow \infty$

$$\begin{aligned}
& \sum_{k=\underline{q}_t+\theta}^{\bar{q}_t} G\left(\frac{1}{\delta_t^{-1}\eta_k\gamma_t}\mathcal{D}_0\right) = \sum_{k=\gamma_t+\theta}^{t-1} G\left(\mathcal{D}_0\frac{k^{-1/\alpha}}{\gamma_t}\right) \\
& = \sum_{k=\gamma_t+\theta}^{t-1} \left|\mathcal{D}_0\frac{k^{-1/\alpha}}{\gamma_t}\right|^\alpha H\left(\left|\mathcal{D}_0\frac{k^{-1/\alpha}}{\gamma_t}\right|^{-1}\right) \geq \mathcal{D}_0^\alpha \gamma_t^{-\alpha} \sum_{k=2\gamma_t}^{t-1} k^{-1} H\left(\left|\mathcal{D}_0\frac{k^{-1/\alpha}}{\gamma_t}\right|^{-1}\right) \\
& \stackrel{(1)}{=} \mathcal{D}_0^\alpha \gamma_t^{-\alpha} \alpha \int_{2^{1/\alpha}\mathcal{D}_0^{-1}\gamma_t^{1+1/\alpha}}^{\mathcal{D}_0^{-1}\gamma_t(t-1)^{1/\alpha}} \frac{H(x)}{x} dx [1 + o(1)] \\
& = \alpha \mathcal{D}_0^\alpha \gamma_t^{-\alpha} \left[L\left(\mathcal{D}_0^{-1}\gamma_t(t-1)^{1/\alpha}\right) - L\left(2^{1/\alpha}\mathcal{D}_0^{-1}\gamma_t^{1+1/\alpha}\right) \right] [1 + o(1)] = \alpha \mathcal{D}_0^\alpha + o(1),
\end{aligned}$$

where $\stackrel{(1)}{=}$ follows from **Lemma 3**, and the last approximation is due to the facts that $L(\cdot)$ is SV, $L(\gamma_t^{1+1/\alpha})$ is SV in γ_t (see BGT 1987; Proposition 1.3.6, Proposition 1.5.7), and $\gamma_t^\alpha \sim L(\gamma_t t^{1/\alpha})$. Since $(t-1)^{1/\alpha} \sim t^{1/\alpha}$ and L is slowly varying, the fixed multiplicative constants and the replacement of $t^{1/\alpha}$ by $(t-1)^{1/\alpha}$ do not affect the preceding asymptotic equivalence. The result follows easily from the observation that $\alpha \mathcal{D}_0^\alpha$ is bounded away from zero.

MI. Suppose that **Assumption MI** holds (i.e. $\delta_t = \kappa_t^{1-m}$, $\eta_t = 1$, $\bar{q}_t = \lfloor \delta \kappa_t \rfloor$, $q_t = \lfloor \delta \kappa_t / 2 \rfloor$ for some $0 < \delta \leq 1$). Recall that $q_t = \bar{q}_t - \underline{q}_t$. In view of (248), as $t \rightarrow \infty$

$$\begin{aligned}
& \sum_{k=\underline{q}_t+\theta}^{\bar{q}_t} G\left(\frac{1}{\delta_t^{-1}\eta_k\gamma_t}\mathcal{D}_0\right) = \sum_{k=\underline{q}_t+\theta}^{\bar{q}_t} G\left(\mathcal{D}_0\frac{\kappa_t^{1-m}}{\gamma_t}\right) \\
& = [\bar{q}_t - (\underline{q}_t + \theta) + 1] G\left(\frac{\mathcal{D}_0}{m_t}\right) \sim q_t G\left(\frac{\mathcal{D}_0}{m_t}\right) \\
& = q_t \left(\frac{m_t}{\mathcal{D}_0}\right)^{-\alpha} H\left(\frac{m_t}{\mathcal{D}_0}\right) \sim \mathcal{D}_0^\alpha \frac{q_t}{\ell_1(\kappa_t) \kappa_t} H\left(\ell_1(\kappa_t)^{1/\alpha} \kappa_t^{1/\alpha}\right) \\
& \sim \mathcal{D}_0^\alpha \frac{q_t}{\kappa_t} = \delta \mathcal{D}_0^\alpha / 2 + o(1),
\end{aligned}$$

where the limit term is bounded away from zero.

(ii) We start with the **BFR** case. As $n \rightarrow \infty$

$$\exp(-r_1 \gamma_n^{r_2}) \gamma_n^2 \delta_n^{-2} \eta_{\underline{q}_n+\theta}^2 = \exp(-r_1 \gamma_n^{r_2}) \gamma_n^2 (\gamma_n + \theta)^{2/\alpha} \rightarrow 0.$$

Next, suppose that **Assumption MI** holds. Once more, using (248) as $n \rightarrow \infty$

$$\begin{aligned}
& \exp(-r_1 \gamma_n^{r_2}) \gamma_n^2 \delta_n^{-2} \eta_{\underline{q}_n+\theta}^2 = \exp(-r_1 \gamma_n^{r_2}) \gamma_n^2 \left(\kappa_n^{1-m}\right)^{-2} \\
& = \exp(-r_1 \gamma_n^{r_2}) m_n^2 \leq C \exp(-r_1 \gamma_n^{r_2}) \left[\ell_1(\kappa_n)^{1/\alpha} \kappa_n^{1/\alpha}\right]^2 \\
& = C \exp\left(-r_1 \left(\ell_1(\kappa_n)^{1/\alpha} \kappa_n^{b/\alpha}\right)^{r_2} + 2 \ln \left[\ell_1(\kappa_n)^{1/\alpha} \kappa_n^{1/\alpha}\right]\right) \rightarrow 0,
\end{aligned}$$

as required. ■

17. Proofs for Section 4

17.1. Supporting Results for Section 4

First, we provide some preliminary technical results. Their proofs appear at the end of this Section.

Lemma 22. Suppose that the conditions of **Lemma SS α M** hold. Then, as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{t=1}^n f\left(\frac{x_t}{\gamma_n^*}\right) \xrightarrow{d} \int_0^1 f(\Lambda_{\alpha,d}(r)) dr.$$

Proposition 4. Let $\tilde{X}_{t,n}(n) := \tilde{X}_{t,n} := \lambda_n^{-1} x_t := \lambda_n^{-1} x_t(n)$, where $\lambda_n > 0$ is some deterministic sequence, possibly $\lambda_n = 1$ for all $n \geq 1$. Suppose that the following hold:

- (i) $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally integrable such that $|f(x)| = O(|x|^{\lambda'})$, as $|x| \rightarrow \infty$, for some $0 < \lambda' < \alpha$;
- (ii) $\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} |\tilde{X}_{t,n}|^{\alpha'} < \infty$, for all $0 < \alpha' < \alpha$;
- (iii) either
 - (a) $\tilde{X}_{t,t}$ has density $\tilde{D}_{n,t}(x)$ such that $\sup_{n \geq n_0, 1 \leq t \leq n} \sup_x \tilde{D}_{n,t}(x) < \infty$, for some $n_0 \geq 1$; or
 - (b) $\tilde{X}_{t,t}$ has density $\tilde{D}_{n,t}(x)$ such that $\sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x \tilde{D}_{n,t}(x) < \infty$, and $\frac{1}{n} \sum_{t=1}^{t'} f(\tilde{X}_t) = o_p(1)$, for all fixed $t' \geq 1$.
- (iv) $\sup_{n \geq 1} \frac{1}{n} \sum_{t=1}^n \frac{\lambda_n}{\lambda_t} < \infty$.

Then for each $\varepsilon > 0$, there is a bounded Lipschitz continuous f_ε such that as $n \rightarrow \infty$ first and then $\varepsilon \rightarrow 0$

$$\frac{1}{n} \sum_{t=1}^n f(\tilde{X}_{t,n}) = \frac{1}{n} \sum_{t=1}^n f_\varepsilon(\tilde{X}_{t,n}) + o_p(1). \quad (264)$$

Suppose that in addition to (i)-(iv), the following holds:

- (v) $\tilde{X}_{t,n} = X_{t,n} + R_{t,n}$, where $R_{t,n}$ such that for each $\epsilon > 0$, $\lim_{n \rightarrow \infty} \sup_{1 \leq t \leq n} P(|R_{t,n}| > \epsilon) = 0$.

Then for each $\varepsilon > 0$, there is a bounded Lipschitz continuous f_ε such that as $n \rightarrow \infty$ first and then $\varepsilon \rightarrow 0$

$$\frac{1}{n} \sum_{t=1}^n f(\tilde{X}_{t,n}) = \frac{1}{n} \sum_{t=1}^n f_\varepsilon(X_{t,n}) + o_p(1). \quad (265)$$

Let $(x)_+ = xI_{\{x>0\}}$. Moreover, for $0 \leq r \leq 1$ and $0 < \mathcal{H} < 1$ let

$$\Lambda(r) := \Lambda_{\alpha,d}(r) = \begin{cases} \sigma_0 \int_{\mathbb{R}} \left[(r-s)_+^{\mathcal{H}-\frac{1}{\alpha}} - (-s)_+^{\mathcal{H}-\frac{1}{\alpha}} \right] dZ_\alpha(s), & \mathcal{H} \neq \frac{1}{\alpha} \\ \sigma_0 Z_\alpha(r), & \mathcal{H} = \frac{1}{\alpha} \end{cases},$$

where $Z_\alpha(r)$ is an α -stable Lévy motion with scale parameter 1 and skewness parameter $\beta_* = \beta I_{\{\alpha \neq 1\}}$, $|\beta| \leq 1$. Recall the connection between the the Hurst index $\mathcal{H}$ and the memory parameter is $\mathcal{H} = d - 1 + 1/\alpha$.

Proposition 5. **(a)** For $0 < r \leq 1$, $\Lambda(r)$ has an L_1 integrable characteristic function and possesses a density $p_{\Lambda(r)}$ such that

$$\int_0^1 \sup_x |p_{\Lambda(r)}(x)| dr < \infty.$$

(b) Suppose that $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is locally integrable and satisfies the growth condition $|f(x)| = O(|x|^{\alpha'})$, $0 < \alpha' < \alpha$, $|x| \rightarrow \infty$. Then

$$\int_0^1 |f(\Lambda(r))| dr < \infty \text{ a.s.}$$

17.2. Proof of Lemma SS_αM and Theorem 6

Proof of Lemma SS_αM . **(a)** First, we present some preliminary results and introduce some notation. Recall that for $t = 1, \dots, n$ we can write

$$x_t = \sum_{k=0}^{\infty} a_{k,t} \xi_{t-k},$$

where

$$a_{k,t} := \sum_{j=0}^{(t-1) \wedge k} \varphi_j c_{k-j} = \begin{cases} \sum_{j=0}^k \varphi_j c_{k-j} =: a_k, & k \leq t-1 \\ \sum_{j=0}^{t-1} \varphi_j c_{k-j} =: a_{k,t}^-, & k > t-1 \end{cases},$$

where under **SNP** and **SNP*** $\varphi_j = 1$ for all j . Moreover, decompose

$$x_t = x_t^+ + x_t^- := \sum_{k=0}^{t-1} a_k \xi_{t-k} + \sum_{k=t}^{\infty} a_{k,t}^- \xi_{t-k}.$$

We will use the convention $x_0^+, x_0^- = 0$.

Let $m_n^* := \gamma_n^* n^{-(1-m)}$ and note that

$$m_n^* \sim n^{1/\alpha} \ell_1(n)^{1/\alpha}, \quad H(m_n^*) \sim \ell_1(n). \quad (266)$$

Let

$$\begin{aligned} \bar{\xi}_{n,k} &:= \xi_k 1\{|\xi_k| \geq m_n^*\} - \mathbf{E}[\xi_k 1\{|\xi_k| \geq m_n^*\}], \\ \underline{\xi}_{n,k} &:= \xi_k 1\{|\xi_k| < m_n^*\} - \mathbf{E}[\xi_k 1\{|\xi_k| < m_n^*\}]. \end{aligned}$$

Set

$$0 < \varepsilon < \left(\frac{1}{m-1} - \alpha \right) \wedge \left(\alpha - \frac{1}{m} \right).$$

For $\alpha < 2$, we will impose the additional restriction, $\varepsilon < 2 - \alpha$, and if $\alpha = 2$ we will use the convention $\alpha + \varepsilon = 2$. These restrictions ensure that for all $\delta > 0$ the following integrals are finite:

$$\begin{aligned} & \int_0^1 \left(\int_x^\infty y^{-m} dy \right)^{\alpha-\varepsilon} dx, \int_0^\delta \left(\int_x^\infty y^{-m} dy \right)^{\alpha\pm\varepsilon} dx, \\ & \int_0^1 \left(\int_x^{\delta+x} y^{-m} dy \right)^{\alpha\pm\varepsilon} dx, \int_0^\infty \left[\int_x^{1+x} [y^{-m} - (\delta+y)^{-m}] dy \right]^{\alpha\pm\varepsilon} dx. \end{aligned}$$

Finally, recall that

$$b := 1 + \alpha(1-m) \text{ and } b_{\varepsilon\pm} = 1 + (\alpha \pm \varepsilon)(1-m).$$

We provide a proof only under **Assumption SNP*-NM** ($m \in (1, 1+1/\alpha)$; $\varphi_j = 1$).

(1.a) x_t^+ : Consider

$$\begin{aligned}
A_{n,t} &:= \mathbf{E} \left| \gamma_n^{*-1} x_t^+ \right|^{\alpha-\varepsilon} = \mathbf{E} \left| \gamma_n^{*-1} \sum_{k=0}^{t-1} a_k \xi_{t-k} \right|^{\alpha-\varepsilon} \\
&\leq_{(1)} C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^n |a_k|^{\alpha-\varepsilon} \mathbf{E} \left| \bar{\xi}_{n,k} \right|^{\alpha-\varepsilon} + C \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=0}^n |a_k|^{\alpha+\varepsilon} \mathbf{E} \left| \underline{\xi}_{n,k} \right|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\
&=: C \left(A_{1n} + A_{2n}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right), \tag{267}
\end{aligned}$$

where $\leq_{(1)}$ follows either from the Loève or the Bahr-Esseen inequalities. Moreover, by Astrauskas (1983; Lemma 1)

$$\mathbf{E} \left| \bar{\xi}_{n,k} \right|^{\alpha-\varepsilon} \leq 4 \mathbf{E} \left[|\xi_k|^{\alpha-\varepsilon} 1 \{ |\xi_k| \geq m_n^* \} \right] \asymp (m_n^*)^{-\varepsilon} h(m_n^*), \tag{268}$$

and

$$\mathbf{E} \left| \underline{\xi}_{n,k} \right|^{\alpha+\varepsilon} \leq 8 \mathbf{E} \left[|\xi_k|^{\alpha+\varepsilon} 1 \{ |\xi_k| < m_n^* \} \right] \asymp (m_n^*)^{\varepsilon} h(m_n^*). \tag{269}$$

In view of the above, as $n \rightarrow \infty$,

$$\begin{aligned}
A_{1n} &\leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^n |a_k|^{\alpha-\varepsilon} (m_n^*)^{-\varepsilon} h(m_n^*) \\
&\leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^n |a_k|^{\alpha-\varepsilon} (m_n^*)^{-\varepsilon} H(m_n^*) \sim_{(1)} C \frac{[n \ell_1(n)]^{-\frac{\varepsilon}{\alpha}} \ell_1(n)}{[n^b \ell_1(n)]^{\frac{\alpha-\varepsilon}{\alpha}}} \sum_{k=0}^n |a_k|^{\alpha-\varepsilon} \\
&= C \frac{n^{-\frac{\varepsilon}{\alpha}} \ell_1(n)^{1-\frac{\varepsilon}{\alpha}}}{n^{b \frac{\alpha-\varepsilon}{\alpha}} \ell_1(n)^{\frac{\alpha-\varepsilon}{\alpha}}} \sum_{k=0}^n |a_k|^{\alpha-\varepsilon} = C \frac{n^{-\frac{\varepsilon}{\alpha}}}{n^{b \frac{\alpha-\varepsilon}{\alpha}}} \sum_{k=0}^n |a_k|^{\alpha-\varepsilon} \\
&\stackrel{(2)}{=} C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^n |a_k|^{\alpha-\varepsilon} = C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^n \left| \sum_{j=1}^k c_j \right|^{\alpha-\varepsilon} \\
&= C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^n \left| \sum_{j=k+1}^{\infty} c_j \right|^{\alpha-\varepsilon} \stackrel{(3)}{=} C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^n \left| \sum_{j=k+1}^{\infty} j^{-m} \right|^{\alpha-\varepsilon} [1 + o(1)] \\
&= C \frac{1}{n} \sum_{k=0}^n \left(\frac{1}{n} \sum_{j=k+1}^{\infty} \left(\frac{j}{n} \right)^{-m} \right)^{\alpha-\varepsilon} [1 + o(1)] \\
&\stackrel{(4)}{=} C \int_0^1 \left(\int_x^{\infty} y^{-m} dy \right)^{\alpha-\varepsilon} dx [1 + o(1)] < C < \infty, \tag{270}
\end{aligned}$$

where $\sim_{(1)}$ follows from (266), and $\stackrel{(2)}{=}$ follows from the fact that

$$\begin{aligned}
b \frac{\alpha-\varepsilon}{\alpha} &= (1 + \alpha - \alpha m) \left(1 - \frac{\varepsilon}{\alpha} \right) = 1 + \alpha - \alpha m - (1 + \alpha - \alpha m) \frac{\varepsilon}{\alpha} \\
&= 1 + \alpha - \alpha m - \left(\frac{\varepsilon}{\alpha} + \varepsilon - \varepsilon m \right) = 1 + \alpha - \alpha m - \frac{\varepsilon}{\alpha} - \varepsilon + \varepsilon m \\
&= 1 + (\alpha - \varepsilon) - (\alpha - \varepsilon) m - \frac{\varepsilon}{\alpha} = b_{\varepsilon^-} - \frac{\varepsilon}{\alpha}.
\end{aligned}$$

Moreover, $=_{(3)}$ and $=_{(4)}$ follow, respectively, from the fact that $c_j \asymp j^{-m}$ and Euler summation. Similarly, we get

$$A_{2n} \leq C \frac{1}{n^{b_\varepsilon+}} \sum_{k=0}^n |a_k|^{\alpha+\varepsilon} = C \int_0^1 \left(\int_x^\infty y^{-m} dy \right)^{\alpha+\varepsilon} dx [1 + o(1)] < C < \infty. \quad (271)$$

Therefore, using (267)-(271) we get

$$\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{*-1} x_t^+ \right|^{\alpha-\varepsilon} = \sup_{n \geq 1, 1 \leq t \leq n} A_{n,t} < \infty,$$

as required.

(1.b) $x_t^+ - x_s^+$: Let $0 < \delta \leq 1$, $t, s = 1, \dots, n$, $0 \leq t - s \leq n\delta$ and consider

$$x_t^+ - x_s^+ = \sum_{l=s+1}^t a_{t-l} \xi_l - \sum_{l=1}^s (a_{t-l} - a_{s-l}) \xi_l,$$

where we are using the convention $\sum_{j=a}^b (\cdot) = 0$ for $a > b$.

Using analogous arguments to those used for (270) and (271) as $n \rightarrow \infty$ first and then $\delta \rightarrow 0$ we get

$$\begin{aligned} A_{n,t,s} &:= \mathbf{E} \left| \gamma_n^{*-1} \sum_{l=s+1}^t a_{t-l} \xi_l \right|^{\alpha-\varepsilon} = \mathbf{E} \left| \gamma_n^{*-1} \sum_{k=0}^{t-(s+1)} a_k \xi_{t-k} \right|^{\alpha-\varepsilon} \\ &\leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^{t-(s+1)} |a_k|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} + C \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=0}^{t-(s+1)} |a_k|^{\alpha+\varepsilon} \mathbf{E} |\xi_{n,k}|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\ &\leq C \left\{ \int_0^\delta \left(\int_x^\infty y^{-m} dy \right)^{\alpha-\varepsilon} dx + \left[\int_0^\delta \left(\int_x^\infty y^{-m} dy \right)^{\alpha+\varepsilon} dx \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right\} [1 + o(1)] = o(1). \end{aligned}$$

Next, consider

$$\begin{aligned} B_{n,t,s} &:= \mathbf{E} \left| \gamma_n^{*-1} \sum_{l=1}^s (a_{t-l} - a_{s-l}) \xi_l \right|^{\alpha-\varepsilon} = \mathbf{E} \left| \gamma_n^{*-1} \sum_{k=0}^{s-1} (a_{t-s+k} - a_k) \xi_{s-k} \right|^{\alpha-\varepsilon} \\ &= \mathbf{E} \left| \gamma_n^{*-1} \sum_{k=0}^{s-1} (a_{t-s+k} - a_k) \xi_k \right|^{\alpha-\varepsilon} \\ &\leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^n |(a_{t-s+k} - a_k)|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} \\ &\quad + C \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=0}^n |(a_{t-s+k} - a_k)|^{\alpha+\varepsilon} \mathbf{E} |\xi_{n,k}|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\ &=: C \left(B_{1n,t,s} + B_{2n,t,s}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right), \end{aligned}$$

Once more, using (268) and (269) as $n \rightarrow \infty$ first, and then $\delta \rightarrow 0$, we have

$$B_{1n,t,s} \leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^n |a_{t-s+k} - a_k|^{\alpha-\varepsilon} (m_n^*)^{-\varepsilon} h(m_n^*)$$

$$\begin{aligned}
&\leq C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^n |a_{t-s+k} - a_k|^{\alpha-\varepsilon} = C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^n \left| \sum_{j=0}^{t-s+k} c_j - \sum_{j=0}^k c_j \right|^{\alpha-\varepsilon} \\
&= C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^n \left| \sum_{j=k+1}^{t-s+k} c_j \right|^{\alpha-\varepsilon} \leq C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^n \left(\sum_{j=k+1}^{t-s+k} j^{-m} \right)^{\alpha-\varepsilon} [1 + o(1)] \\
&\leq C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^n \left(\sum_{j=k+1}^{\lfloor n\delta \rfloor + k} j^{-m} \right)^{\alpha-\varepsilon} [1 + o(1)] = C \frac{1}{n} \sum_{k=0}^n \left[\frac{1}{n} \sum_{j=k+1}^{\lfloor n\delta \rfloor + k} \left(\frac{j}{n} \right)^{-m} \right]^{\alpha-\varepsilon} [1 + o(1)] \\
&= C \int_0^1 \left[\int_x^{\delta+x} y^{-m} dy \right]^{\alpha-\varepsilon} dx [1 + o(1)] = o(1),
\end{aligned}$$

where the last equality follows from dominated convergence. Similarly, as $n \rightarrow \infty$ first and then $\delta \rightarrow 0$

$$B_{2n,t,s} \leq C \int_0^1 \left[\int_x^{\delta+x} y^{-m} dy \right]^{\alpha+\varepsilon} dx = o(1).$$

In view of the above,

$$\begin{aligned}
&\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{0 \leq r, s \leq 1 \\ 0 \leq r-s \leq \delta}} \mathbf{E} \left| \gamma_n^{*-1} x_{\lfloor nr \rfloor}^+ - \gamma_n^{*-1} x_{\lfloor ns \rfloor}^+ \right|^{\alpha-\varepsilon} \\
&= \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{1 \leq t, s \leq n \\ 0 \leq t-s \leq \delta n}} \mathbf{E} \left| \gamma_n^{*-1} x_t^+ - \gamma_n^{*-1} x_s^+ \right|^{\alpha-\varepsilon} = 0.
\end{aligned}$$

(2a) x_t^- : Write

$$\begin{aligned}
x_t^- &= \sum_{k=t}^{\infty} a_{k,t}^- \xi_{t-k} = \sum_{l=-\infty}^0 a_{t-l,t}^- \xi_l. \\
&[l = t - k \Leftrightarrow k = t - l]
\end{aligned}$$

Using arguments analogous to those used below for the increment process, it can be shown that

$$\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left| \gamma_n^{*-1} x_t^- \right|^{\alpha-\varepsilon} < \infty.$$

(2b) $x_t^- - x_s^-$: Let $0 < \delta \leq 1$, $t, s = 1, \dots, n$, $0 \leq t-s \leq n\delta$. The increment

$$\begin{aligned}
x_t^- - x_s^- &= \sum_{l=-\infty}^0 a_{t-l,t}^- \xi_l - \sum_{l=-\infty}^0 a_{s-l,s}^- \xi_l = \sum_{l=-\infty}^0 (a_{t-l,t}^- - a_{s-l,s}^-) \xi_l \\
&[k = s - l - j \Rightarrow j = s - l - k] \\
&= \sum_{l=-\infty}^0 \left[\sum_{j=0}^{t-1} c_{t-l-j} - \sum_{j=0}^{s-1} c_{s-l-j} \right] \xi_l = \sum_{l=-\infty}^0 \left[\sum_{k=s-l-(t-1)}^{s-l} c_{t-s+k} - \sum_{k=s-l-(s-1)}^{s-l} c_k \right] \xi_l \\
[r &= -l]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=-\infty}^0 \left[\sum_{k=s-t-l+1}^{s-l} c_{t-s+k} - \sum_{k=-l+1}^{s-l} c_k \right] \xi_l = \sum_{r=0}^{\infty} \left[\sum_{k=s-t+r+1}^{s+r} c_{t-s+k} - \sum_{k=r+1}^{s+r} c_k \right] \xi_{-r} \\
&= \sum_{r=0}^{\infty} \left[\sum_{k=-(t-s)+r+1}^{s+r} c_{t-s+k} - \sum_{k=r+1}^{s+r} c_k \right] \xi_{-r} = \sum_{r=0}^{\infty} \left[\sum_{k=-(t-s)+r+1}^r c_{t-s+k} + \sum_{k=r+1}^{s+r} c_{t-s+k} - \sum_{k=r+1}^{s+r} c_k \right] \xi_{-r} \\
&= \sum_{r=0}^{\infty} \left[\sum_{k=-(t-s)+r+1}^r c_{t-s+k} + \sum_{k=r+1}^{s+r} c_{t-s+k} - \sum_{k=r+1}^{s+r} c_k \right] \xi_{-r} \\
&= \sum_{k=0}^{\infty} \left[\sum_{j=-(t-s)+k+1}^k c_{t-s+j} \right] \xi_{-k} + \sum_{k=0}^{\infty} \left[\sum_{j=k+1}^{s+k} c_{t-s+j} - \sum_{j=k+1}^{s+k} c_j \right] \xi_{-k}
\end{aligned}$$

First, consider

$$\begin{aligned}
C_{n,t,s} &:= \mathbf{E} \left| \gamma_n^{*-1} \sum_{k=0}^{\infty} \left(\sum_{j=-(t-s)+k+1}^k c_{t-s+j} \right) \xi_{-k} \right|^{\alpha-\varepsilon} \\
&\leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^{\infty} \left| \sum_{j=-(t-s)+k+1}^k c_{t-s+j} \right|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} \\
&+ C \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=0}^{\infty} \left| \sum_{j=-(t-s)+k+1}^k c_{t-s+j} \right|^{\alpha+\varepsilon} \mathbf{E} |\xi_{n,k}|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\
&=: C \left(C_{1n,t,s} + C_{2n,t,s}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right).
\end{aligned}$$

The first term above, as $n \rightarrow \infty$ first and then $\delta \rightarrow 0$

$$\begin{aligned}
C_{1n,t,s} &:= C \frac{1}{n^{b_{\varepsilon}^-}} \sum_{k=0}^{\infty} \left| \sum_{j=-(t-s)+k+1}^k c_{t-s+j} \right|^{\alpha-\varepsilon} \leq C \frac{1}{n^{b_{\varepsilon}^-}} \sum_{k=0}^{\infty} \left| \sum_{j=-(t-s)+k+1}^k c_{t-s+j} \right|^{\alpha-\varepsilon} \\
&= C \frac{1}{n^{b_{\varepsilon}^-}} \sum_{k=0}^{\infty} \left| \sum_{j=t-s+[-(t-s)+k+1]}^{t-s+k} c_j \right|^{\alpha-\varepsilon} = C \frac{1}{n^{b_{\varepsilon}^-}} \sum_{k=0}^{\infty} \left| \sum_{j=k+1}^{t-s+k} c_j \right|^{\alpha-\varepsilon} \\
&\leq C \frac{1}{n^{b_{\varepsilon}^-}} \sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{\lfloor n\delta \rfloor + k} j^{-m} \right)^{\alpha-\varepsilon} [1 + o(1)] = C \frac{1}{n} \sum_{k=0}^{\infty} \left(\frac{1}{n} \sum_{j=k+1}^{\lfloor n\delta \rfloor + k} \left(\frac{j}{n} \right)^{-m} \right)^{\alpha-\varepsilon} [1 + o(1)] \\
&= C \int_0^{\infty} \left(\int_x^{\delta+x} y^{-m} dy \right)^{\alpha-\varepsilon} dx = o(1).
\end{aligned}$$

Similarly,

$$C_{2n,t,s} \leq C \int_0^{\infty} \left(\int_x^{\delta+x} y^{-m} dy \right)^{\alpha+\varepsilon} dx = o(1).$$

Finally, consider

$$D_{n,t,s} := \mathbf{E} \left| \gamma_n^{*-1} \sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{s+k} c_{t-s+j} - \sum_{j=k+1}^{s+k} c_j \right) \xi_{-k} \right|^{\alpha-\varepsilon}$$

$$\begin{aligned}
&\leq C \gamma_n^{-(\alpha-\varepsilon)} \sum_{k=0}^{\infty} \left| \sum_{j=k+1}^{s+k} c_{t-s+j} - \sum_{j=k+1}^{s+k} c_j \right|^{\alpha-\varepsilon} \mathbf{E} |\bar{\xi}_{n,k}|^{\alpha-\varepsilon} \\
&+ C \left[\gamma_n^{-(\alpha+\varepsilon)} \sum_{k=0}^n \left| \sum_{j=k+1}^{s+k} c_{t-s+j} - \sum_{j=k+1}^{s+k} c_j \right|^{\alpha+\varepsilon} \mathbf{E} |\underline{\xi}_{n,k}|^{\alpha+\varepsilon} \right]^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \\
&\leq C \left(D_{1n,t,s} + D_{2n,t,s}^{\frac{\alpha-\varepsilon}{\alpha+\varepsilon}} \right).
\end{aligned}$$

As $n \rightarrow \infty$ first and then $\delta \rightarrow 0$, the first term

$$\begin{aligned}
D_{1n,t,s} &\leq C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^{\infty} \left| \sum_{j=k+1}^{s+k} c_{t-s+j} - \sum_{j=k+1}^{s+k} c_j \right|^{\alpha-\varepsilon} \\
&= C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^{\infty} \left| \sum_{j=k+1}^{s+k} (t-s+j)^{-m} - \sum_{j=k+1}^{s+k} j^{-m} \right|^{\alpha-\varepsilon} [1+o(1)] \\
&= C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^{\infty} \left| \sum_{j=k+1}^{s+k} j^{-m} - \sum_{j=k+1}^{s+k} (t-s+j)^{-m} \right|^{\alpha-\varepsilon} [1+o(1)] \\
&\leq C \frac{1}{n^{b_{\varepsilon^-}}} \sum_{k=0}^{\infty} \left[\sum_{j=k+1}^{s+k} j^{-m} - \sum_{j=k+1}^{s+k} ([n\delta] + j)^{-m} \right]^{\alpha-\varepsilon} [1+o(1)] \\
&\leq C \frac{1}{n} \sum_{k=0}^{\infty} \left[\frac{1}{n} \sum_{j=k+1}^{n+k} \left\{ \left(\frac{j}{n} \right)^{-m} - \left(\frac{[n\delta] + j}{n} \right)^{-m} \right\} \right]^{\alpha-\varepsilon} [1+o(1)] \\
&\leq C \int_0^{\infty} \left[\int_x^{1+x} \left[y^{-m} - (\delta + y)^{-m} \right] dy \right]^{\alpha-\varepsilon} dx [1+o(1)] = o(1).
\end{aligned}$$

Finally, using identical arguments as those used above we get

$$D_{2n,t,s} \leq C \int_0^{\infty} \left[\int_x^{1+x} \left[y^{-m} - (\delta + y)^{-m} \right] dy \right]^{\alpha+\varepsilon} dx = o(1).$$

Hence,

$$\begin{aligned}
&\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{0 \leq r, s \leq 1 \\ 0 \leq r-s \leq \delta}} \mathbf{E} \left| \gamma_n^{*-1} x_{[nr]}^- - \gamma_n^{*-1} x_{[ns]}^- \right|^{\alpha-\varepsilon} \\
&= \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{1 \leq t, s \leq n \\ 0 \leq t-s \leq \delta n}} \mathbf{E} \left| \gamma_n^{*-1} x_t^- - \gamma_n^{*-1} x_s^- \right|^{\alpha-\varepsilon} = 0.
\end{aligned}$$

The proof of part (a) is complete.

(b) From part (a) we have

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{0 \leq r, s \leq 1 \\ 0 \leq r-s \leq \delta}} \mathbf{E} \left| \gamma_n^{*-1} x_{[nr]} - \gamma_n^{*-1} x_{[ns]} \right|^{\alpha-\varepsilon} = 0 \quad (272)$$

Set $p = \alpha - \varepsilon$, $\Delta := \gamma_n^{*-1} x_{[nr]} - \gamma_n^{*-1} x_{[ns]}$ and $X := \gamma_n^{*-1} x_{[ns]}$.

Recall that f is Lipschitz and bounded i.e. $|f(X)| \leq M < \infty$. First, assume that $p \leq 1$. Then we have

$$|f(X + \Delta) - f(X)| \leq C_{Lip} |\Delta| \wedge 2M \leq C_{Lip}^p (2M)^{1-p} |\Delta|^p$$

To see this, first suppose that $C_{Lip} |\Delta| \leq 2M$. Then

$$C_{Lip} |\Delta| \wedge 2M = C_{Lip} |\Delta| \leq (C_{Lip} |\Delta|)^p (2M)^{1-p} = C_{Lip}^p (2M)^{1-p} |\Delta|^p.$$

Next, suppose that $C_{Lip} |\Delta| > 2M$. Then

$$C_{Lip} |\Delta| \wedge 2M = 2M = (2M)^p (2M)^{1-p} \leq (C_{Lip} |\Delta|)^p (2M)^{1-p} = C_{Lip}^p (2M)^{1-p} |\Delta|^p.$$

Hence,

$$\mathbf{E} |f(X + \Delta) - f(X)| \leq C_{Lip}^p (2M)^{1-p} \mathbf{E} |\Delta|^p \quad (273)$$

Next, suppose the $p > 1$. Then by Lyapunov's inequality

$$\mathbf{E} |f(X + \Delta) - f(X)| \leq C_{Lip} \mathbf{E} |\Delta| \leq C_{Lip} \{\mathbf{E} |\Delta|^p\}^{1/p}. \quad (274)$$

The result follows from (272)-(274). ■

Proof of Theorem 6. First, using analogous arguments to those used for **Lemma 21** -see also Jeganathan (2008; Section 3) it can be shown that there is some $t_0 \geq 1$ such that $(\gamma_t^*)^{-1} x_t$ has density $\tilde{D}_t(x)$ such that

$$\sup_{x \geq t_0} \sup_x \tilde{D}_t(x) < \infty. \quad (275)$$

If in addition **Assumption INN** holds with $\theta = 1$, then we can set $t_0 = 1$. By **Lemma SS $_{\alpha}$ M**,

$$\sup_{n \geq 1} \sup_{1 \leq t \leq n} \mathbf{E} \left| \frac{x_t}{\gamma_n^*} \right|^{\alpha'} < \infty, \quad (276)$$

for all $0 < \alpha' < \alpha$. Moreover, γ_n^* is regularly varying of index $0 < \mathcal{H} < 1$. Hence, standard arguments e.g. DK2021 (proof of Theorem 3.2), show that

$$\frac{1}{n} \sum_{t=1}^n \frac{\gamma_n^*}{\gamma_t^*} < C < \infty \quad (277)$$

We will only prove the result under condition (iv.a). Fix $\varepsilon > 0$. In view (275)-(277) and **Proposition 4**, there some bounded Lipschitz continuous f_ε such that

$$\int_{|x| \leq \varepsilon^{-1}} |f_\varepsilon(x) - f(x)| dx < \varepsilon,$$

$f_\varepsilon(x) = 0$ for $|x| > \varepsilon^{-1}$, and as $n \rightarrow \infty$ first and then $\varepsilon \rightarrow 0$

$$\frac{1}{n} \sum_{t=1}^n f\left(\frac{x_t}{\gamma_n^*}\right) \stackrel{(1)}{=} \frac{1}{n} \sum_{t=t_0}^n f\left(\frac{x_t}{\gamma_n^*}\right) + o_p(1) = \frac{1}{n} \sum_{t=1}^n f_\varepsilon\left(\frac{x_t}{\gamma_n^*}\right) + o_p(1), \quad (278)$$

where $\stackrel{(1)}{=}$ follows from condition (iv.a). Next, by **Lemma 22** we get

$$\frac{1}{n} \sum_{t=1}^n f_\varepsilon\left(\frac{x_t}{\gamma_n^*}\right) \xrightarrow{d} \int_0^1 f_\varepsilon(\Lambda(r)) dr. \quad (279)$$

Next, as $\varepsilon \rightarrow 0$ we get

$$\begin{aligned}
& \mathbf{E} \left| \int_0^1 f_\varepsilon(\Lambda(r)) dr - \int_0^1 f(\Lambda(r)) dr \right| \leq \int_0^1 \mathbf{E} |f_\varepsilon(\Lambda(r)) - f(\Lambda(r))| dr \\
& = \int_0^1 \int_{\mathbb{R}} |f_\varepsilon(x) - f(x)| p_{\Lambda(r)}(x) dx dr \\
& = \int_0^1 \int_{|x| > \varepsilon^{-1}} |f_\varepsilon(x) - f(x)| p_{\Lambda(r)}(x) dx dr + \int_0^1 \int_{|x| \leq \varepsilon^{-1}} |f_\varepsilon(x) - f(x)| p_{\Lambda(r)}(x) dx dr \\
& = \int_0^1 \int_{|x| > \varepsilon^{-1}} |f(x)| p_{\Lambda(r)}(x) dx dr + \int_0^1 \sup_x p_{\Lambda(r)}(x) dr \int_{|x| \leq \varepsilon^{-1}} |f_\varepsilon(x) - f(x)| dx \\
& \leq C \int_0^1 \int_{|x| > \varepsilon^{-1}} |x|^{\alpha'} p_{\Lambda(r)}(x) dx dr + \varepsilon \int_0^1 \sup_x p_{\Lambda(r)}(x) dr \\
& \leq_{(1)} C \int_0^1 r^{\alpha \mathcal{H}} dr \int_{|x| > \varepsilon^{-1}} |x|^{\alpha'} |x|^{-1-\alpha} dx + \varepsilon \int_0^1 \sup_x p_{\Lambda(r)}(x) dr \rightarrow 0. \tag{280}
\end{aligned}$$

Combining (278) - (280) we get

$$\frac{1}{n} \sum_{t=1}^n f\left(\frac{x_t}{\gamma_n^*}\right) \xrightarrow{d} \int_0^1 f(\Lambda(r)) dr,$$

as required.

Finally, suppose that there is $\tilde{f} : \mathbb{R} \rightarrow \bar{\mathbb{R}}$ such that $\tilde{f} = f$ a.e. and

$$\frac{1}{n} \sum_{t=1}^{t_0-1} \tilde{f}\left(\frac{x_t}{\gamma_n^*}\right) = o_p(1).$$

Using the limit result established in the first part of this proof, as $n \rightarrow \infty$, we get

$$\frac{1}{n} \sum_{t=1}^n \tilde{f}\left(\frac{x_t}{\gamma_n^*}\right) = \frac{1}{n} \sum_{t=t_0}^n \tilde{f}\left(\frac{x_t}{\gamma_n^*}\right) + o_p(1) \stackrel{a.s.}{=}_{(2)} \frac{1}{n} \sum_{t=t_0}^n f\left(\frac{x_t}{\gamma_n^*}\right) \xrightarrow{d} \int_0^1 f(\Lambda(r)) dr,$$

where $\stackrel{a.s.}{=}_{(2)}$ follows from (275). ■

17.3. Proofs of Supporting Results for Section 4

Proof of Lemma 22. By Astrauskas (1983) and Kasahara and Maejima (1988), we have

$$\frac{1}{\gamma_n^*} x_{[nr]} \xrightarrow{f.d.d.} \Lambda_{\alpha,d}(r).$$

In view of the above and **Lemma SS _{α} M**, the result follows directly from Theorem 1 of Gikhman and Skorokhod (1969, p. 485). ■

Proof of Proposition 4. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally integrable. It follows from Folland (1999) that for each $\varepsilon > 0$, there is a Lipschitz continuous f_ε such that

$$\int_{|x| \leq \varepsilon^{-1}} |f(x) - f_\varepsilon(x)| dx < \varepsilon, \text{ and } f_\varepsilon(x) = 0, |x| > \varepsilon^{-1}.$$

Write

$$f(x) = f(x)1_{\{|x| \leq \varepsilon^{-1}\}} + f(x)1_{\{|x| > \varepsilon^{-1}\}} =: f_{1,\varepsilon}(x) + f_{2,\varepsilon}(x).$$

Choose ε small enough such that $|f_{2,\varepsilon}(x)| \leq C|x|^{\lambda'}$. Choose α' such that $\lambda' < \alpha' < \alpha$. By condition (iii), $|\tilde{X}_t(n)|^{\lambda'}$ is uniformly integrable. Therefore, as $\varepsilon \rightarrow 0$

$$\frac{1}{n} \mathbf{E} \sum_{t=1}^n |f_{2,\varepsilon}(\tilde{X}_{t,n}(n))| \leq C \sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} |\tilde{X}_{t,n}(n)|^{\lambda'} 1_{\{| \tilde{X}_{t,n}(n) |^{\lambda'} > \varepsilon^{-\lambda'}\}} = o(1). \quad (281)$$

Next, suppose that condition (iii.b) holds. An analogous derivation holds under (iii.a). As $n \rightarrow \infty$

$$\frac{1}{n} \sum_{t=1}^n f(\tilde{X}_{t,n}(n)) = \frac{1}{n} \sum_{t=t_0}^n f(\tilde{X}_{t,n}(n)) + o_p(1). \quad (282)$$

In view of (281) and (282), as $n \rightarrow \infty$ first and then $\varepsilon \rightarrow 0$,

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n f(\tilde{X}_{t,n}(n)) &= \frac{1}{n} \sum_{t=t_0}^n f_{1,\varepsilon}(\tilde{X}_{t,n}(n)) + \frac{1}{n} \sum_{t=t_0}^n f_{2,\varepsilon}(\tilde{X}_{t,n}(n)) + o_p(1) \\ &= \frac{1}{n} \sum_{t=t_0}^n f_{1,\varepsilon}(\tilde{X}_{t,n}(n)) + o_p(1). \end{aligned}$$

Next, note that $\tilde{D}_{n,t}(x)$ is the density of $\tilde{X}_{t,t}(n) = \lambda_t^{-1} x_t(n)$. Hence,

$$\begin{aligned} \frac{1}{n} \sum_{t=t_0}^n \mathbf{E} |f_{1,\varepsilon}(\tilde{X}_{t,n}(n)) - f_\varepsilon(\tilde{X}_{t,n}(n))| &= \frac{1}{n} \sum_{t=t_0}^n \int_{\mathbb{R}} \left| f_{1,\varepsilon} \left(\frac{\lambda_t}{\lambda_n} x \right) - f_\varepsilon \left(\frac{\lambda_t}{\lambda_n} x \right) \right| \tilde{D}_{n,t}(x) dx \\ &\leq \sup_{n \geq n_0, t_0 \leq t \leq n} \sup_x \tilde{D}_{n,t}(x) \int_{\mathbb{R}} |f_{1,\varepsilon}(x) - f_\varepsilon(x)| dx \frac{1}{n} \sum_{t=1}^n \frac{\lambda_n}{\lambda_t} \leq C\varepsilon \rightarrow 0, \end{aligned}$$

as required for (264).

Next, we show (265). Using the fact that f_ε is Lipschitz and bounded, as $n \rightarrow \infty$,

$$\frac{1}{n} \sum_{t=1}^n \mathbf{E} |f_\varepsilon(\tilde{X}_{t,n}) - f_\varepsilon(X_{t,n})| \leq C \sup_{1 \leq t \leq n} \mathbf{E} (1 \wedge |R_{t,n}|) \rightarrow 0.$$

In view of the above and (264), we get (265).

Finally, the result for $\tilde{f} : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ follows easily from the fact that $\tilde{f}(\tilde{X}_{t,n}) = f(\tilde{X}_{t,n})$ a.s. for all $1 \leq t \leq n$. ■

Proof of Proposition 5. (a) We only consider the case $\mathcal{H} \neq \frac{1}{\alpha}$. Moreover, without loss of generality set $\sigma_0 = 1$. Let

$$\mathcal{K}_r(s) := (r-s)_+^{\mathcal{H}-\frac{1}{\alpha}} - (-s)_+^{\mathcal{H}-\frac{1}{\alpha}}.$$

Under our Assumptions, it follows from Proposition 3.4.1 in Samorodnitsky and Taqqu (1994), that the characteristic function of $\Lambda(r)$ is

$$\mathbf{E} e^{i\lambda \Lambda(r)} = \exp \left\{ - \int_{\mathbb{R}} |\lambda \mathcal{K}_r(s)|^\alpha \left\{ 1 - i\tilde{\beta} \operatorname{sgn}[\lambda \mathcal{K}_r(s)] \right\} ds \right\},$$

where

$$\tilde{\beta} = \beta_* \tan \left(\frac{\pi \alpha}{2} \right),$$

$\beta_* := \beta I_{\{\alpha \neq 1\}}$ and $|\beta| \leq 1$ the skewness of the Lévy motion $Z_\alpha(r)$, see e.g., Example 3.3 in Samorodnitsky and Taqqu (1994). We will prove the result for $\alpha \neq 1$. The case $\alpha = 1$ follows analogously, since the modulus of the characteristic function has the same form required below.

We start with the case $\mathcal{H} \neq \frac{1}{\alpha}$. Consider

$$\begin{aligned} \int_{\mathbb{R}} |\mathcal{K}_r(s)|^\alpha ds &= \int_{\mathbb{R}} \left| (r-s)_+^{\mathcal{H}-\frac{1}{\alpha}} - (-s)_+^{\mathcal{H}-\frac{1}{\alpha}} \right|^\alpha ds \\ &= r^{\alpha\mathcal{H}} \int_{\mathbb{R}} \left[(1-u)_+^{\mathcal{H}-\frac{1}{\alpha}} - (-u)_+^{\mathcal{H}-\frac{1}{\alpha}} \right]^\alpha du = C_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}} \end{aligned}$$

where $0 < C_{\mathcal{H},\alpha} < \infty$. In view of the above for $r > 0$

$$\begin{aligned} \int_{\mathbb{R}} |\mathbf{E} e^{i\lambda\Lambda(r)}| d\lambda &= \int_{\mathbb{R}} \left| \exp \left\{ - \int_{\mathbb{R}} |\lambda \mathcal{K}_r(s)|^\alpha \left\{ 1 - i\tilde{\beta} \operatorname{sgn}[\lambda \mathcal{K}_r(s)] \right\} ds \right\} \right| d\lambda \\ &\leq \int_{\mathbb{R}} \exp \left\{ - \int_{\mathbb{R}} |\lambda \mathcal{K}_r(s)|^\alpha ds \right\} d\lambda = \int_{\mathbb{R}} \exp \left\{ - C_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}} |\lambda|^\alpha \right\} d\lambda \\ &= 2 \int_0^\infty \exp \left\{ - C_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}} \lambda^\alpha \right\} d\lambda \end{aligned}$$

$$\begin{aligned} \left[\mu = C_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}} \lambda^\alpha \iff \lambda = \left(C_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}} \right)^{-1/\alpha} \mu^{1/\alpha} \iff \frac{d\lambda}{d\mu} = \frac{1}{\alpha} \left(C_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}} \right)^{-1/\alpha} \mu^{1/\alpha-1} \right] \\ = \frac{2}{\alpha} \left(C_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}} \right)^{-1/\alpha} \int_0^\infty \mu^{1/\alpha-1} e^{-\mu} d\mu = \frac{2}{\alpha} C_{\mathcal{H},\alpha} r^{-\mathcal{H}} \Gamma(\alpha^{-1}) < \infty, \end{aligned}$$

where Γ denotes the gamma function.

Next, in view of the fact that the Hurst index is $0 < \mathcal{H} < 1$ we have

$$\int_0^1 r^{-\mathcal{H}} dr = \frac{1}{1-\mathcal{H}} < \infty.$$

Hence,

$$\int_0^1 \int_{\mathbb{R}} |\mathbf{E} e^{i\lambda\Lambda(r)}| d\lambda dr < \infty.$$

In view of the integrability of $\mathbf{E} e^{i\lambda\Lambda(r)}$, it follows from Fourier inversion that $\Lambda(r)$ possesses a density, $p_{\Lambda(r)}(x)$ say, such that

$$p_{\Lambda(r)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\lambda x} \mathbf{E} e^{i\lambda\Lambda(r)} d\lambda.$$

Hence,

$$\int_0^1 \sup_x |p_{\Lambda(r)}(x)| dr \leq \frac{1}{2\pi} \int_0^1 \int_{\mathbb{R}} |\mathbf{E} e^{i\lambda\Lambda(r)}| d\lambda dr < \infty,$$

as required.

Finally, for $\mathcal{H} = \frac{1}{\alpha}$ note that

$$\begin{aligned} |\mathbf{E} e^{i\lambda\Lambda(r)}| &= \exp \left\{ - \int_0^r |\lambda|^\alpha ds \right\} \\ &= e^{-r|\lambda|^\alpha} = e^{-r^{\alpha\mathcal{H}}|\lambda|^\alpha}. \end{aligned}$$

Hence, the result follows from analogous arguments as those used above.

(b) We will show the result for $\mathcal{H} \neq \frac{1}{\alpha}$. The proof for the case $\mathcal{H} = \frac{1}{\alpha}$ is analogous to the one provided below and therefore omitted. Recall that

$$\mathbf{E} e^{i\lambda\Lambda(r)} = \exp \left\{ -\sigma_0^\alpha \int_{\mathbb{R}} |\lambda \mathcal{K}_r(s)|^\alpha \left\{ 1 - i\tilde{\beta} \operatorname{sgn} [\lambda \mathcal{K}_r(s)] \right\} ds \right\}.$$

A change of variables gives

$$\begin{aligned} \int_{\mathbb{R}} |\mathcal{K}_r(s)|^\alpha ds &= \int_{\mathbb{R}} \left| (r-s)_+^{\mathcal{H}-\frac{1}{\alpha}} - (-s)_+^{\mathcal{H}-\frac{1}{\alpha}} \right|^\alpha ds \\ &= r^{\alpha\mathcal{H}} \int_{\mathbb{R}} \left[(1-u)_+^{\mathcal{H}-\frac{1}{\alpha}} - (-u)_+^{\mathcal{H}-\frac{1}{\alpha}} \right]^\alpha du \\ &= r^{\alpha\mathcal{H}} \int_{\mathbb{R}} |\mathcal{K}_1(s)|^\alpha ds =: C_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}}. \end{aligned}$$

Similarly,

$$\int_{\mathbb{R}} |\mathcal{K}_r(s)|^\alpha \operatorname{sgn} [\mathcal{K}_r(s)] ds = r^{\alpha\mathcal{H}} \int_{\mathbb{R}} |\mathcal{K}_1(s)|^\alpha \operatorname{sgn} [\mathcal{K}_1(s)] ds =: D_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}}.$$

In view of the above,

$$\begin{aligned} \mathbf{E} e^{i\lambda\Lambda(r)} &= \exp \left\{ -\sigma_0^\alpha \int_{\mathbb{R}} |\lambda \mathcal{K}_r(s)|^\alpha ds - i\tilde{\beta} \int_{\mathbb{R}} |\lambda \mathcal{K}_r(s)|^\alpha \operatorname{sgn} [\lambda \mathcal{K}_r(s)] ds \right\} \\ &= \exp \left\{ -\sigma_0^\alpha \int_{\mathbb{R}} |\lambda \mathcal{K}_r(s)|^\alpha ds \left[1 - i\tilde{\beta} \frac{\int_{\mathbb{R}} |\lambda \mathcal{K}_r(s)|^\alpha \operatorname{sgn} [\lambda \mathcal{K}_r(s)] ds}{\int_{\mathbb{R}} |\lambda \mathcal{K}_r(s)|^\alpha ds} \right] \right\} \\ &= \exp \left\{ -\sigma_0^\alpha \int_{\mathbb{R}} |\mathcal{K}_r(s)|^\alpha ds |\lambda|^\alpha \left[1 - i\tilde{\beta} \operatorname{sgn} (\lambda) \frac{\int_{\mathbb{R}} |\mathcal{K}_r(s)|^\alpha \operatorname{sgn} [\mathcal{K}_r(s)] ds}{\int_{\mathbb{R}} |\mathcal{K}_r(s)|^\alpha ds} \right] \right\} \\ &= \exp \left\{ -\sigma_0^\alpha C_{\mathcal{H},\alpha} r^{\alpha\mathcal{H}} |\lambda|^\alpha \left[1 - i\beta_* \frac{D_{\mathcal{H},\alpha}}{C_{\mathcal{H},\alpha}} \tan \left(\frac{\pi\alpha}{2} \right) \operatorname{sgn} (\lambda) \right] \right\}. \quad (283) \end{aligned}$$

Using Ibragimov and Linnik (1971) Theorems 2.4.1-2.4.3 together with (283), for $0 < \alpha < 2$ we get

$$p_{\Lambda(r)}(x) \sim K_\alpha \sigma_0^\alpha (C_{\mathcal{H},\alpha} \pm \beta_* D_{\mathcal{H},\alpha}) r^{\alpha\mathcal{H}} |x|^{-1-\alpha}, \quad \pm x \rightarrow \infty.$$

Moreover, for $\alpha = 2$, the density $p_{\Lambda_\alpha(r)}(x)$ is Gaussian. Hence, there is some $\delta_1 > 0$ and $0 < C_1 < \infty$ such that

$$p_{\Lambda(r)}(x) \leq C_1 |x|^{-1-\alpha}$$

for all $|x| > \delta_1$. In addition, due to the assumption $|f(x)| = O(|x|^{\alpha'})$, $0 < \alpha' < \alpha$, there is some $\delta_2 > 0$ and $0 < C_2 < \infty$ such that

$$|f(x)| \leq C_2 |x|^{\alpha'},$$

for all $|x| > \delta_2$. Set $C_* = C_1 \vee C_2$ and $\delta_* = \delta_1 \vee \delta_2$. In view of the above,

$$\begin{aligned} \mathbf{E} \int_0^1 |f(\Lambda(r))| dr &= \int_{\mathbb{R}} \int_0^1 |f(x)| p_{\Lambda(r)}(x) dr dx \\ &= \int_{|x| \leq \delta_*} \int_0^1 |f(x)| p_{\Lambda(r)}(x) dr dx + \int_{|x| > \delta_*} \int_0^1 |f(x)| p_{\Lambda(r)}(x) dr dx \\ &\leq \int_{|x| \leq \delta_*} |f(x)| dx \int_0^1 \sup_x p_{\Lambda(r)}(x) dr + C_* \int_{|x| > \delta_*} |x|^{\alpha'} |x|^{-1-\alpha} dx \int_0^1 dr < \infty. \end{aligned}$$

Hence,

$$\int_0^1 |f(\Lambda(r))| dr < \infty \text{ a.s.},$$

as required. ■

18. Proofs for Sections 5-6

Proof of Theorem 7. In view of **Theorem 4**, the result follows by arguments analogous to those used in the proofs of Thm 3.1 in Wang and Phillips (2009), and Thm. 4.2 in DK2021. ■

Proof of Theorem 8. In view of **Theorem 4**, the result follows by arguments analogous to those used in the proof of Thm. 5.1 in DK2021. ■

Proof of Theorem 9. We start with some preliminary results. Write

$$\begin{aligned}\hat{m}(x) - m(x) &= \frac{\sum_{t=1}^n K_{n,t-1}(x) u_t}{\sum_{t=1}^n K_{n,t-1}(x)} + \frac{\sum_{t=1}^n K_{n,t-1}(x) [m(x_{t-1}) - m(x)]}{\sum_{t=1}^n K_{n,t-1}(x)} \\ &=: S_n + B_n.\end{aligned}\quad (284)$$

By **Theorem 2** and **Theorem 5** it follows that the rescaled score term

$$\begin{aligned}&\left(\frac{\gamma_n}{h_n n}\right)^{\frac{1}{\alpha_u}-1} S_n = \frac{\left(\frac{\gamma_n}{h_n n}\right)^{1/\alpha_u} \sum_{t=1}^n K_{n,t-1}(x) u_t}{\left(\frac{\gamma_n}{h_n n}\right) \sum_{t=1}^n K_{n,t-1}(x)} \\ &\xrightarrow{d} \frac{1}{\varrho(0) \int_{\mathbb{R}} K(x) dx} \cdot S_{\alpha_u} \left\{ \sigma_u^{\alpha_u} \varrho(0) \int_{\mathbb{R}} |K(x)|^{\alpha_u} dx, \beta_u^* \frac{\int_{\mathbb{R}} K(x)^{\langle \alpha_u \rangle} dx}{\int_{\mathbb{R}} |K(x)|^{\alpha_u} dx} \right\} \\ &= {}_d S_{\alpha_u} \left\{ \frac{\sigma_u^{\alpha_u} \varrho(0)^{1-\alpha_u} \int_{\mathbb{R}} |K(x)|^{\alpha_u} dx}{\left(\int_{\mathbb{R}} K(x) dx\right)^{\alpha_u}}, \beta_u^* \frac{\int_{\mathbb{R}} K(x)^{\langle \alpha_u \rangle} dx}{\int_{\mathbb{R}} |K(x)|^{\alpha_u} dx} \right\} = S_{\alpha_u} \{\tau_1, \beta_u^* \tau_2\}.\end{aligned}\quad (285)$$

Next, recall that under our assumptions, $\gamma_t^{-1} x_t$ as density function $\mathcal{D}_{n,t}(u)$ (cf. **Lemma 21**) such that

$$\sup_{n \geq 1, 1 \leq t \leq n} \sup_u \mathcal{D}_{n,t}(u) < \infty$$

and $\frac{1}{n} \sum_{t=1}^n \frac{\gamma_n}{\gamma_t} < C < \infty$. For the latter see the proof of **Theorem 5**. In view of these facts, consider

$$\begin{aligned}\mathbf{E} \left| \sum_{t=1}^n K_{n,t-1}(x) [m(x_{t-1}) - m(x)] \right| &\leq \sum_{t=1}^n \mathbf{E} \left| K\left(\frac{x_t - x}{h_n}\right) [m(x_{t-1}) - m(x)] \right| \\ &= \sum_{t=1}^n \int_{\mathbb{R}} \left| K\left(\frac{\gamma_t u - x}{h_n}\right) [m(\gamma_t u) - m(x)] \right| \mathcal{D}_{n,t}(u) du \\ &\quad \left[z = \frac{\gamma_t u - x}{h_n} \iff u = \gamma_t^{-1} (h_n z + x) \right] \\ &= \sum_{t=1}^n \frac{h_n}{\gamma_t} \int_{\mathbb{R}} |K(z) [m(h_n z + x) - m(x)]| \mathcal{D}_t(\gamma_t^{-1} (h_n z + x)) dz \\ &\leq C h_n^{1+\nu} \int_{\mathbb{R}} |K(z)| \bar{m}(x, z) dz \sum_{t=1}^n \gamma_t^{-1} \leq C h_n^{1+\nu} \sum_{t=1}^n \gamma_t^{-1}.\end{aligned}\quad (286)$$

Using **Theorem 2**,

$$|B_n| = \left| \frac{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x) [m(x_{t-1}) - m(x)]}{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x)} \right|$$

$$\begin{aligned}
&= \left| \frac{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x) [m(x_{t-1}) - m(x)]}{\varrho(0) \int_{\mathbb{R}} K(x) dx} \right| + o_p(1) \leq C \frac{h_n^{1+\nu} \gamma_n}{nh_n} \sum_{t=1}^n \gamma_t^{-1} + o_p(1) \\
&= Ch_n^\nu \frac{1}{n} \sum_{t=1}^n \frac{\gamma_n}{\gamma_t} + o_p(1) = O_p(h_n^\nu) = o_p(1).
\end{aligned} \tag{287}$$

(a) Next, we show part (a). Suppose that $\alpha_u > 1$. Consider the numerator of the 'bias' term B_n Combining (284)-(286),

$$\hat{m}(x) - m(x) = O_p(h_n^\nu) + O_p\left(\left(\frac{\gamma_n}{nh_n}\right)^{1-1/\alpha_u}\right) = o_p(1),$$

as required.

Next, we show the weak limit result of part (a). Using (286),

$$\begin{aligned}
&\left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \sum_{t=1}^n \mathbf{E} \left| K\left(\frac{x_t - x}{h_n}\right) [m(x_{t-1}) - m(x)] \right| \leq C \left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \sum_{t=1}^n \frac{h_n^{1+\nu}}{\gamma_t} \\
&= Ch_n^{1+\nu} \left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \gamma_n^{-1} \sum_{t=1}^n \frac{\gamma_n}{\gamma_t} = Ch_n^{1+\nu} \left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \frac{n}{\gamma_n} \frac{1}{n} \sum_{t=1}^n \frac{\gamma_n}{\gamma_t} \\
&= Ch_n^\nu \left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \frac{nh_n}{\gamma_n} \frac{1}{n} \sum_{t=1}^n \frac{\gamma_n}{\gamma_t} = Ch_n^\nu \left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \frac{nh_n}{\gamma_n} \frac{1}{n} \sum_{t=1}^n \frac{\gamma_n}{\gamma_t} \leq Ch_n^\nu \left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}-1}.
\end{aligned}$$

Therefore, using the upper bound shown above together with **Theorem 4** and (285) we get

$$\begin{aligned}
&\left(\frac{nh_n}{\gamma_n}\right)^{1-\frac{1}{\alpha_u}} [\hat{m}(x) - m(x)] = \frac{\left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \sum_{t=1}^n K_{n,t-1}(x) [m(x_{t-1}) - m(x)]}{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x)} \\
&+ \frac{\left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \sum_{t=1}^n K_{n,t-1}(x) u_t}{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x)} = O_p\left(h_n^\nu \left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}-1}\right) + \frac{\left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \sum_{t=1}^n K_{n,t-1}(x) u_t}{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x)} \\
&= \frac{\left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \sum_{t=1}^n K_{n,t-1}(x) u_t}{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x)} + o_p(1) \xrightarrow{d} S_{\alpha_u} \{\tau_1, \beta_u^* \tau_2\},
\end{aligned}$$

as required.

(b) Suppose that $\alpha_u = 1$. Using (285) together with (287)

$$\hat{m}(x) - m(x) = O_p(h_n^\nu) + \frac{\left(\frac{\gamma_n}{nh_n}\right) \sum_{t=1}^n K_{n,t-1}(x) u_t}{\left(\frac{\gamma_n}{nh_n}\right) \sum_{t=1}^n K_{n,t-1}(x)} \xrightarrow{d} S_{\alpha_u} \{\tau_1, \beta_u^* \tau_2\},$$

as required.

(c) Suppose that $\alpha_u < 1$. Using (285) together with (287)

$$\begin{aligned}
&\left(\frac{nh_n}{\gamma_n}\right)^{1-\frac{1}{\alpha_u}} [\hat{m}(x) - m(x)] = \frac{\left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \sum_{t=1}^n K_{n,t-1}(x) [m(x_{t-1}) - m(x)]}{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x)} \\
&+ \frac{\left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \sum_{t=1}^n K_{n,t-1}(x) u_t}{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x)} = O_p\left(h_n^\nu \left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}-1}\right) + \frac{\left(\frac{\gamma_n}{nh_n}\right)^{\frac{1}{\alpha_u}} \sum_{t=1}^n K_{n,t-1}(x) u_t}{\frac{\gamma_n}{nh_n} \sum_{t=1}^n K_{n,t-1}(x)}
\end{aligned}$$

$$\xrightarrow{d} S_{\alpha_u} \{ \tau_1, \beta_u^* \tau_2 \},$$

as required. ■

Proof of Theorem 10. (a) Recall that x_t which is linear process of the form

$$x_t = \sum_{s=0}^{\infty} c_s \xi_{t-s},$$

which exists a.s. In particular, x_t is a causal measurable transformation of the i.i.d. sequence $\{\xi_j\}_{-\infty}^t$. Therefore, x_t is strictly stationary and ergodic; see e.g. Straumann (2005, Corollary 2.1.3). Let g be a $\mathcal{B}(\overline{\mathbb{R}})$ -measurable function. Then, $\{g(x_t)\}_{t=1}^n$ is also strictly stationary and ergodic; see, e.g., Straumann (2005, Corollary 2.1.1). Moreover, if $\mathbf{E}|g(x_t)| < \infty$, then by ergodic theorem (see, e.g., Shiryaev, 1996, p. 409-410)

$$\frac{1}{n} \sum_{t=1}^n g(x_t) \xrightarrow{a.s.} \mathbf{E}g(x_t), \quad (288)$$

as $n \rightarrow \infty$. Next, consider the WIV estimator

$$\sqrt{n} (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) = \left[\frac{1}{n} \sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right]^{-1} \frac{1}{\sqrt{n}} \sum_{t=1}^n \mathbf{w}_{t-1} u_t. \quad (289)$$

Note that $f, w_0, w_1, w_0 f, w_1 f, w_0^2, w_1^2$ all satisfy the conditions imposed on g ; that is, they are $\mathcal{B}(\overline{\mathbb{R}})$ -measurable and have finite absolute moments. Let $\mathbf{j} \in \mathbb{R}^2$, $\mathbf{j} \neq \mathbf{0}$. Consequently,

$$\left[\frac{1}{n} \sum_{t=1}^n \mathbf{A}(x_t), \frac{1}{n} \sum_{t=1}^n (\mathbf{j}' \mathbf{w}_{t-1})^2 \right] \xrightarrow{p} [\mathbf{E}(\mathbf{A}(x_t)), \mathbf{E}(\mathbf{j}' \mathbf{w}_{t-1})^2], \quad (290)$$

and

$$\mathbf{E}_{t-1} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n \mathbf{j}' \mathbf{w}_{t-1} u_t \right)^2 = \frac{\sigma_u^2}{n} \sum_{t=1}^n (\mathbf{j}' \mathbf{w}_{t-1})^2 \xrightarrow{p} \sigma_u^2 \mathbf{E}(\mathbf{j}' \mathbf{w}_{t-1})^2. \quad (291)$$

To verify the corresponding Lindeberg condition, fix $\epsilon > 0$. Moreover, set $J_t := \mathbf{j}' \mathbf{w}_{t-1}$. Since $\mathbf{E} \|\mathbf{w}_{t-1}\|^{2+\delta} < \infty$, for some $\delta > 0$. we have $\mathbf{E} J_t^{2+\delta} < \infty$. Therefore, dominated convergence yields

$$\begin{aligned} \mathbf{E} \mathbf{E}_{t-1} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n J_t u_t I_{\{|J_t| > \sqrt{n}\epsilon\}} \right)^2 &= \frac{\sigma_u^2}{n} \mathbf{E} \sum_{t=1}^n J_t^2 I_{\{|J_t| > \sqrt{n}\epsilon\}} \\ &= \mathbf{E} J_t^2 I_{\{|J_t| > \sqrt{n}\epsilon\}} = o(1), \end{aligned} \quad (292)$$

as $n \rightarrow \infty$. Using (289)-(292) together with the Cramér–Wold device and Hall and Heyde (1980, Corollary 3.1) we get

$$\sqrt{n} (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) \xrightarrow{d} [\mathbf{E}(\mathbf{A}(x_t))]^{-1} N[\mathbf{0}, \sigma_u^2 \mathbf{E}(\mathbf{W}(x_t))], \quad (293)$$

as required.

(b) Next, we show the consistency of

$$\tilde{\sigma}_w^2 = \frac{\sum_{t=1}^n \tilde{u}_t^2 w_u(x_{t-1})}{\sum_{t=1}^n w_u(x_{t-1})}.$$

In view of the fact that $w_u(x_{t-1})$ is strictly stationary and ergodic with a finite mean we obtain

$$\frac{1}{n} \sum_{t=1}^n w_u(x_{t-1}) \xrightarrow{p} \mathbf{E}[w_u(x_{t-1})]. \quad (294)$$

Next, consider

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n \tilde{u}_t^2 w_u(x_{t-1}) &= \frac{1}{n} \sum_{t=1}^n u_t^2 w_u(x_{t-1}) \\ &+ \frac{1}{n} \sum_{t=1}^n \left[(\tilde{\mathbf{b}}_{WIV} - \mathbf{b})' \mathbf{x}_{t-1} \right]^2 w_u(x_{t-1}) - \frac{2}{n} \sum_{t=1}^n (\tilde{\mathbf{b}}_{WIV} - \mathbf{b})' \mathbf{x}_{t-1} u_t w_u(x_{t-1}) \end{aligned} \quad (295)$$

By the Bahr-Essen inequality for martingales, see e.g. Pinelis (2015, Thm 1.1), for some $1 < p < 2$, the

$$\begin{aligned} \mathbf{E} \left| \frac{1}{n} \sum_{t=1}^n (u_t^2 - \sigma_u^2) w_u(x_{t-1}) \right|^p &\leq \frac{1}{n^p} \sum_{t=1}^n \mathbf{E} \left[|u_t^2 - \sigma_u^2|^p w_u(x_{t-1})^p \right] \\ &\leq \frac{C}{n^p} \sum_{t=1}^n \mathbf{E} w_u(x_{t-1})^p \leq \frac{C}{n^{p-1}} = o(1), \end{aligned} \quad (296)$$

The second term

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n \left[(\tilde{\mathbf{b}}_{WIV} - \mathbf{b})' \mathbf{x}_{t-1} \right]^2 w_u(x_{t-1}) &\leq \|\tilde{\mathbf{b}}_{WIV} - \mathbf{b}\|^2 \frac{1}{n} \sum_{t=1}^n \|\mathbf{x}_{t-1}\|^2 w_u(x_{t-1}) \\ &\leq C \|\tilde{\mathbf{b}}_{WIV} - \mathbf{b}\|^2 \frac{1}{n} \sum_{t=1}^n (1 + |f(x_{t-1})|)^2 w_u(x_{t-1}) = O_p(n^{-1}), \end{aligned} \quad (297)$$

where the last equality above follows from the $\sqrt{n}$ -consistency of the WIV estimator and the fact that $\mathbf{E} f(x_{t-1})^2 w_u(x_{t-1}) < \infty$. Similarly, the third term

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n \left| (\tilde{\mathbf{b}}_{WIV} - \mathbf{b})' \mathbf{x}_{t-1} u_t w_u(x_{t-1}) \right| \\ \leq \frac{1}{n} \sum_{t=1}^n \left| (\tilde{\mathbf{b}}_{WIV} - \mathbf{b})' \mathbf{x}_{t-1} u_t w_u(x_{t-1}) \right| \\ \leq \|\tilde{\mathbf{b}}_{WIV} - \mathbf{b}\| \frac{1}{n} \sum_{t=1}^n |f(x_{t-1}) w_u(x_{t-1}) u_t| = O_p(n^{-1/2}), \end{aligned} \quad (298)$$

where we have used the fact that

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n \mathbf{E} |f(x_{t-1}) w_u(x_{t-1}) u_t| &\leq \frac{1}{n} \sum_{t=1}^n \mathbf{E} |f(x_{t-1}) w_u(x_{t-1})| \mathbf{E}_{t-1} |u_t| \\ &\leq \frac{1}{n} \sum_{t=1}^n \mathbf{E} |f(x_{t-1}) w_u(x_{t-1})| [\mathbf{E}_{t-1} u_t^2]^{1/2} \leq \frac{|\sigma_u|}{n} \sum_{t=1}^n \mathbf{E} |f(x_{t-1}) w_u(x_{t-1})| = O(1). \end{aligned}$$

The result follows from (294)-(298).

Finally, the limit distribution of the t-statistic follows easily from part (a) and the consistency of $\tilde{\sigma}_w^2$. Indeed, set $\mathbf{e}_2' := [0, 1]$ and

$$Q_n := n^{-1} \mathbf{e}_2' \left[\sum_{t=1}^n \mathbf{A}(x_{t-1}) \right]^{-1} \sum_{t=1}^n \mathbf{W}(x_{t-1}) \left[\sum_{t=1}^n \mathbf{A}'(x_{t-1}) \right]^{-1} \mathbf{e}_2.$$

Then by the ergodic theorem

$$Q_n \xrightarrow{p} \mathbf{e}'_2 \mathbf{E}[\mathbf{A}(x_{t-1})] \mathbf{E}[\mathbf{W}(x_{t-1})] \{\mathbf{E}[\mathbf{A}'(x_{t-1})]\}^{-1} \mathbf{e}_2. \quad (299)$$

Combining (293), (299) and the consistency of $\tilde{\sigma}_u^2$ we get

$$\tilde{t}_{WIV} = \frac{\sqrt{n} \mathbf{e}'_2 (\tilde{\mathbf{b}}_{WIV} - \mathbf{b})}{\sqrt{\tilde{\sigma}_u^2 Q_n}} \xrightarrow{d} N[0, 1],$$

as required. ■

Proof of Theorem 11. (a) Set $C_n = \sqrt{n} \text{diag}\{1, k_f(\gamma_n)\}$ and $D_n = \sqrt{n} \text{diag}\{k_{w_0}(\gamma_n), k_{w_1}(\gamma_n)\}$. By **Theorem 3**,

$$\begin{aligned} C_n (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) &= C_n \left[\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right]^{-1} D_n D_n^{-1} \sum_{t=1}^n \mathbf{w}_{t-1} u_t, \\ &= \left\{ D_n^{-1} \left[\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right] C_n^{-1} \right\}^{-1} D_n \sum_{t=1}^n \mathbf{w}_{t-1} u_t =: Q_n^{-1} \cdot D_n \sum_{t=1}^n \mathbf{w}_{t-1} u_t \\ &\xrightarrow{d} \left[\int_{\mathbb{R}} \mathbf{A}_H(x) \varrho(x) dx \right]^{-1} \cdot MN \left[\mathbf{0}, \sigma_u^2 \int_{\mathbb{R}} \mathbf{W}_H(x) \varrho(x) dx \right], \end{aligned} \quad (300)$$

as required.

(b) Set $\mathbf{e}'_2 := [0, 1]$. Using the limit distribution result of part (a) above, the numerator of $\tilde{t}_{WLS}$ after scaling by $\sqrt{n} k_f(\gamma_n)$ is

$$\begin{aligned} \sqrt{n} k_f(\gamma_n) (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) &= \mathbf{e}'_2 C_n (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) \\ &\xrightarrow{d} MN \left[0, \sigma_u^2 \mathbf{e}'_2 \left[\int_{\mathbb{R}} \mathbf{A}_H(x) \varrho(x) dx \right]^{-1} \int_{\mathbb{R}} \mathbf{W}_H(x) \varrho(x) dx \left[\int_{\mathbb{R}} \mathbf{A}'_H(x) \varrho(x) dx \right]^{-1} \mathbf{e}_2 \right]. \end{aligned}$$

In view of **Theorem 3**, the normalizing term

$$\begin{aligned} &n k_f(\gamma_n)^2 \mathbf{e}'_2 \left(\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right)^{-1} \left(\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{w}'_{t-1} \right) \left(\sum_{t=1}^n \mathbf{x}_{t-1} \mathbf{w}'_{t-1} \right)^{-1} \mathbf{e}_2 \\ &= n k_f(\gamma_n)^2 \mathbf{e}'_2 C_n^{-1} C_n \left(\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right)^{-1} D_n D_n^{-1} \left(\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{w}'_{t-1} \right) D_n^{-1} D_n \left(\sum_{t=1}^n \mathbf{x}_{t-1} \mathbf{w}'_{t-1} \right)^{-1} C_n C_n^{-1} \mathbf{e}_2 \\ &= \mathbf{e}'_2 C_n \left(\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right)^{-1} D_n D_n^{-1} \left(\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{w}'_{t-1} \right) D_n^{-1} D_n \left(\sum_{t=1}^n \mathbf{x}_{t-1} \mathbf{w}'_{t-1} \right)^{-1} C_n \mathbf{e}_2 \\ &\xrightarrow{d} \mathbf{e}'_2 \left[\int_{\mathbb{R}} \mathbf{A}_H(x) \varrho(x) dx \right]^{-1} \int_{\mathbb{R}} \mathbf{W}_H(x) \varrho(x) dx \left[\int_{\mathbb{R}} \mathbf{A}'_H(x) \varrho(x) dx \right]^{-1} \mathbf{e}_2. \end{aligned}$$

To complete the proof, it suffices to show that $\tilde{\sigma}_u^2 = \sigma_u^2 + o_p(1)$. This can be established using arguments analogous to those used for the proof of **Theorem 13(b)**. ■

Proof of Theorem 12. Let C_n, D_n, Q_n be as in (300). Then by **Theorem 3**, as $n \rightarrow \infty$,

$$Q_n^{-1} \xrightarrow{p} Q^{-1} := \left[\int_{\mathbb{R}} \mathbf{A}_H(x) \varrho(x) dx \right]^{-1}. \quad (301)$$

Next, fix $\mathbf{v} = (v_0, v_1)' \in \mathbb{R}^2$. Since w_0, w_1 are **HF**, the score term is

$$\begin{aligned} \mathbf{v}' D_n^{-1} \sum_{t=1}^n \mathbf{w}_{t-1} u_t &= n^{-1/\alpha_u} \sum_{t=1}^n \left[v_0 \frac{w_0(x_{t-1})}{k_{w_0}(\gamma_n)} + v_1 \frac{w_1(x_{t-1})}{k_{w_1}(\gamma_n)} \right] u_t \\ &= n^{-1/\alpha_u} \sum_{t=1}^n \mathbf{v}' \mathbf{w}(\gamma_n^{-1} x_{t-1}) u_t. \end{aligned}$$

We verify the conditions on $\mathbf{v}' \mathbf{w}(\cdot)$ required by **Theorem 5**. For all $\alpha_u > 0$, there exists $C_{\mathbf{v}, \alpha_u} < \infty$ such that

$$|\mathbf{v}' \mathbf{w}(x)|^{\alpha_u} \leq C_{\mathbf{v}, \alpha_u} (|w_0(x)|^{\alpha_u} + |w_1(x)|^{\alpha_u})$$

By condition (iv) and **Definition 4**, each $|w_i|^{\alpha_u}$ is locally integrable. In view of the above, $|\mathbf{v}' \mathbf{w}(x)|^{\alpha_u}$ is locally integrable. Moreover, setting $\lambda = \max\{\lambda_0, \lambda_1\} < \alpha$, condition (iv) gives

$$|\mathbf{v}' \mathbf{w}(x)|^{\alpha_u} = O(|x|^\lambda),$$

as $|x| \rightarrow \infty$. Next, we show condition (iii) of **Theorem 5** i.e. the positivity requirement $\int_{\mathbb{R}} |\mathbf{v}' \mathbf{w}(x)|^{\alpha_u} \varrho(x) dx \neq 0$. Indeed, if $\int_{\mathbb{R}} |\mathbf{v}' \mathbf{w}(x)|^{\alpha_u} \varrho(x) dx = 0$ for some $\mathbf{v} \neq 0$, then $\mathbf{v}' \mathbf{w}(x) = 0$ for $\varrho(x) dx$ -almost every x . In view of the fact $\mathbf{A}_H(x) = \mathbf{w}(x)(1, f_H(x))$, this implies $\mathbf{v}' Q = 0$, contradicting condition (v) of **Theorem 12**.

Thus, by **Theorem 5** as $n \rightarrow \infty$,

$$\mathbf{v}' D_n^{-1} \sum_{t=1}^n \mathbf{w}_{t-1} u_t \xrightarrow{d} S_{\alpha_u} \left\{ \sigma_u^{\alpha_u} \int_{\mathbb{R}} |\mathbf{v}' \mathbf{w}(x)|^{\alpha_u} \varrho(x) dx, \beta_u^* \frac{\int_{\mathbb{R}} |\mathbf{v}' \mathbf{w}(x)|^{\alpha_u} \varrho(x) dx}{\int_{\mathbb{R}} |\mathbf{v}' \mathbf{w}(x)|^{\alpha_u} \varrho(x) dx} \right\}. \quad (302)$$

The requisite result follows from (301) and (302). ■

Proof of Theorem 13. (a) Set $C_n^* = \sqrt{n} \text{diag}\{1, k_f(\gamma_n^*)\}$ and $D_n^* = \sqrt{n} \text{diag}\{k_{w_0}(\gamma_n^*), k_{w_1}(\gamma_n^*)\}$. Write

$$\begin{aligned} C_n^* (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) &= C_n^* \left[\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right]^{-1} D_n^* D_n^{*-1} \sum_{t=1}^n \mathbf{w}_{t-1} u_t \\ &= \left\{ D_n^{*-1} \left[\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right] C_n^{*-1} \right\}^{-1} D_n^{*-1} \sum_{t=1}^n \mathbf{w}_{t-1} u_t. \end{aligned}$$

Furthermore,

$$D_n^{*-1} \left[\sum_{t=1}^n \mathbf{w}_{t-1} \mathbf{x}'_{t-1} \right] C_n^{*-1} \xrightarrow{d} \int_0^1 \mathbf{A}_H(\Lambda_{\alpha, d}(r)) dr. \quad (303)$$

Next, for $i = \{0, 1\}$,

$$\begin{aligned} \mathbf{E} \left| \frac{1}{\sqrt{n} k_{w_i}(\gamma_n^*)} \sum_{t=1}^n w_i(x_{t-1}) u_t \right|^2 &\leq 2 \mathbf{E} \sum_{t=1}^n \left| \frac{1}{\sqrt{n} k_{w_i}(\gamma_n^*)} w_i(x_{t-1}) u_t \right|^2 \\ &= \frac{2}{n} \mathbf{E} \sum_{t=1}^n \left| \frac{1}{k_{w_i}(\gamma_n^*)} w_i(x_{t-1}) u_t \right|^2 = \frac{2}{n} \mathbf{E} \sum_{t=1}^n \left| \frac{1}{k_{w_i}(\gamma_n^*)} w_i(x_{t-1}) \right|^2 \mathbf{E}_{t-1} u_t^2 \\ &= \frac{2}{n} \mathbf{E} \sum_{t=1}^n \left| \frac{1}{k_{w_i}(\gamma_n^*)} w_i(x_{t-1}) \right|^2 \mathbf{E}_{t-1} u_t^2 \leq \frac{\sigma^2}{n} \mathbf{E} \sum_{t=1}^n \left| \frac{1}{k_{w_i}(\gamma_n^*)} w_i(x_{t-1}) \right|^2 \end{aligned}$$

$$\leq_{(1)} \frac{\sigma^2}{n} \mathbf{E} \sum_{t=1}^n \left| \frac{1}{k_{w_i}(\gamma_n^*)} w_i(x_{t-1}) \right|^2 = O(1), \quad (304)$$

where $\leq_{(1)}$ follows from **Lemma SS _{α}** M. Hence,

$$C_n^* (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) = O_p(1),$$

as required.

(b) Next, we show the consistency of

$$\tilde{\sigma}_w^2 = \frac{\sum_{t=1}^n \tilde{u}_t^2 w_u(x_{t-1})}{\sum_{t=1}^n w_u(x_{t-1})}.$$

Recall that $\mathbf{x}_{t-1} := [1, f(x_{t-1})]'$. Then

$$\tilde{u}_t = y_t - \mathbf{x}_{t-1}' \tilde{\mathbf{b}}_{WIV} = \mathbf{x}_{t-1}' (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) + u_t$$

For a matrix $\mathbf{A} = [a_{ij}]$, set $\|\mathbf{A}\| = \sum_{ij} |a_{ij}|$. By **Theorem 3** as, $n \rightarrow \infty$,

$$\frac{1}{nk_{w_u}(\gamma_n^*)} \sum_{t=1}^n \|\mathbf{x}_{t-1}' C_n^{*-1}\| w_u(x_{t-1}) \xrightarrow{d} \int_0^1 [(1 + |f_H|) w_u] (\Lambda_{\alpha,d}(r)) dr, \quad (305)$$

$$\frac{1}{nk_{w_u}(\gamma_n^*)} \sum_{t=1}^n \|\mathbf{x}_{t-1}' C_n^{*-1}\|^2 w_u(x_{t-1}) \xrightarrow{d} \int_0^1 [(1 + |f_H|)^2 w_u] (\Lambda_{\alpha,d}(r)) dr, \quad (306)$$

and

$$\frac{1}{nk_{w_u}(\gamma_n^*)} \sum_{t=1}^n w_u(x_{t-1}) \xrightarrow{d} \int_0^1 w_u (\Lambda_{\alpha,d}(r)) dr. \quad (307)$$

Moreover, by the Bahr-Essen inequality for martingales, see e.g. Pinelis (2015, Thm 1.1), for some $1 < p < 2$,

$$\begin{aligned} \mathbf{E} \left| \frac{1}{nk_{w_u}(\gamma_n^*)} \sum_{t=1}^n (u_t^2 - \sigma_u^2) w_u(x_{t-1}) \right|^p &\leq \frac{1}{n^p k_{w_u}(\gamma_n^*)^p} \sum_{t=1}^n \mathbf{E} [|u_t^2 - \sigma_u^2|^p w_u(x_{t-1})^p] \\ &\leq \frac{C}{n^p k_{w_u}(\gamma_n^*)^p} \sum_{t=1}^n \mathbf{E} w_u(x_{t-1})^p \leq_{(2)} \frac{C}{n^{p-1}} = o(1), \end{aligned} \quad (308)$$

where $\leq_{(2)}$ follows from **Lemma SS _{α}** M.

Next, consider the numerator of $\tilde{\sigma}_u^2$

$$\begin{aligned} \frac{1}{nk_{w_u}(\gamma_n^*)} \sum_{t=1}^n \tilde{u}_t^2 w_u(x_{t-1}) &= \frac{1}{nk_{w_u}(\gamma_n^*)} \sum_{t=1}^n u_t^2 w_u(x_{t-1}) + 2 \frac{1}{nk_{w_u}(\gamma_n^*)} \sum_{t=1}^n \mathbf{x}_{t-1}' C_n^{*-1} C_n^* (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) w_u(x_{t-1}) \\ &\quad + \frac{1}{nk_{w_u}(\gamma_n^*)} \sum_{t=1}^n [\mathbf{x}_{t-1}' C_n^{*-1} C_n^* (\tilde{\mathbf{b}}_{WIV} - \mathbf{b})]^2 w_u(x_{t-1}) \\ &=: \frac{1}{nk_{w_u}(\gamma_n^*)} \sum_{t=1}^n u_t^2 w_u(x_{t-1}) + 2R_{1n} + R_{2n}. \end{aligned}$$

In view of (305)-(308), we get

$$R_{1n} \leq \left\| \sqrt{n} C_n^* (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) \right\| \frac{1}{n^{3/2} k_{w_u}(\gamma_n^*)} \sum_{t=1}^n \|\mathbf{x}_{t-1}' C_n^{*-1}\| w_u(x_{t-1}) = O_p(n^{-1/2}),$$

$$\begin{aligned}
R_{2n} &\leq \left\| \sqrt{n} C_n^* (\tilde{\mathbf{b}}_{WIV} - \mathbf{b}) \right\|^2 \frac{1}{n^2 k_{w_u}(\gamma_n^*)} \sum_{t=1}^n \left\| \mathbf{x}'_{t-1} C_n^{*-1} \right\|^2 w_u(x_{t-1}) = O_p(n^{-1}), \\
&\quad \frac{1}{n k_{w_u}(\gamma_n^*)} \sum_{t=1}^n u_t^2 w_u(x_{t-1}) \\
&= \frac{\sigma_u^2}{n k_{w_u}(\gamma_n^*)} \sum_{t=1}^n w_u(x_{t-1}) + o_p(1) \xrightarrow{d} \sigma_u^2 \int_0^1 w_u(\Lambda_{\alpha,d}(r)) dr.
\end{aligned}$$

Combining the above we have

$$\tilde{\sigma}_w^2 \xrightarrow{p} \sigma_u^2,$$

as required. ■

19. Proofs for Section 7

19.1. Supporting Results for Section 7

Lemma 23. Suppose that the following hold:

- (i) **Assumption INN** holds with $\theta = 1$;
- (ii) x_t satisfies **Assumption SP**;
- (iii) $z_t := z_t(n)$ is defined by (36) and $\rho_{z,n}$ by (37).

Then

$$\sup_{n \geq 1, 1 \leq t \leq n} \int_{\mathbb{R}} |\lambda| \left| \mathbf{E} \left(e^{i\lambda z_t} \right) \right| d\lambda < \infty.$$

19.2. Proofs of Main Results in Section 7

Proof of Theorem 14. (a) Let g locally integrable and suppose that for some $0 < \lambda < \alpha$, $g(x) = O(|x|^\lambda)$, as $|x| \rightarrow \infty$. We will show that

$$\frac{1}{n} \sum_{t=1}^n g(z_t) \xrightarrow{p} \mathbf{E} g(x_t), \quad (309)$$

as $n \rightarrow \infty$. Indeed, since the elements of $\mathbf{A}(x)$ and $\mathbf{W}(x)$ satisfy the same local integrability and growth conditions as $g(x)$, the result follows by arguments analogous to those used in the proof of **Theorem 10**.

First, by **Lemma 23** under **Assumption SP**, $z_t := z_t(n)$ has a density $D_{z_t(n)}(x)$ that is uniformly bounded i.e. density

$$\sup_{n \geq 1, 1 \leq t \leq n} \sup_x D_{z_t(n)}(x) < \infty. \quad (310)$$

Furthermore, the arguments used for the proof of **Lemma 23** show that x_t has also a density $D(x)$ say, such that

$$\sup_x D(x) < \infty. \quad (311)$$

Next, it is easy to show that for all $0 < \alpha' < \alpha$,

$$\sup_{t \geq 1} \mathbf{E} |x_t|^{\alpha'} < \infty. \quad (312)$$

In addition, it follows from **Lemma 2** that as $n \rightarrow \infty$,

$$\sup_{1 \leq t \leq n} \mathbf{E} |z_t(n) - x_t|^{\alpha'} \rightarrow 0. \quad (313)$$

Next, let g locally integrable such that for some $0 < \lambda < \alpha$, $g(x) = O(|x|^\lambda)$, as $|x| \rightarrow \infty$. It follows from (310)-(313) and **Proposition 4** that for each $\epsilon > 0$ there is a bounded and Lipschitz g_ϵ such that

$$g_\epsilon(x) = 0, \quad |x| > \epsilon^{-1}, \quad (314)$$

$$\int_{|x| \leq \epsilon^{-1}} |g_\epsilon(x) - g(x)| dx < \epsilon, \quad (315)$$

and as $n \rightarrow \infty$ and the $\epsilon \rightarrow 0$

$$\frac{1}{n} \sum_{t=1}^n g(z_t) = \frac{1}{n} \sum_{t=1}^n g_\epsilon(x_t) + o_p(1). \quad (316)$$

Since, $\{g_\epsilon(x_t)\}_{t=1}^n$ is strictly stationary and ergodic, see, e.g. (288), by the ergodic theorem we get

$$\frac{1}{n} \sum_{t=1}^n g_\epsilon(x_t) \xrightarrow{a.s.} \mathbf{E} g_\epsilon(x_t), \quad (317)$$

as $n \rightarrow \infty$.

Next, in view of (311) as $\epsilon \rightarrow 0$,

$$\begin{aligned} |\mathbf{E} g_\epsilon(x_t) - \mathbf{E} g(x_t)| &\leq \mathbf{E} |g_\epsilon(x_t) - g(x_t)| = \int_{\mathbb{R}} |g_\epsilon(x) - g(x)| D(x) dx \\ &=_{(1)} \int_{|x| \leq \epsilon^{-1}} |g_\epsilon(x) - g(x)| D(x) dx + \int_{|x| > \epsilon^{-1}} |g(x)| D(x) dx \\ &\leq \sup_x D(x) \int_{|x| \leq \epsilon^{-1}} |g_\epsilon(x) - g(x)| dx + \mathbf{E} g(x_t) I_{\{|x_t| > \epsilon^{-1}\}} \\ &\leq \sup_x D(x) \int_{|x| \leq \epsilon^{-1}} |g_\epsilon(x) - g(x)| dx + \mathbf{E} g(x_t) I_{\{|x_t| > \epsilon^{-1}\}} \\ &\leq_{(2)} \sup_x D(x) \cdot \epsilon + C \mathbf{E} |x_t|^\lambda I_{\{|x_t| > \epsilon^{-1}\}} =_{(3)} 0, \end{aligned} \quad (318)$$

where $=_{(1)}$, $\leq_{(2)}$ follow from (314) and (315) respectively. Finally, $=_{(3)}$ follows from dominated convergence i.e. it is possible to find $\lambda < \alpha' < \alpha$ such that $\mathbf{E} |x_t|^{\alpha'} < C < \infty$. Combining (316)-(318) we get (309).

(b) Finally, for the IVX t-statistic, as $n \rightarrow \infty$,

$$\tilde{t}_{IVX} = \tilde{t}_{WLS} + o_p(1) \xrightarrow{d} N(0, 1),$$

where the first relation follows from part (a), while the weak convergence is a direct consequence of Theorem (10). ■

Proof of Theorem 16. Under our Assumptions, z_t is of the form

$$z_t = \sum_{j=1}^t \rho_{z,n}^{t-j} v_j,$$

and therefore exactly a MI process with localizing coefficients $\rho_{z,n}$. Moreover, setting $\mu_n = \gamma_{z,n}$ it follows from **Lemma 1**,

$$\frac{1}{\mu_n} z_n \xrightarrow{d} X^+,$$

where X^+ is some α -stable variate. Set $D_n = \sqrt{n} \text{diag}\{k_{w_0}(\mu_n), k_{w_1}(\mu_n)\}$. In view of the above, **Theorem 1**, we obtain, for as $n \rightarrow \infty$,

$$\begin{aligned} D_n^{-1} \sum_{t=1}^n \mathbf{W}(z_{t-1}) D_n^{-1} &\xrightarrow{p} \int_{\mathbb{R}} \mathbf{W}_H(x) \varrho(x) dx \\ &= \int_{\mathbb{R}} \begin{bmatrix} w_{0,H}^2 & w_{0,H} w_{1,H} \\ w_{0,H} w_{1,H} & w_{1,H}^2 \end{bmatrix} (x) \varrho(x) dx, \end{aligned} \quad (319)$$

where $\varrho(x)$ is the density function of X^+ . Moreover,

Under H_0 ,

$$\begin{aligned} \hat{t}_{IVX} &= \frac{1}{\sqrt{\hat{\sigma}^2 \sum_{t=1}^n \bar{w}_1(z_{t-1})^2}} \sum_{t=1}^n w_1(z_{t-1}) \bar{u}_t \\ &= \frac{1}{\sqrt{\hat{\sigma}^2 \sum_{t=1}^n \bar{w}_1(z_{t-1})^2}} \sum_{t=1}^n w_1(z_{t-1}) \left(u_t - \frac{\sum_{j=1}^n u_j w_0(z_{j-1})}{\sum_{j=1}^n w_0(z_{j-1})} \right). \end{aligned} \quad (320)$$

Set $e'_2 := [0, 1]$. In view of (319), **Theorem 3** and the Cramér–Wold device, we obtain

$$\begin{aligned} e'_2 D_n^{-1} \sum_{t=1}^n \mathbf{W}(z_{t-1}) \bar{u}_t &= \frac{1}{\sqrt{n} k_{w_1}(\mu_n)} \sum_{t=1}^n w_1(z_{t-1}) \left(u_t - \frac{\sum_{j=1}^n u_j w_0(z_{j-1})}{\sum_{j=1}^n w_0(z_{j-1})} \right) \\ &= \begin{bmatrix} -\frac{\frac{1}{n k_{w_1}(\mu_n)} \sum_{t=1}^n w_1(z_{t-1})}{\frac{1}{n k_{w_0}(\mu_n)} \sum_{t=1}^n w_0(z_{t-1})}, & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n} k_{w_0}(\mu_n)} \sum_{t=1}^n w_0(z_{t-1}) u_t \\ \frac{1}{\sqrt{n} k_{w_1}(\mu_n)} \sum_{t=1}^n w_1(z_{t-1}) u_t \end{bmatrix} \\ &=: \begin{bmatrix} -a_n, & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{n} k_{w_0}(\mu_n)} \sum_{t=1}^n w_0(z_{t-1}) u_t \\ \frac{1}{\sqrt{n} k_{w_1}(\mu_n)} \sum_{t=1}^n w_1(z_{t-1}) u_t \end{bmatrix} \\ &\xrightarrow{d} \begin{bmatrix} -\frac{\int_{\mathbb{R}} w_{1,H}(x) \varrho(x) dx}{\int_{\mathbb{R}} w_{0,H}(x) \varrho(x) dx}, & 1 \end{bmatrix} \cdot N \left[\mathbf{0}, \sigma_u^2 \int_{\mathbb{R}} \mathbf{W}_H(x) \varrho(x) dx \right] \\ &=: [-a_H, 1] \cdot N \left[\mathbf{0}, \sigma_u^2 \int_{\mathbb{R}} \mathbf{W}_H(x) \varrho(x) dx \right] \\ &= N \left[0, \sigma_u^2 \int_{\mathbb{R}} [w_{1,H}(x) - a_H w_{0,H}]^2 \varrho(x) dx \right]. \end{aligned} \quad (321)$$

Next, note that

$$\bar{w}_1(z_{t-1}) = w_1(z_{t-1}) - \frac{\sum_{j=1}^n w_1(z_{j-1})}{\sum_{j=1}^n w_0(z_{j-1})} w_1(z_{t-1}).$$

Consequently,

$$\frac{1}{k_{w_1}(\mu_n)} \bar{w}_1(z_{t-1}) = \frac{1}{k_{w_1}(\mu_n)} w_1(z_{t-1}) - a_n \frac{1}{k_{w_0}(\mu_n)} w_1(z_{t-1}).$$

Therefore,

$$\frac{1}{n k_{w_1}^2(\mu_n)} \sum_{t=1}^n \bar{w}_1(z_{t-1})^2 = \begin{bmatrix} -a_n, & 1 \end{bmatrix} D_n^{-1} \sum_{t=1}^n \mathbf{W}(z_{t-1}) D_n^{-1} \begin{bmatrix} -a_n, & 1 \end{bmatrix}'$$

$$\xrightarrow{p} \int_{\mathbb{R}} [w_{1,H}(x) - a_H w_{0,H}]^2 \varrho(x) dx > 0. \quad (322)$$

Finally, in view of (320)-(322) and the fact that $\hat{\sigma}^2 = \sigma_u^2 + o_p(1)$, we obtain

$$\hat{t}_{IVX} = \frac{\frac{1}{\sqrt{n}k_{w_1}(\mu_n)} \sum_{t=1}^n w_1(z_{t-1}) \bar{u}_t}{\left[\frac{\hat{\sigma}^2}{nk_{w_1}^2(\mu_n)} \sum_{t=1}^n \bar{w}_1(z_{t-1})^2 \right]^{1/2}} \xrightarrow{d} N(0, 1),$$

as required. ■

Before we prove **Theorem 15**, we introduce some notation.

Definition 6. Suppose that x_t satisfies either **Assumption SNP** or **Assumption MI**. Define the random array $\tilde{z}_t(n)$ by

$$\tilde{z}_t := \tilde{z}_t(n) := \begin{cases} \sum_{j=1}^t \rho_{z,n}^{t-j} v_j, & \text{under } \mathbf{Assumption SNP} \\ \text{or } \mathbf{Assumption MI} \text{ with } \kappa_n/\kappa_{z,n} \rightarrow \infty; \\ x_t, & \text{under } \mathbf{Assumption MI} \text{ with } \kappa_{z,n}/\kappa_n \rightarrow \infty. \end{cases}$$

Proof of Theorem 15. Set

$$R_t(n) := z_t - \tilde{z}_t.$$

By **Lemma 2**, for all $1 < \alpha' < \alpha$

$$\lim_{n \rightarrow \infty} \sup_{1 \leq t \leq n} \mathbf{E} \left| \frac{1}{\mu_n} R_t(n) \right|^{\alpha'} = 0, \quad (323)$$

where μ_n is given in **Definition 5**. Moreover, note that $\tilde{z}_t$ is a MI process. Therefore, by **Lemma 1**,

$$\frac{1}{\mu_n} \tilde{z}_n \xrightarrow{d} X^+,$$

where X^+ is some α -stable variate. Set $D_n = \sqrt{n} \text{diag}\{k_{w_0}(\mu_n), k_{w_1}(\mu_n)\}$. In view (323), **Theorem 1** and the fact that $\mathbf{W}(x)$ is **AHF** $\{z_t, \mu_n\}$ and **LH**, as $n \rightarrow \infty$ we obtain

$$\begin{aligned} D_n^{-1} \sum_{t=1}^n \mathbf{W}(z_{t-1}) D_n^{-1} &= \frac{1}{n} \sum_{t=1}^n \mathbf{W}_H(\mu_n^{-1} \tilde{z}_{t-1}) + o_p(1) \\ &\xrightarrow{p} \int_{\mathbb{R}} \mathbf{W}_H(x) \varrho(x) dx, \end{aligned} \quad (324)$$

where $\varrho(x)$ is the density of X^+ . In view of (324), the remainder of the proof follows by arguments analogous to those used in the proof of **Theorem 16**. ■

19.3. Proofs of Technical Results for Section 7

Proof of Lemma 2. Proof of Lemma 2. We start with two decompositions of z_t .

- Recall that under assumptions $x_0 = 0$. Summation by parts gives

$$z_t = x_t - \rho_{z,n}^{t-1} x_0 - (1 - \rho_{z,n}) \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j$$

$$\begin{aligned}
&= x_t - \rho_{z,n}^{t-1} x_0 - (1 - \{1 - \kappa_{z,n}^{-1}\}) \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j \\
&= x_t - \frac{1}{\kappa_{z,n}} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j.
\end{aligned} \tag{325}$$

Hence,

$$\frac{1}{\gamma_n} z_t = \frac{1}{\gamma_n} x_t - \frac{1}{\gamma_n \kappa_{z,n}} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j. \tag{326}$$

- Alternatively,

$$\begin{aligned}
z_t &= \sum_{j=1}^t \rho_{z,n}^{t-j} \Delta x_j = \sum_{j=1}^t \rho_{z,n}^{t-j} v_j - \frac{1}{k_n} \sum_{j=1}^t \rho_{z,n}^{t-j} x_{j-1} \\
&= \sum_{j=1}^t \rho_{z,n}^{t-j} v_j - \frac{1}{k_n} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{\gamma_{z,n}} z_t &= \frac{1}{\gamma_{z,n}} \sum_{j=1}^t \rho_{z,n}^{t-j} \Delta x_j \\
&= \frac{1}{\gamma_{z,n}} \sum_{j=1}^t \rho_{z,n}^{t-j} v_j - \frac{1}{\gamma_{z,n} k_n} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j.
\end{aligned}$$

Facts:

- $\gamma_n = R_{\mathcal{H}}(\kappa_n) := \kappa_n^{\mathcal{H}} \varsigma(\kappa_n)$, where $\varsigma(\cdot)$ is slowly varying -i.e. γ_n is regularly varying in κ_n of (Hurst) index $0 < \mathcal{H} < 1$.
- $\gamma_{z,n} = R_{\mathcal{H}}(\kappa_{z,n}) := \kappa_{z,n}^{\mathcal{H}} \varsigma(\kappa_{z,n})$ -i.e. γ_n is regularly varying in $\kappa_{z,n}$ of (Hurst) index $0 < \mathcal{H} < 1$.

(a) Case $\frac{\kappa_n}{\kappa_{z,n}} \rightarrow \infty$: The result follows analogous arguments to those used in part (b) below.

(b) Case $\frac{\kappa_{z,n}}{\kappa_n} \rightarrow \infty$: Recall that we can decompose x_t as

$$x_t = x_t^+ + x_t^-,$$

where

$$x_t^+ = \sum_{k=0}^{t-1} a_k \xi_{t-k}, \quad x_t^- = \sum_{k=t}^{\infty} a_{k,t}^- \xi_{t-k}.$$

Hence, we can write

$$\sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j = \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j^+ + \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j^-.$$

For $1 < \alpha \leq 2$, let

$$0 < \epsilon < (\alpha - 1) \wedge \left[(2 - \alpha) I_{\{\alpha < 2\}} + I_{\{\alpha = 2\}} \right]$$

and set

$$\begin{aligned} \alpha_{\epsilon^-} &:= \alpha - \epsilon \text{ and} \\ \alpha_{\epsilon^+} &:= \begin{cases} \alpha + \epsilon, & \alpha < 2 \\ \alpha, & \alpha = 2 \end{cases}. \end{aligned}$$

- x_t^+ : Write

$$\begin{aligned} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j^+ &= \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} \sum_{k=0}^{j-1} a_k \xi_{j-k} \\ &\quad [l = j - k \rightarrow k = j - l] \\ &= \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} \sum_{l=1}^j a_{j-l} \xi_l = \sum_{l=1}^{t-1} \left(\sum_{j=l}^{t-1} \rho_{z,n}^{t-j-1} \sum_{i=0}^{j-l} \rho_n^{j-l-i} c_i \right) \xi_l \\ &\quad \left. \begin{matrix} 1 \leq j \leq t-1 \\ 1 \leq l \leq j \end{matrix} \right\} \rightarrow \left. \begin{matrix} 1 \leq l \leq t-1 \\ l \leq j \leq t-1 \end{matrix} \right\} \\ &= \sum_{l=1}^{t-1} \left(\sum_{j=l}^{t-1} \rho_{z,n}^{t-j-1} \sum_{i=0}^{j-l} \rho_n^{j-l-i} c_i \right) \xi_l \\ &=: \sum_{l=1}^{t-1} A_{nl} \xi_l =: \mathcal{A}_{n,t}. \end{aligned}$$

Recall that that under **Assumption MI-SM**,

$$\gamma_n \sim \kappa_n^{1/\alpha} \ell_1(\kappa_n)^{1/\alpha}, \quad H(\gamma_n) \sim \ell_1(\kappa_n). \quad (327)$$

Let

$$\begin{aligned} \bar{\xi}_{n,l} &:= \xi_l 1\{|\xi_l| \geq \gamma_n\} - \mathbf{E}[\xi_l 1\{|\xi_l| \geq \gamma_n\}], \\ \underline{\xi}_{n,l} &:= \xi_l 1\{|\xi_l| < \gamma_n\} - \mathbf{E}[\xi_l 1\{|\xi_l| < \gamma_n\}]. \end{aligned}$$

Note that as $n \rightarrow \infty$, cf. Astrauskas (1983, Lemma 1),

$$\mathbf{E} \left| \bar{\xi}_{n,l} \right|^{\alpha_{\epsilon^-}} \leq Ch(\gamma_n) \gamma_n^{-\epsilon} [1 + o(1)],$$

and

$$\mathbf{E} \left| \underline{\xi}_{n,l} \right|^{\alpha_{\epsilon^+}} \leq C [h(\gamma_n) \gamma_n^{\epsilon} 1\{\alpha \neq 2\} + H(\gamma_n) 1\{\alpha = 2\}] [1 + o(1)],$$

where recall that $h(\gamma_n)/H(\gamma_n) < C$. We explicitly consider only the case $\alpha \neq 2$. The proof for the case $\alpha = 2$ follows from analogous arguments to those used below. Write

$$\begin{aligned} \mathbf{E} \left| \frac{1}{\gamma_n \kappa_{z,n}} \mathcal{A}_{n,t} \right|^{\alpha_{\epsilon^-}} &\leq \mathbf{E} \left| \frac{1}{\gamma_n \kappa_{z,n}} \sum_{l=1}^{t-1} A_{nl} \xi_l \right|^{\alpha_{\epsilon^-}} \\ &\leq 2\mathbf{E} \left| \frac{1}{\gamma_n \kappa_{z,n}} \sum_{l=1}^{t-1} A_{nl} \bar{\xi}_{n,l} \right|^{\alpha_{\epsilon^-}} + 2\mathbf{E} \left| \frac{1}{\gamma_n \kappa_{z,n}} \sum_{l=1}^{t-1} A_{nl} \underline{\xi}_{n,l} \right|^{\alpha_{\epsilon^-}} \end{aligned}$$

$$\begin{aligned}
&\leq 2\mathbf{E} \left| \frac{1}{\gamma_n \kappa_{z,n}} \sum_{l=1}^{t-1} A_{nl} \bar{\xi}_{n,l} \right|^{\alpha_{\epsilon}^-} + 2 \left[\mathbf{E} \left| \frac{1}{\gamma_n \kappa_{z,n}} \sum_{l=1}^{t-1} A_{nl} \xi_{n,l} \right|^{\alpha_{\epsilon}^+} \right]^{\frac{\alpha_{\epsilon}^-}{\alpha_{\epsilon}^+}} \\
&\leq_{(1)} 2 \left(\frac{1}{\gamma_n \kappa_{z,n}} \right)^{\alpha_{\epsilon}^-} \sum_{l=1}^{t-1} |A_{nl}|^{\alpha_{\epsilon}^-} \mathbf{E} |\bar{\xi}_{n,l}|^{\alpha_{\epsilon}^-} + 2 \left[\left(\frac{1}{\gamma_n \kappa_{z,n}} \right)^{\alpha_{\epsilon}^+} \sum_{l=1}^{t-1} |A_{nl}|^{\alpha_{\epsilon}^+} \mathbf{E} |\xi_{n,l}|^{\alpha_{\epsilon}^+} \right]^{\frac{\alpha_{\epsilon}^-}{\alpha_{\epsilon}^+}} \\
&\leq C \left(\frac{1}{\gamma_n \kappa_{z,n}} \right)^{\alpha_{\epsilon}^-} \sum_{l=1}^{t-1} |A_{nl}|^{\alpha_{\epsilon}^-} h(\gamma_n) \gamma_n^{-\epsilon} [1 + o(1)] \\
&\quad + C \left[\left(\frac{1}{\gamma_n \kappa_{z,n}} \right)^{\alpha_{\epsilon}^+} \sum_{l=1}^{t-1} |A_{nl}|^{\alpha_{\epsilon}^+} h(\gamma_n) \gamma_n^{\epsilon} [1 + o(1)] \right]^{\frac{\alpha_{\epsilon}^-}{\alpha_{\epsilon}^+}} \\
&\leq C \left(\frac{1}{\gamma_n^{\alpha}} \right) \sum_{l=1}^{t-1} \frac{1}{\kappa_{z,n}^{\frac{\alpha_{\epsilon}^-}{\alpha}}} |A_{nl}|^{\alpha_{\epsilon}^-} h(\gamma_n) [1 + o(1)] + C \left[\frac{1}{\gamma_n^{\alpha}} \sum_{l=1}^{t-1} \frac{1}{\kappa_{z,n}^{\frac{\alpha_{\epsilon}^+}{\alpha}}} |A_{nl}|^{\alpha_{\epsilon}^+} h(\gamma_n) [1 + o(1)] \right]^{\frac{\alpha_{\epsilon}^-}{\alpha_{\epsilon}^+}} \\
&\leq C \frac{1}{\gamma_n^{\alpha}} \sum_{l=1}^{t-1} \frac{1}{\kappa_{z,n}^{\frac{\alpha_{\epsilon}^-}{\alpha}}} |A_{nl}|^{\alpha_{\epsilon}^-} H(\gamma_n) [1 + o(1)] + C \left[\frac{1}{\gamma_n^{\alpha}} \sum_{l=1}^{t-1} \frac{1}{\kappa_{z,n}^{\frac{\alpha_{\epsilon}^+}{\alpha}}} |A_{nl}|^{\alpha_{\epsilon}^+} H(\gamma_n) [1 + o(1)] \right]^{\frac{\alpha_{\epsilon}^-}{\alpha_{\epsilon}^+}} \\
&\leq C \frac{1}{\kappa_n} \sum_{l=1}^{t-1} \frac{1}{\kappa_{z,n}^{\frac{\alpha_{\epsilon}^-}{\alpha}}} |A_{nl}|^{\alpha_{\epsilon}^-} [1 + o(1)] + C \left[\frac{1}{\kappa_n} \sum_{l=1}^{t-1} \frac{1}{\kappa_{z,n}^{\frac{\alpha_{\epsilon}^+}{\alpha}}} |A_{nl}|^{\alpha_{\epsilon}^+} [1 + o(1)] \right]^{\frac{\alpha_{\epsilon}^-}{\alpha_{\epsilon}^+}},
\end{aligned}$$

where $\leq_{(1)}$ follows from Bahr-Esseen inequality if the exponents are greater than one and the Loève inequality when there less or equal to unity.

Next, set

$$\alpha_{\epsilon} := \{\alpha_{\epsilon}^+, \alpha_{\epsilon}^-\}.$$

Moreover,

$$R_{1nt} := \frac{1}{\kappa_n} \sum_{l=1}^{t-1} \frac{1}{\kappa_{z,n}^{\alpha_{\epsilon}}} |A_{nl}|^{\alpha_{\epsilon}} = \frac{1}{\kappa_n} \frac{1}{\kappa_{z,n}^{\alpha_{\epsilon}}} \sum_{l=1}^{t-1} \left| \sum_{j=l}^{t-1} \rho_{z,n}^{t-j-1} \sum_{i=0}^{j-l} \rho_n^{j-l-i} c_i \right|^{\alpha_{\epsilon}}.$$

The summand

$$\begin{aligned}
&\sum_{j=l}^{t-1} \rho_{z,n}^{t-j-1} \sum_{i=0}^{j-l} \rho_n^{j-l-i} c_i \\
&\quad [k = j - l \rightarrow j = k + l] \\
&= \sum_{k=0}^{t-1-l} \rho_{z,n}^{t-(k+l)-1} \sum_{i=0}^{k+l-l} \rho_n^{k+l-l-i} c_i = \sum_{k=0}^{t-1-l} \rho_{z,n}^{t-1-l-k} \sum_{i=0}^k \rho_n^{k-i} c_i. \tag{328}
\end{aligned}$$

Set

$$B_n(h) := \begin{cases} \sum_{l=0}^h \rho_{z,n}^{h-l} \rho_n^l, & h \geq 0 \\ 0, & h < 0 \end{cases}.$$

Note that for $p \geq 1$, and n large enough such that $0 < \rho_{z,n}, \rho_n < 1$ and $q \in \mathbb{Z}$,

$$\begin{aligned}
\left[\sum_{h=0}^{\infty} B_n(q+h)^p \right]^{1/p} &= \left[\sum_{h=q}^{\infty} B_n(h)^p I_{\{h \geq 0\}} \right]^{1/p} \leq \left[\sum_{h=0}^{\infty} B_n(h)^p \right]^{1/p} \\
&= \left[\sum_{h=0}^{\infty} \left(\sum_{l=0}^h \rho_{z,n}^{h-l} \rho_n^l \right)^p \right]^{1/p} = \left[\sum_{h=0}^{\infty} \left[\sum_{l=0}^{\infty} \rho_{z,n}^{h-l} \rho_n^l I_{\{h \geq l\}} \right]^p \right]^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\leq_{(1)} \sum_{l=0}^{\infty} \left[\sum_{h=0}^{\infty} \left[\rho_{z,n}^{h-l} \rho_n^l I_{\{h \geq l\}} \right]^p \right]^{1/p} = \sum_{l=0}^{\infty} \rho_n^l \left[\sum_{h=0}^{\infty} \left[\rho_{z,n}^{p(h-l)} I_{\{h \geq l\}} \right] \right]^{1/p} \\
&\quad [j = h - l] \\
&= \sum_{l=0}^{\infty} \rho_n^l \left[\sum_{j=-l}^{\infty} \rho_{z,n}^{pj} I_{\{j \geq 0\}} \right]^{1/p} = \sum_{l=0}^{\infty} \rho_n^l \left[\sum_{j=0}^{\infty} \rho_{z,n}^{pj} \right]^{1/p} \leq C \kappa_n (\kappa_{z,n})^{1/p}, \quad (329)
\end{aligned}$$

where $\leq_{(1)}$ follows from Minkowski's inequality. Moreover, we have used the fact that

$$\sum_{l=0}^{\infty} \rho_n^l = \frac{1}{1 - \rho_n} = \kappa_n,$$

and that as $n \rightarrow \infty$,

$$\sum_{j=0}^{\infty} \rho_{z,n}^{pj} = \frac{1}{1 - (1 - \kappa_{z,n}^{-1})^p} \sim \frac{\kappa_{z,n}}{p}.$$

First, assume that $1 < \alpha \leq 2$. In this case $\alpha_\epsilon > 1$. In view of (328),

$$\begin{aligned}
\kappa_n \kappa_{z,n}^{\alpha_\epsilon} R_{1nt} &= \sum_{l=1}^{t-1} \left| \sum_{k=0}^{t-1-l} \rho_{z,n}^{t-1-l-k} \sum_{i=0}^k \rho_n^{k-i} c_i \right|^{\alpha_\epsilon} \\
&\quad [j = t - 1 - l] \\
&= \sum_{j=0}^{t-2} \left| \sum_{k=0}^j \rho_{z,n}^{j-k} \sum_{i=0}^k \rho_n^{k-i} c_i \right|^{\alpha_\epsilon} = \sum_{j=0}^{t-2} \left| \sum_{k=0}^j \rho_{z,n}^{j-k} \sum_{i=0}^k \rho_n^{k-i} c_i \right|^{\alpha_\epsilon} \\
&\quad \left. \begin{matrix} 0 \leq k \leq j \\ 0 \leq i \leq k \end{matrix} \right\} \rightarrow \left. \begin{matrix} 0 \leq i \leq j \\ i \leq k \leq j \end{matrix} \right\} \\
&= \sum_{j=0}^{t-2} \left| \sum_{i=0}^j c_i \sum_{k=i}^j \rho_{z,n}^{j-k} \rho_n^{k-i} \right|^{\alpha_\epsilon} \\
&\quad l = k - i \rightarrow k = l + i \\
&= \sum_{j=0}^{t-2} \left| \sum_{i=0}^j c_i \sum_{k=0}^{j-i} \rho_{z,n}^{j-(l+i)} \rho_n^l \right|^{\alpha_\epsilon} = \sum_{j=0}^{t-2} \left| \sum_{i=0}^j c_i \sum_{l=0}^{j-i} \rho_{z,n}^{j-i-l} \rho_n^l \right|^{\alpha_\epsilon} \\
&= \sum_{j=0}^{t-2} \left| \sum_{i=0}^j c_i B_n(j-i) \right|^{\alpha_\epsilon} \leq \sum_{j=0}^{t-2} \left| \sum_{i=0}^j |c_i| B_n(j-i) \right|^{\alpha_\epsilon} \quad (330) \\
&= \left\{ \left[\sum_{j=0}^{t-2} \left| \sum_{i=0}^{\infty} |c_i| B_n(j-i) \right|^{\alpha_\epsilon} \right]^{1/\alpha_\epsilon} \right\}^{\alpha_\epsilon} \leq_{(2)} \left\{ \sum_{i=0}^{\infty} \left[\sum_{j=0}^{t-2} [|c_i| B_n(j-i)]^{\alpha_\epsilon} \right]^{1/\alpha_\epsilon} \right\}^{\alpha_\epsilon} \\
&\leq \left\{ \sum_{i=0}^{\infty} |c_i| \left[\sum_{j=0}^t B_n(j-i)^{\alpha_\epsilon} \right]^{1/\alpha_\epsilon} \right\}^{\alpha_\epsilon} \leq \left\{ \sum_{i=0}^{\infty} |c_i| \left[\sum_{j=0}^{\infty} B_n(j-i)^{\alpha_\epsilon} \right]^{1/\alpha_\epsilon} \right\}^{\alpha_\epsilon} \\
&\leq_{(3)} C \left\{ \sum_{i=0}^{\infty} |c_i| \kappa_n \kappa_{z,n}^{1/\alpha_\epsilon} \right\}^{\alpha_\epsilon}, \quad (331)
\end{aligned}$$

where $\leq_{(2)}$ follows from Minkowski's inequality and (329). In view of (331), as $n \rightarrow \infty$, we get

$$\begin{aligned} R_{1nt} &\leq \frac{1}{\kappa_n \kappa_{z,n}^{\alpha_\epsilon}} \left\{ \sum_{i=0}^{\infty} |c_i| \left[\sum_{j=0}^t B_n(j-i)^{\alpha_\epsilon} \right]^{1/\alpha_\epsilon} \right\}^{\alpha_\epsilon} \leq \frac{C}{\kappa_n \kappa_{z,n}^{\alpha_\epsilon}} \left\{ \sum_{i=0}^{\infty} |c_i| \kappa_n \kappa_{z,n}^{1/\alpha_\epsilon} \right\}^{\alpha_\epsilon} \\ &\leq \frac{C}{\kappa_n \kappa_{z,n}^{\alpha_\epsilon}} \kappa_n^{\alpha_\epsilon} \kappa_{z,n} = C \frac{\kappa_n^{\alpha_\epsilon-1}}{\kappa_{z,n}^{\alpha_\epsilon-1}} \rightarrow 0. \end{aligned} \quad (332)$$

Next suppose that $0 < \alpha \leq 1$. This case $0 < \alpha_{\epsilon^-} < 1$, while α_{ϵ^+} can be either greater or smaller than unity. Therefore, we either have $\alpha_\epsilon > 1$ or $\alpha_\epsilon \leq 1$. For the former case, the result follows using the same arguments as those used for establishing (332). Therefore, let's assume that $0 < \alpha_\epsilon \leq 1$. In view of (330),

$$\begin{aligned} \kappa_n \kappa_{z,n}^{\alpha_\epsilon} R_{1nt} &\leq \sum_{j=0}^{t-2} \left| \sum_{i=0}^j |c_i| B_n(j-i) \right|^{\alpha_\epsilon} \\ &\leq \sum_{j=0}^{t-2} \sum_{i=0}^j |c_i|^{\alpha_\epsilon} B_n(j-i)^{\alpha_\epsilon} \leq \sum_{i=0}^{\infty} |c_i|^{\alpha_\epsilon} \sum_{h=0}^{\infty} B_n(h)^{\alpha_\epsilon}. \end{aligned}$$

Next, note that

$$\begin{aligned} \sum_{h=0}^{\infty} B_n^{\alpha_\epsilon}(h) &= \sum_{h=0}^{\infty} \left[\sum_{l=0}^h \rho_{z,n}^{h-l} \rho_n^l \right]^{\alpha_\epsilon} = \sum_{h=0}^{\infty} \left[\rho_{z,n}^h \sum_{l=0}^h \left(\frac{\rho_n}{\rho_{z,n}} \right)^l \right]^{\alpha_\epsilon} \\ &\leq \sum_{h=0}^{\infty} \left[\rho_{z,n}^h \sum_{l=0}^{\infty} \left(\frac{\rho_n}{\rho_{z,n}} \right)^l \right]^{\alpha_\epsilon} \leq \sum_{h=0}^{\infty} \rho_{z,n}^{\alpha_\epsilon h} \left[\sum_{l=0}^{\infty} \left(\frac{\rho_n}{\rho_{z,n}} \right)^l \right]^{\alpha_\epsilon} \\ &\leq \sum_{h=0}^{\infty} \rho_{z,n}^{\alpha_\epsilon h} \left[\sum_{l=0}^{\infty} \left(\frac{\rho_n}{\rho_{z,n}} \right)^l \right]^{\alpha_\epsilon} = \frac{1}{1 - \rho_{z,n}^{\alpha_\epsilon}} \left[\frac{1}{1 - \frac{\rho_n}{\rho_{z,n}}} \right]^{\alpha_\epsilon} \\ &= \frac{1}{1 - \rho_{z,n}^{\alpha_\epsilon}} \left[\frac{\rho_{z,n}}{\rho_{z,n} - \rho_n} \right]^{\alpha_\epsilon} = \frac{1}{1 - \rho_{z,n}^{\alpha_\epsilon}} \left[\frac{\rho_{z,n}}{\rho_{z,n} - \rho_n} \right]^{\alpha_\epsilon} = \frac{1}{1 - \rho_{z,n}^{\alpha_\epsilon}} \left[\frac{\rho_{z,n}}{\kappa_n^{-1} - \kappa_{z,n}^{-1}} \right]^{\alpha_\epsilon} \\ &= \frac{1}{1 - \rho_{z,n}^{\alpha_\epsilon}} \left[\frac{\kappa_n \rho_{z,n}}{1 - \kappa_n \kappa_{z,n}^{-1}} \right]^{\alpha_\epsilon} \sim \frac{\kappa_{z,n}}{\alpha_\epsilon} \kappa_n^{\alpha_\epsilon}. \end{aligned}$$

Hence,

$$R_{1nt} \leq C \frac{1}{\kappa_n \kappa_{z,n}^{\alpha_\epsilon}} \frac{\kappa_{z,n}}{\alpha_\epsilon} \kappa_n^{\alpha_\epsilon} \rightarrow \infty$$

- x_t^- : Recall that

$$x_t^- = \sum_{k=t}^{\infty} a_{k,t}^-(n) \xi_{t-k} = \sum_{k=t}^{\infty} \left(\sum_{i=0}^{t-1} \rho_n^i c_{k-i} \right) \xi_{t-k}$$

Hence,

$$\begin{aligned} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j^- &= \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} \sum_{k=j}^{\infty} \left(\sum_{i=0}^{j-1} \rho_n^i c_{k-i} \right) \xi_{j-k} \\ &\quad [-l = j - k \rightarrow l = k - j \rightarrow k = l + j] \\ &= \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} \sum_{l=0}^{\infty} \left(\sum_{i=0}^{j-1} \rho_n^i c_{l+j-i} \right) \xi_{-l} \end{aligned}$$

$$= \sum_{l=0}^{\infty} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} \left(\sum_{i=0}^{j-1} \rho_n^i c_{l+j-i} \right) \xi_{-l}.$$

Set

$$R_{2nt} := \frac{1}{\kappa_n \kappa_{z,n}^{\alpha_\epsilon}} \sum_{l=0}^{\infty} \left| \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} \left(\sum_{i=0}^{j-1} \rho_n^i c_{l+j-i} \right) \right|^{\alpha_\epsilon}.$$

Using arguments analogous to those employed in the proof for x_t^+ it is easily seen that it suffices to establish

$$\sup_{1 \leq t \leq n} R_{2nt} = o(1),$$

to obtain the requisite result. Set $I_l(r) := I_{\{l+1 \leq r \leq l+t-1\}}$ and consider

$$\begin{aligned} \kappa_n \kappa_{z,n}^{\alpha_\epsilon} R_{2nt} &:= \sum_{l=0}^{\infty} \left| \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} \left(\sum_{i=0}^{j-1} \rho_n^i c_{l+j-i} \right) \right|^{\alpha_\epsilon} \\ &= \sum_{l=0}^{\infty} \left| \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} \sum_{q=1}^j \rho_n^{j-q} c_{l+q} \right|^{\alpha_\epsilon} \\ &\quad \left. \begin{matrix} 1 \leq j \leq t-1 \\ 1 \leq q \leq j \end{matrix} \right\} \rightarrow \left. \begin{matrix} 1 \leq q \leq t-1 \\ q \leq j \leq t-1 \end{matrix} \right\} \\ &= \sum_{l=0}^{\infty} \left| \sum_{q=1}^{t-1} c_{l+q} \sum_{j=q}^{t-1} \rho_{z,n}^{t-j-1} \rho_n^{j-q} \right|^{\alpha_\epsilon} \\ &\quad [r = j - q \iff j = r + q] \\ &= \sum_{l=0}^{\infty} \left| \sum_{q=1}^{t-1} c_{l+q} \sum_{r=0}^{t-1-q} \rho_{z,n}^{t-1-q-r} \rho_n^r \right|^{\alpha_\epsilon} =: \sum_{l=0}^{\infty} \left| \sum_{q=1}^{t-1} c_{l+q} B_n(t-1-q) \right|^{\alpha_\epsilon} \\ &\quad [r = l + q \iff q = r - l] \\ &= \sum_{l=0}^{\infty} \left| \sum_{r=l+1}^{l+t-1} c_r B_n(l+t-1-r) \right|^{\alpha_\epsilon} = \sum_{l=0}^{\infty} \left| \sum_{r=0}^{\infty} c_r B_n(l+t-1-r) I_l(r) \right|^{\alpha_\epsilon} \\ &=: \left\{ \left[\sum_{l=0}^{\infty} \left| \sum_{r=0}^{\infty} c_r B_n(l+t-1-r) I_l(r) \right|^{\alpha_\epsilon} \right]^{\frac{1}{\alpha_\epsilon}} \right\}^{\alpha_\epsilon} \leq_{(4)} \left\{ \sum_{r=0}^{\infty} \left[\sum_{l=0}^{\infty} |I_l(r)| |c_r B_n(l+t-1-r)|^{\alpha_\epsilon} \right]^{\frac{1}{\alpha_\epsilon}} \right\}^{\alpha_\epsilon} \\ &= \left\{ \sum_{r=0}^{\infty} |c_r| \left[\sum_{l=0}^{\infty} |I_l(r)| |B_n(l+t-1-r)|^{\alpha_\epsilon} \right]^{\frac{1}{\alpha_\epsilon}} \right\}^{\alpha_\epsilon} \leq \left\{ \sum_{r=0}^{\infty} |c_r| \left[\sum_{l=0}^{\infty} |B_n(l+t-1-r)|^{\alpha_\epsilon} \right]^{\frac{1}{\alpha_\epsilon}} \right\}^{\alpha_\epsilon} \\ &\leq_{(5)} C \left\{ \sum_{r=0}^{\infty} |c_r| \kappa_n(\kappa_{z,n})^{\frac{1}{\alpha_\epsilon}} \right\}^{\alpha_\epsilon}, \tag{333} \end{aligned}$$

where $\leq_{(4)}$ follows from Mikowski's inequality and $\leq_{(5)}$ from (329). In view of (333), as $n \rightarrow \infty$, we get

$$R_{2nt} \leq C \kappa_n^{-1} \kappa_{z,n}^{-\alpha_\epsilon} \left\{ \sum_{r=0}^{\infty} |c_r| \kappa_n(\kappa_{z,n})^{\frac{1}{\alpha_\epsilon}} \right\}^{\alpha_\epsilon} \leq C \kappa_n^{-1} \kappa_{z,n}^{-\alpha_\epsilon} \left\{ \kappa_n(\kappa_{z,n})^{\frac{1}{\alpha_\epsilon}} \right\}^{\alpha_\epsilon}$$

$$= C\kappa_n^{-1}\kappa_{z,n}^{-\alpha_\epsilon}\kappa_n^{\alpha_\epsilon}\kappa_{z,n} = C\kappa_{z,n}^{1-\alpha_\epsilon}\kappa_n^{\alpha_\epsilon-1} = o(1).$$

(c) First, recall that $\gamma_{z,n} = R_{\mathcal{H}}(\kappa_{z,n}) := \kappa_{z,n}^{\mathcal{H}} \varsigma(\kappa_{z,n})$ -i.e. γ_n is regularly varying in $\kappa_{z,n}$ of (Hurst) index $0 < \mathcal{H} < 1$. Notice that by **Lemma FS $_{\alpha}$ A**, for all $0 < \alpha' < \alpha$,

$$\sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left| \frac{1}{\gamma_{z,n}} \sum_{j=1}^t \rho_{z,n}^{t-j} x_j \right|^{\alpha'} = \sup_{n \geq 1, 1 \leq t \leq n} \mathbf{E} \left| \frac{1}{\gamma_{z,n}} \sum_{j=1}^t \rho_{z,n}^{t-j} v_j \right|^{\alpha'} < \infty.$$

In view of this, as $n \rightarrow \infty$,

$$\begin{aligned} \sup_{1 \leq t \leq n} \mathbf{E} \left| \frac{1}{\kappa_{z,n}} \sum_{j=1}^t \rho_{z,n}^{t-j} x_j \right|^{\alpha'} &= \left(\frac{\gamma_{z,n}}{\kappa_{z,n}} \right)^{\alpha'} \sup_{1 \leq t \leq n} \mathbf{E} \left| \frac{1}{\gamma_{z,n}} \sum_{j=1}^t \rho_{z,n}^{t-j} x_j \right|^{\alpha'} \\ &\leq C \left(\frac{\gamma_{z,n}}{\kappa_{z,n}} \right)^{\alpha'} = C \left(\frac{R_{\mathcal{H}}(\kappa_{z,n})}{\kappa_{z,n}} \right)^{\alpha'} = o(1). \end{aligned}$$

In view of the above and (325), as $n \rightarrow \infty$, we get

$$\sup_{1 \leq t \leq n} \mathbf{E} |z_t - x_t|^{\alpha'} = \sup_{1 \leq t \leq n} \mathbf{E} \left| \frac{1}{\kappa_{z,n}} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j \right|^{\alpha'} = o(1).$$

The proof is complete. $\blacksquare$

19.4. Proofs of Supporting Results for Section 7

Proof of Lemma 23. Recall that $\{\xi_t\}$ is i.i.d. with characteristic function

$$\psi_{\xi}(\lambda) := \mathbf{E} \left(e^{i\lambda \xi_t} \right).$$

Moreover, we can write

$$z_t = x_t - \frac{1}{\kappa_{z,n}} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j =: x_t - z_t^-. \quad (334)$$

Since

$$x_t = \sum_{k=0}^{\infty} c_k \xi_{t-k},$$

the characteristic function of the first term is

$$\phi_{x_t}(\lambda) := \mathbf{E} \left(e^{i\lambda x_t} \right) = \prod_{k=0}^{\infty} \psi_{\xi}(\lambda c_k). \quad (335)$$

Next, consider z_t^- . Write

$$\begin{aligned} z_t^- &:= \frac{1}{\kappa_{z,n}} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} x_j = \frac{1}{\kappa_{z,n}} \sum_{j=1}^{t-1} \rho_{z,n}^{t-j-1} \sum_{i=0}^{\infty} c_i \xi_{j-i} \\ &= \frac{1}{\kappa_{z,n}} \sum_{h=0}^{t-2} \rho_{z,n}^h \sum_{i=0}^{\infty} c_i \xi_{t-h-1-i} = \frac{1}{\kappa_{z,n}} \sum_{i=0}^{\infty} \sum_{h=0}^{t-2} \rho_{z,n}^h c_i \xi_{t-h-1-i} \\ &\quad [k = h + 1 + i \iff h = k - i - 1] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\kappa_{z,n}} \sum_{i=0}^{\infty} \sum_{k=i+1}^{t+i-1} \rho_{z,n}^{k-i-1} c_i \xi_{t-k} = \frac{1}{\kappa_{z,n}} \sum_{i=0}^{\infty} \sum_{k=i+1}^{t+i-1} \rho_{z,n}^{k-i-1} c_i \xi_{t-k} \\
&\left[\begin{array}{c} 0 \leq i \leq \infty \\ i+1 \leq k \leq t-2+i+1 \end{array} \right] \iff \left[\begin{array}{c} 1 \leq k \leq \infty \\ 0 \vee (k+1-t) \leq i \leq k-1 \end{array} \right] \\
&= \frac{1}{\kappa_{z,n}} \sum_{k=1}^{\infty} \sum_{i=0 \vee (k+1-t)}^{k-1} \rho_{z,n}^{k-i-1} c_i \xi_{t-k} =: \frac{1}{\kappa_{z,n}} \sum_{k=1}^{\infty} b_{t,k} \xi_{t-k}. \tag{336}
\end{aligned}$$

Hence, the characteristic function of z_t^- is

$$\phi_{z_t^-}(\lambda) = \prod_{k=1}^{\infty} \psi_{\xi} \left(\lambda \frac{b_{t,k}}{\kappa_{z,n}} \right).$$

Specifically, after a change of variables we can write the coefficients $b_{t,k}$ as

$$\begin{aligned}
b_{t,k} &= \sum_{i=0 \vee (k+1-t)}^{k-1} \rho_{z,n}^{k-i-1} c_i \\
&[l = k - i - 1 \iff i = k - l - 1] \\
&= \sum_{l=0}^{k-1-(0 \vee (k+1-t))} \rho_{z,n}^l c_{k-1-l} = \sum_{l=0}^{(k-1-0) \wedge (k-1-(k+1-t))} \rho_{z,n}^l c_{k-1-l} \\
&= \sum_{l=0}^{(k-1) \wedge (t-2)} \rho_{z,n}^l c_{k-1-l}. \tag{337}
\end{aligned}$$

Next, we find an expression for the characteristic function of z_t . Using (334) and (336) we obtain

$$z_t = c_0 \xi_t + \sum_{k=1}^{\infty} \left(c_k - \frac{1}{\kappa_{z,n}} b_{t,k} \right) \xi_{t-k}.$$

Consequently,

$$\phi_{z_t}(\lambda) = \psi_{\xi}(\lambda c_0) \prod_{k=1}^{\infty} \psi_{\xi} \left(\lambda \left[c_k - \frac{1}{\kappa_{z,n}} b_{t,k} \right] \right). \tag{338}$$

For $k \geq 0$, write

$$\tilde{c}_{k,t} := \begin{cases} c_0, & k = 0, \\ c_k - \frac{1}{\kappa_{z,n}} b_{t,k}, & k \geq 1 \end{cases}.$$

In particular, using (337) note that for $k \geq 1$,

$$\tilde{c}_{k,t} = c_k - \frac{1}{\kappa_{z,n}} b_{t,k} = c_k - \frac{1}{\kappa_{z,n}} \sum_{l=0}^{(k-1) \wedge (t-2)} \rho_{z,n}^l c_{k-1-l}.$$

Under **Assumption SP**, there is at least one coefficient in the sequence $\{c_k\}_{k \geq 0}$ that is nonzero. Let $k_* \geq 0$ the minimal index for which $c_{k_*} \neq 0$. In view of the above we have,

$$\tilde{c}_{k_*,t} = c_{k_*} \neq 0.$$

By virtue of (338), we get

$$\begin{aligned}
&\int_{\mathbb{R}} |\lambda| |\phi_{z_t}(\lambda)| d\lambda = \int_{\mathbb{R}} |\lambda| \left| \prod_{k=0}^{\infty} \psi_{\xi}(\lambda \tilde{c}_{k,t}) \right| d\lambda \\
&\leq \int_{\mathbb{R}} |\lambda| |\psi_{\xi}(\lambda \tilde{c}_{k_*,t})| d\lambda = |\tilde{c}_{k_*,t}|^{-2} \int_{\mathbb{R}} |\lambda| |\psi_{\xi}(\lambda)| d\lambda < \infty,
\end{aligned}$$

uniformly in $1 \leq t \leq n$, $n \geq 1$, as required. ■

20. Simulation Results

This section collects the simulation evidence discussed in Section 8. The four sets of size tables use the eight stability parameters in the order

$$\alpha = 2, 1.99, 1.8, 1.6, 1.4, 1.2, 1, 0.8.$$

Tables 6, 7, 8, 9, 10, 11, 12, and 13 report empirical size under DGP1 for a linear regression function with the fixed weight $w(x) = x^{(0.5)}$.

Tables 14, 15, 16, 17, 18, 19, 20, and 21 report empirical size under DGP1 for a linear regression function with alternative weight functions.

Tables 22, 23, 24, 25, 26, 27, 28, and 29 report empirical size under DGP1 for the nonlinear regression specification.

Tables 30, 31, 32, 33, 34, 35, 36, and 37 report empirical size under DGP2 for a linear regression function with alternative weight functions.

Figures 1, 2, 3, and 4 report empirical power for the linear regression specification with weight exponents 0, 0.5, 0.75, 1, respectively. Figures 5, 6, 7, and 8 report the corresponding size-adjusted power. Figure 9 reports size-adjusted power for the nonlinear specification. Each size-adjusted-power figure continues on a second page.

Empirical Size

Table 6: Empirical size (nominal size 5%): $[\alpha, \nu] = [2, 2.1]$; **DGP1**; $[f(x), w(x)] = [x, x^{(0.5)}]$; $d_* = 0.5$

n	d	W-IVX					WLS					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.10	0.084	0.082	0.080	0.082	0.082	0.083	0.083	0.081	0.081	0.083	0.062	0.063	0.065	0.064	0.061
500	-0.10	0.080	0.078	0.083	0.078	0.081	0.080	0.077	0.084	0.078	0.082	0.055	0.060	0.062	0.060	0.058
1000	-0.10	0.077	0.075	0.077	0.073	0.076	0.077	0.076	0.077	0.073	0.077	0.054	0.059	0.058	0.052	0.055
250	0.10	0.081	0.082	0.082	0.084	0.083	0.082	0.084	0.082	0.085	0.084	0.062	0.065	0.065	0.066	0.062
500	0.10	0.077	0.077	0.081	0.079	0.082	0.078	0.078	0.081	0.080	0.083	0.056	0.059	0.063	0.058	0.058
1000	0.10	0.079	0.078	0.076	0.076	0.077	0.079	0.078	0.076	0.077	0.078	0.054	0.057	0.058	0.055	0.054
250	0.30	0.079	0.081	0.078	0.084	0.080	0.094	0.091	0.084	0.093	0.093	0.068	0.069	0.071	0.072	0.068
500	0.30	0.075	0.076	0.078	0.075	0.078	0.086	0.085	0.080	0.082	0.089	0.062	0.063	0.066	0.060	0.061
1000	0.30	0.078	0.076	0.075	0.076	0.073	0.087	0.081	0.078	0.080	0.081	0.060	0.061	0.059	0.059	0.059
250	0.50	0.085	0.084	0.083	0.086	0.083	0.128	0.112	0.094	0.116	0.128	0.103	0.093	0.082	0.096	0.103
500	0.50	0.077	0.077	0.079	0.076	0.078	0.116	0.103	0.091	0.102	0.119	0.093	0.085	0.076	0.083	0.094
1000	0.50	0.079	0.077	0.075	0.074	0.076	0.117	0.102	0.084	0.097	0.112	0.092	0.083	0.070	0.079	0.088
250	0.70	0.080	0.083	0.085	0.082	0.080	0.196	0.155	0.107	0.157	0.196	0.190	0.148	0.100	0.151	0.191
500	0.70	0.073	0.075	0.080	0.073	0.071	0.188	0.149	0.099	0.145	0.186	0.183	0.143	0.092	0.137	0.183
1000	0.70	0.074	0.071	0.076	0.072	0.075	0.188	0.142	0.097	0.140	0.180	0.182	0.137	0.089	0.137	0.177
250	0.90	0.067	0.070	0.083	0.076	0.067	0.283	0.207	0.119	0.214	0.282	0.302	0.216	0.115	0.217	0.300
500	0.90	0.059	0.067	0.080	0.063	0.059	0.280	0.203	0.110	0.201	0.278	0.300	0.211	0.105	0.205	0.298
1000	0.90	0.061	0.061	0.078	0.061	0.060	0.279	0.197	0.106	0.193	0.269	0.297	0.202	0.104	0.203	0.293
250	1.10	0.064	0.065	0.086	0.071	0.064	0.349	0.243	0.127	0.249	0.353	0.381	0.257	0.121	0.259	0.384
500	1.10	0.058	0.065	0.080	0.061	0.059	0.346	0.238	0.116	0.233	0.343	0.377	0.250	0.113	0.246	0.373
1000	1.10	0.059	0.058	0.075	0.059	0.057	0.341	0.234	0.111	0.229	0.331	0.376	0.250	0.110	0.246	0.369

Table 7: Empirical size (nominal size 5%): $[\alpha, \nu] = [1.99, 2.1]$; **DGP1**; $[f(x), w(x)] = [x, x^{(0.5)}]$; $d_* = 0.497$

n	d	W-IVX					WLS					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.103	0.080	0.081	0.083	0.081	0.082	0.081	0.082	0.083	0.082	0.083	0.060	0.062	0.067	0.066	0.060
500	-0.103	0.081	0.079	0.080	0.080	0.076	0.081	0.079	0.080	0.079	0.078	0.056	0.057	0.063	0.060	0.054
1000	-0.103	0.076	0.075	0.074	0.075	0.079	0.076	0.075	0.074	0.075	0.078	0.053	0.055	0.056	0.057	0.053
250	0.097	0.079	0.080	0.084	0.082	0.082	0.081	0.082	0.085	0.084	0.084	0.060	0.063	0.068	0.068	0.059
500	0.097	0.078	0.078	0.078	0.077	0.076	0.079	0.078	0.078	0.077	0.077	0.057	0.057	0.062	0.060	0.054
1000	0.097	0.075	0.076	0.074	0.077	0.078	0.076	0.077	0.075	0.078	0.079	0.052	0.054	0.056	0.057	0.052
250	0.297	0.077	0.079	0.086	0.081	0.079	0.088	0.088	0.089	0.090	0.092	0.065	0.067	0.073	0.073	0.065
500	0.297	0.076	0.076	0.076	0.076	0.073	0.088	0.084	0.079	0.084	0.083	0.061	0.062	0.065	0.064	0.058
1000	0.297	0.073	0.075	0.073	0.075	0.073	0.083	0.080	0.077	0.081	0.082	0.057	0.060	0.059	0.059	0.055
250	0.497	0.077	0.080	0.087	0.082	0.078	0.121	0.107	0.099	0.110	0.122	0.095	0.090	0.086	0.093	0.100
500	0.497	0.078	0.079	0.077	0.076	0.075	0.119	0.102	0.089	0.102	0.113	0.094	0.086	0.076	0.086	0.089
1000	0.497	0.073	0.076	0.075	0.071	0.077	0.109	0.100	0.084	0.095	0.110	0.086	0.081	0.069	0.076	0.085
250	0.697	0.078	0.079	0.088	0.082	0.076	0.190	0.153	0.110	0.152	0.191	0.182	0.146	0.100	0.145	0.184
500	0.697	0.074	0.074	0.078	0.076	0.072	0.188	0.148	0.097	0.147	0.182	0.182	0.143	0.093	0.143	0.178
1000	0.697	0.068	0.074	0.075	0.069	0.069	0.180	0.144	0.093	0.137	0.178	0.175	0.139	0.086	0.131	0.173
250	0.897	0.063	0.073	0.083	0.074	0.066	0.278	0.210	0.118	0.205	0.279	0.298	0.216	0.115	0.212	0.298
500	0.897	0.062	0.069	0.077	0.065	0.060	0.279	0.205	0.111	0.199	0.272	0.297	0.211	0.107	0.206	0.290
1000	0.897	0.056	0.062	0.073	0.059	0.058	0.270	0.196	0.106	0.195	0.273	0.292	0.203	0.100	0.204	0.293
250	1.097	0.062	0.071	0.083	0.072	0.064	0.340	0.246	0.127	0.242	0.341	0.372	0.259	0.122	0.257	0.374
500	1.097	0.057	0.065	0.079	0.061	0.058	0.343	0.240	0.118	0.236	0.335	0.377	0.253	0.115	0.249	0.364
1000	1.097	0.054	0.058	0.076	0.058	0.056	0.335	0.232	0.113	0.234	0.341	0.373	0.247	0.108	0.249	0.373

Table 8: Empirical size (nominal size 5%): $[\alpha, \nu] = [1.8, 2.1]$; **DGP1**; $[f(x), w(x)] = [x, x^{(0.5)}]$; $d_* = 0.444$

n	d	W-IVX					WLS					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.156	0.074	0.077	0.081	0.077	0.081	0.075	0.078	0.081	0.079	0.082	0.055	0.060	0.064	0.059	0.055
500	-0.156	0.078	0.073	0.078	0.077	0.076	0.078	0.075	0.079	0.078	0.077	0.051	0.053	0.057	0.058	0.052
1000	-0.156	0.073	0.071	0.077	0.074	0.072	0.073	0.071	0.078	0.074	0.073	0.050	0.050	0.057	0.050	0.050
250	0.044	0.078	0.078	0.079	0.079	0.079	0.080	0.080	0.080	0.080	0.082	0.056	0.060	0.061	0.059	0.056
500	0.044	0.077	0.074	0.079	0.079	0.075	0.079	0.076	0.080	0.079	0.077	0.051	0.054	0.057	0.057	0.053
1000	0.044	0.074	0.073	0.078	0.072	0.073	0.076	0.073	0.079	0.072	0.074	0.051	0.050	0.058	0.048	0.049
250	0.244	0.077	0.078	0.079	0.079	0.078	0.089	0.086	0.083	0.088	0.090	0.059	0.062	0.063	0.062	0.059
500	0.244	0.070	0.072	0.076	0.077	0.073	0.080	0.079	0.079	0.083	0.082	0.053	0.057	0.060	0.059	0.057
1000	0.244	0.071	0.068	0.077	0.069	0.071	0.078	0.074	0.079	0.074	0.077	0.051	0.052	0.058	0.051	0.052
250	0.444	0.075	0.077	0.080	0.079	0.079	0.115	0.102	0.093	0.104	0.116	0.081	0.079	0.072	0.079	0.084
500	0.444	0.075	0.075	0.077	0.077	0.075	0.108	0.098	0.088	0.100	0.109	0.074	0.072	0.070	0.074	0.077
1000	0.444	0.071	0.071	0.073	0.069	0.071	0.102	0.093	0.080	0.087	0.099	0.068	0.066	0.065	0.063	0.067
250	0.644	0.078	0.080	0.084	0.079	0.078	0.176	0.141	0.107	0.140	0.174	0.155	0.126	0.093	0.125	0.156
500	0.644	0.072	0.074	0.081	0.077	0.074	0.164	0.136	0.102	0.135	0.168	0.143	0.121	0.091	0.120	0.147
1000	0.644	0.068	0.071	0.074	0.070	0.065	0.155	0.128	0.093	0.127	0.156	0.137	0.113	0.082	0.110	0.138
250	0.844	0.065	0.074	0.084	0.073	0.063	0.259	0.193	0.122	0.193	0.254	0.271	0.191	0.115	0.195	0.262
500	0.844	0.060	0.066	0.080	0.065	0.061	0.253	0.193	0.114	0.187	0.251	0.260	0.193	0.109	0.191	0.261
1000	0.844	0.052	0.063	0.075	0.063	0.056	0.245	0.183	0.106	0.182	0.246	0.252	0.183	0.101	0.182	0.252
250	1.044	0.058	0.068	0.087	0.068	0.057	0.327	0.233	0.133	0.235	0.324	0.358	0.243	0.126	0.246	0.348
500	1.044	0.056	0.062	0.081	0.059	0.055	0.325	0.231	0.123	0.229	0.326	0.352	0.240	0.116	0.241	0.350
1000	1.044	0.051	0.056	0.074	0.057	0.051	0.318	0.225	0.117	0.225	0.314	0.345	0.234	0.112	0.235	0.343

Table 9: Empirical size (nominal size 5%): $[\alpha, \nu] = [1.6, 2.1]$; **DGP1**; $[f(x), w(x)] = [x, x^{(0.5)}]$; $d_* = 0.375$

n	d	W-IVX					WLS					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.225	0.077	0.073	0.074	0.075	0.076	0.078	0.075	0.075	0.077	0.078	0.052	0.053	0.056	0.054	0.051
500	-0.225	0.071	0.071	0.075	0.072	0.071	0.073	0.072	0.076	0.072	0.073	0.051	0.051	0.054	0.053	0.048
1000	-0.225	0.067	0.068	0.071	0.074	0.070	0.068	0.068	0.071	0.074	0.071	0.047	0.045	0.048	0.050	0.046
250	-0.025	0.078	0.076	0.077	0.076	0.076	0.081	0.078	0.079	0.078	0.078	0.053	0.053	0.057	0.056	0.052
500	-0.025	0.071	0.073	0.076	0.072	0.072	0.073	0.074	0.076	0.075	0.074	0.050	0.050	0.056	0.052	0.050
1000	-0.025	0.070	0.068	0.074	0.072	0.071	0.072	0.069	0.075	0.074	0.073	0.047	0.046	0.048	0.049	0.048
250	0.175	0.075	0.074	0.075	0.073	0.073	0.085	0.082	0.079	0.079	0.082	0.054	0.054	0.059	0.055	0.052
500	0.175	0.069	0.070	0.072	0.070	0.069	0.075	0.079	0.076	0.077	0.078	0.050	0.051	0.054	0.053	0.050
1000	0.175	0.069	0.068	0.071	0.068	0.070	0.075	0.073	0.074	0.073	0.076	0.046	0.047	0.049	0.049	0.049
250	0.375	0.077	0.073	0.077	0.076	0.073	0.107	0.098	0.089	0.097	0.104	0.066	0.066	0.065	0.065	0.064
500	0.375	0.067	0.072	0.074	0.069	0.070	0.094	0.092	0.086	0.091	0.096	0.059	0.060	0.062	0.063	0.056
1000	0.375	0.067	0.066	0.071	0.068	0.068	0.095	0.083	0.082	0.082	0.095	0.055	0.054	0.056	0.056	0.056
250	0.575	0.080	0.078	0.077	0.080	0.076	0.155	0.130	0.102	0.131	0.153	0.118	0.100	0.083	0.099	0.114
500	0.575	0.071	0.074	0.077	0.074	0.070	0.142	0.125	0.099	0.121	0.140	0.106	0.096	0.080	0.095	0.105
1000	0.575	0.068	0.070	0.074	0.070	0.070	0.137	0.116	0.091	0.115	0.139	0.101	0.086	0.076	0.087	0.102
250	0.775	0.068	0.073	0.077	0.074	0.066	0.230	0.180	0.114	0.177	0.225	0.219	0.168	0.107	0.165	0.214
500	0.775	0.059	0.066	0.074	0.069	0.058	0.212	0.172	0.110	0.170	0.215	0.203	0.162	0.102	0.161	0.205
1000	0.775	0.055	0.062	0.072	0.062	0.056	0.212	0.163	0.108	0.158	0.211	0.200	0.155	0.097	0.151	0.203
250	0.975	0.056	0.065	0.080	0.065	0.055	0.295	0.222	0.131	0.222	0.296	0.314	0.226	0.124	0.227	0.307
500	0.975	0.052	0.060	0.076	0.060	0.050	0.284	0.220	0.127	0.216	0.291	0.300	0.224	0.120	0.220	0.310
1000	0.975	0.049	0.054	0.073	0.054	0.050	0.286	0.208	0.121	0.200	0.285	0.298	0.211	0.112	0.207	0.301

Table 10: Empirical size (nominal size 5%): $[\alpha, \nu] = [1.4, 2.1]$; **DGP1**; $[f(x), w(x)] = [x, x^{(0.5)}]$; $d_* = 0.286$

n	d	W-IVX					WLS					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.314	0.065	0.069	0.075	0.071	0.068	0.067	0.070	0.076	0.073	0.070	0.045	0.050	0.051	0.049	0.043
500	-0.314	0.066	0.069	0.072	0.069	0.063	0.067	0.070	0.073	0.071	0.065	0.048	0.050	0.051	0.049	0.048
1000	-0.314	0.061	0.064	0.068	0.068	0.065	0.062	0.065	0.069	0.069	0.066	0.043	0.042	0.044	0.045	0.044
250	-0.114	0.065	0.069	0.074	0.071	0.065	0.069	0.072	0.076	0.073	0.069	0.046	0.050	0.050	0.050	0.044
500	-0.114	0.065	0.068	0.071	0.069	0.065	0.067	0.071	0.073	0.072	0.066	0.047	0.049	0.050	0.049	0.047
1000	-0.114	0.062	0.063	0.067	0.068	0.065	0.064	0.065	0.069	0.069	0.067	0.042	0.042	0.043	0.044	0.044
250	0.086	0.066	0.068	0.073	0.071	0.064	0.073	0.077	0.076	0.078	0.073	0.048	0.050	0.051	0.052	0.044
500	0.086	0.066	0.066	0.069	0.069	0.064	0.073	0.071	0.073	0.076	0.070	0.047	0.048	0.050	0.049	0.048
1000	0.086	0.062	0.064	0.065	0.065	0.063	0.067	0.068	0.069	0.068	0.068	0.041	0.043	0.043	0.044	0.043
250	0.286	0.067	0.074	0.072	0.070	0.067	0.092	0.092	0.085	0.090	0.093	0.052	0.056	0.055	0.056	0.050
500	0.286	0.067	0.069	0.067	0.072	0.067	0.088	0.084	0.079	0.088	0.087	0.051	0.050	0.053	0.053	0.048
1000	0.286	0.063	0.066	0.068	0.062	0.059	0.085	0.078	0.076	0.078	0.077	0.045	0.046	0.048	0.045	0.045
250	0.486	0.073	0.077	0.074	0.074	0.072	0.132	0.119	0.096	0.113	0.132	0.078	0.080	0.069	0.076	0.078
500	0.486	0.068	0.069	0.070	0.076	0.068	0.123	0.107	0.091	0.110	0.124	0.075	0.069	0.066	0.073	0.072
1000	0.486	0.067	0.067	0.070	0.066	0.061	0.119	0.103	0.090	0.101	0.108	0.070	0.066	0.062	0.064	0.061
250	0.686	0.067	0.077	0.077	0.072	0.069	0.191	0.160	0.116	0.150	0.193	0.157	0.133	0.098	0.127	0.158
500	0.686	0.064	0.067	0.071	0.073	0.064	0.181	0.146	0.104	0.150	0.185	0.147	0.121	0.088	0.127	0.149
1000	0.686	0.061	0.064	0.070	0.065	0.057	0.175	0.141	0.103	0.145	0.166	0.146	0.116	0.087	0.118	0.132
250	0.886	0.054	0.067	0.077	0.065	0.057	0.261	0.203	0.133	0.197	0.263	0.256	0.199	0.123	0.189	0.259
500	0.886	0.053	0.059	0.069	0.059	0.051	0.254	0.190	0.121	0.193	0.248	0.247	0.183	0.113	0.188	0.245
1000	0.886	0.048	0.052	0.066	0.056	0.046	0.243	0.183	0.117	0.187	0.236	0.242	0.177	0.110	0.183	0.233

Table 11: Empirical size (nominal size 5%): $[\alpha, \nu] = [1.2, 2.1]$; **DGP1**; $[f(x), w(x)] = [x, x^{(0.5)}]$; $d_* = 0.167$

n	d	W-IVX					WLS					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.433	0.066	0.065	0.067	0.067	0.061	0.067	0.066	0.069	0.069	0.063	0.046	0.047	0.050	0.049	0.046
500	-0.433	0.062	0.064	0.068	0.060	0.058	0.062	0.065	0.068	0.061	0.060	0.042	0.046	0.046	0.044	0.044
1000	-0.433	0.060	0.061	0.062	0.060	0.058	0.061	0.062	0.062	0.060	0.058	0.042	0.043	0.041	0.041	0.040
250	-0.233	0.063	0.063	0.067	0.067	0.061	0.065	0.065	0.069	0.069	0.064	0.045	0.045	0.050	0.048	0.044
500	-0.233	0.061	0.062	0.067	0.059	0.057	0.063	0.066	0.069	0.061	0.060	0.041	0.045	0.045	0.042	0.042
1000	-0.233	0.060	0.059	0.060	0.060	0.058	0.062	0.061	0.060	0.061	0.060	0.041	0.042	0.040	0.040	0.039
250	-0.033	0.060	0.063	0.067	0.067	0.061	0.063	0.066	0.069	0.070	0.064	0.045	0.044	0.050	0.048	0.044
500	-0.033	0.059	0.063	0.065	0.059	0.057	0.063	0.066	0.064	0.063	0.061	0.041	0.044	0.044	0.043	0.042
1000	-0.033	0.061	0.059	0.058	0.061	0.057	0.063	0.060	0.061	0.061	0.060	0.041	0.042	0.040	0.041	0.041
250	0.167	0.062	0.064	0.065	0.065	0.061	0.077	0.078	0.078	0.077	0.077	0.046	0.045	0.049	0.049	0.045
500	0.167	0.059	0.062	0.059	0.060	0.059	0.075	0.073	0.069	0.071	0.073	0.042	0.044	0.044	0.044	0.043
1000	0.167	0.058	0.059	0.059	0.058	0.058	0.071	0.071	0.068	0.070	0.072	0.040	0.043	0.041	0.041	0.043
250	0.367	0.066	0.066	0.069	0.068	0.065	0.104	0.097	0.093	0.101	0.102	0.056	0.057	0.058	0.059	0.055
500	0.367	0.064	0.065	0.062	0.065	0.062	0.099	0.093	0.081	0.094	0.099	0.049	0.050	0.048	0.051	0.052
1000	0.367	0.061	0.059	0.064	0.061	0.061	0.091	0.086	0.082	0.086	0.094	0.047	0.047	0.049	0.049	0.049
250	0.567	0.066	0.069	0.074	0.074	0.069	0.149	0.127	0.111	0.129	0.146	0.101	0.088	0.079	0.090	0.097
500	0.567	0.064	0.067	0.067	0.067	0.063	0.138	0.119	0.098	0.123	0.139	0.086	0.082	0.070	0.084	0.087
1000	0.567	0.064	0.062	0.068	0.063	0.061	0.131	0.112	0.097	0.115	0.135	0.081	0.073	0.069	0.074	0.084
250	0.767	0.057	0.064	0.078	0.067	0.059	0.205	0.167	0.129	0.170	0.208	0.178	0.146	0.110	0.148	0.184
500	0.767	0.051	0.059	0.068	0.058	0.051	0.195	0.156	0.117	0.161	0.197	0.172	0.136	0.100	0.137	0.174
1000	0.767	0.052	0.057	0.066	0.056	0.050	0.185	0.152	0.114	0.151	0.192	0.163	0.129	0.098	0.132	0.168

Table 12: Empirical size (nominal size 5%): $[\alpha, \nu] = [1.0, 2.1]$; **DGP1**; $[f(x), w(x)] = [x, x^{(0.5)}]$; $d_* = 0$

n	d	W-IVX					WLS					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.60	0.054	0.061	0.063	0.057	0.054	0.055	0.061	0.064	0.058	0.055	0.043	0.044	0.050	0.045	0.042
500	-0.60	0.051	0.056	0.057	0.056	0.051	0.052	0.057	0.058	0.057	0.051	0.040	0.041	0.044	0.043	0.039
1000	-0.60	0.055	0.051	0.056	0.056	0.049	0.055	0.051	0.056	0.056	0.049	0.043	0.037	0.043	0.042	0.040
250	-0.40	0.052	0.061	0.063	0.058	0.053	0.055	0.062	0.064	0.060	0.056	0.042	0.043	0.049	0.045	0.041
500	-0.40	0.049	0.056	0.056	0.058	0.050	0.052	0.058	0.058	0.059	0.051	0.039	0.040	0.044	0.043	0.038
1000	-0.40	0.055	0.051	0.055	0.057	0.049	0.057	0.052	0.056	0.058	0.050	0.041	0.037	0.042	0.041	0.039
250	-0.20	0.052	0.060	0.061	0.055	0.052	0.056	0.062	0.065	0.058	0.057	0.040	0.041	0.047	0.044	0.041
500	-0.20	0.049	0.055	0.057	0.058	0.049	0.053	0.058	0.060	0.062	0.052	0.039	0.041	0.044	0.041	0.036
1000	-0.20	0.055	0.051	0.054	0.056	0.049	0.057	0.052	0.056	0.059	0.052	0.040	0.037	0.041	0.041	0.038
250	0.00	0.055	0.057	0.061	0.056	0.053	0.056	0.061	0.063	0.058	0.056	0.041	0.043	0.047	0.044	0.043
500	0.00	0.051	0.056	0.056	0.057	0.048	0.055	0.059	0.058	0.059	0.051	0.040	0.042	0.044	0.042	0.036
1000	0.00	0.051	0.051	0.052	0.053	0.051	0.053	0.055	0.054	0.056	0.053	0.041	0.037	0.040	0.040	0.039
250	0.20	0.056	0.059	0.061	0.058	0.057	0.081	0.077	0.076	0.077	0.081	0.044	0.044	0.048	0.045	0.043
500	0.20	0.057	0.058	0.057	0.056	0.052	0.073	0.075	0.071	0.071	0.073	0.041	0.043	0.044	0.040	0.039
1000	0.20	0.054	0.054	0.053	0.055	0.052	0.072	0.067	0.068	0.071	0.070	0.041	0.038	0.042	0.041	0.040
250	0.40	0.060	0.065	0.065	0.065	0.064	0.107	0.099	0.094	0.101	0.112	0.056	0.055	0.058	0.057	0.057
500	0.40	0.058	0.062	0.062	0.057	0.059	0.097	0.096	0.088	0.091	0.103	0.050	0.054	0.051	0.049	0.050
1000	0.40	0.057	0.057	0.057	0.057	0.056	0.094	0.087	0.084	0.092	0.096	0.050	0.047	0.049	0.049	0.046
250	0.60	0.061	0.065	0.069	0.067	0.066	0.150	0.130	0.115	0.132	0.156	0.097	0.089	0.084	0.090	0.101
500	0.60	0.058	0.063	0.065	0.060	0.060	0.139	0.125	0.106	0.122	0.144	0.091	0.085	0.074	0.082	0.090
1000	0.60	0.057	0.058	0.062	0.061	0.055	0.134	0.118	0.101	0.121	0.135	0.085	0.075	0.071	0.078	0.082

Table 13: Empirical size (nominal size 5%): $[\alpha, \nu] = [0.80, 2.1]$; **DGP1**; $[f(x), w(x)] = [x, x^{(0.5)}]$; $d_* = -0.250$

n	d	W-IVX					WLS					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.85	0.051	0.055	0.056	0.054	0.049	0.051	0.055	0.056	0.054	0.049	0.044	0.047	0.044	0.045	0.043
500	-0.85	0.048	0.053	0.053	0.047	0.048	0.049	0.053	0.053	0.047	0.048	0.039	0.043	0.045	0.041	0.042
1000	-0.85	0.046	0.049	0.047	0.048	0.047	0.045	0.049	0.047	0.048	0.047	0.037	0.041	0.040	0.041	0.037
250	-0.65	0.051	0.054	0.052	0.055	0.050	0.052	0.055	0.053	0.055	0.051	0.043	0.046	0.043	0.044	0.042
500	-0.65	0.048	0.051	0.052	0.049	0.048	0.049	0.052	0.053	0.049	0.049	0.038	0.042	0.043	0.040	0.041
1000	-0.65	0.046	0.048	0.048	0.048	0.046	0.046	0.048	0.048	0.048	0.047	0.037	0.039	0.039	0.040	0.037
250	-0.45	0.051	0.054	0.053	0.054	0.048	0.054	0.055	0.055	0.056	0.051	0.042	0.046	0.042	0.044	0.043
500	-0.45	0.048	0.053	0.052	0.047	0.048	0.050	0.055	0.053	0.048	0.049	0.037	0.042	0.042	0.039	0.039
1000	-0.45	0.046	0.048	0.048	0.047	0.047	0.047	0.049	0.049	0.047	0.048	0.035	0.038	0.037	0.039	0.036
250	-0.25	0.049	0.053	0.052	0.053	0.050	0.053	0.057	0.056	0.057	0.054	0.040	0.043	0.040	0.043	0.042
500	-0.25	0.047	0.052	0.053	0.046	0.047	0.051	0.055	0.056	0.050	0.050	0.036	0.041	0.042	0.038	0.039
1000	-0.25	0.046	0.048	0.046	0.045	0.046	0.048	0.050	0.049	0.047	0.049	0.034	0.037	0.037	0.037	0.035
250	-0.05	0.048	0.053	0.051	0.051	0.049	0.051	0.054	0.052	0.053	0.053	0.041	0.042	0.040	0.043	0.041
500	-0.05	0.044	0.050	0.052	0.046	0.045	0.048	0.053	0.053	0.049	0.050	0.037	0.041	0.042	0.038	0.039
1000	-0.05	0.042	0.046	0.047	0.045	0.043	0.045	0.047	0.048	0.046	0.047	0.034	0.037	0.037	0.036	0.035
250	0.15	0.051	0.053	0.049	0.054	0.052	0.070	0.070	0.066	0.071	0.069	0.043	0.043	0.040	0.044	0.042
500	0.15	0.046	0.052	0.053	0.049	0.047	0.062	0.065	0.068	0.063	0.064	0.036	0.041	0.044	0.038	0.039
1000	0.15	0.045	0.046	0.051	0.047	0.044	0.059	0.061	0.061	0.060	0.058	0.035	0.036	0.037	0.037	0.036
250	0.35	0.056	0.059	0.056	0.060	0.055	0.101	0.092	0.086	0.095	0.097	0.051	0.051	0.049	0.052	0.049
500	0.35	0.051	0.054	0.056	0.052	0.051	0.092	0.089	0.086	0.086	0.090	0.043	0.048	0.048	0.044	0.045
1000	0.35	0.051	0.054	0.055	0.054	0.050	0.087	0.083	0.081	0.084	0.085	0.042	0.040	0.043	0.043	0.041

Table 14: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [2, 2.1]$; $f(x) = x$; $d_* = 0.5$; **DGP1**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.10	0.074	0.075	0.072	0.084	0.080	0.082	0.062	0.067	0.059	0.058	0.070	0.061
500	-0.10	0.070	0.069	0.067	0.080	0.083	0.081	0.053	0.064	0.058	0.057	0.064	0.056
1000	-0.10	0.066	0.067	0.068	0.077	0.077	0.076	0.053	0.059	0.051	0.054	0.058	0.054
250	0.10	0.073	0.074	0.073	0.081	0.082	0.083	0.060	0.067	0.058	0.059	0.070	0.061
500	0.10	0.071	0.071	0.070	0.077	0.081	0.082	0.054	0.064	0.056	0.056	0.063	0.055
1000	0.10	0.062	0.067	0.068	0.079	0.076	0.077	0.055	0.058	0.051	0.055	0.058	0.054
250	0.30	0.074	0.077	0.077	0.079	0.078	0.080	0.059	0.067	0.059	0.058	0.071	0.062
500	0.30	0.070	0.074	0.072	0.075	0.078	0.078	0.055	0.066	0.056	0.057	0.063	0.058
1000	0.30	0.064	0.068	0.069	0.078	0.075	0.073	0.056	0.061	0.051	0.056	0.063	0.055
250	0.50	0.082	0.081	0.084	0.085	0.083	0.083	0.066	0.072	0.064	0.065	0.076	0.066
500	0.50	0.078	0.073	0.081	0.077	0.079	0.078	0.059	0.071	0.061	0.064	0.067	0.064
1000	0.50	0.072	0.071	0.074	0.079	0.075	0.076	0.060	0.070	0.058	0.065	0.066	0.065
250	0.70	0.084	0.085	0.084	0.080	0.085	0.080	0.077	0.078	0.077	0.076	0.082	0.074
500	0.70	0.077	0.079	0.080	0.073	0.080	0.071	0.068	0.077	0.070	0.073	0.072	0.073
1000	0.70	0.073	0.079	0.072	0.074	0.076	0.075	0.069	0.074	0.067	0.070	0.073	0.072
250	0.90	0.081	0.088	0.077	0.067	0.083	0.067	0.074	0.080	0.076	0.073	0.082	0.072
500	0.90	0.071	0.081	0.073	0.059	0.080	0.059	0.068	0.080	0.071	0.068	0.075	0.069
1000	0.90	0.064	0.079	0.063	0.061	0.078	0.060	0.063	0.074	0.068	0.065	0.071	0.068
250	1.10	0.074	0.091	0.074	0.064	0.086	0.064	0.076	0.086	0.077	0.077	0.086	0.077
500	1.10	0.068	0.083	0.069	0.058	0.080	0.059	0.072	0.083	0.074	0.073	0.078	0.073
1000	1.10	0.061	0.080	0.059	0.059	0.075	0.057	0.069	0.076	0.074	0.067	0.072	0.073

Table 15: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.99, 2.1]$; $f(x) = x$; $d_* = 0.497$; **DGP1**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.103	0.069	0.071	0.071	0.080	0.083	0.082	0.057	0.067	0.057	0.057	0.065	0.058
500	-0.103	0.070	0.071	0.069	0.081	0.080	0.076	0.054	0.063	0.056	0.058	0.061	0.053
1000	-0.103	0.066	0.062	0.065	0.076	0.074	0.079	0.056	0.057	0.051	0.053	0.056	0.052
250	0.097	0.071	0.071	0.072	0.079	0.084	0.082	0.057	0.069	0.055	0.056	0.064	0.058
500	0.097	0.069	0.068	0.069	0.078	0.078	0.076	0.054	0.062	0.055	0.057	0.060	0.053
1000	0.097	0.066	0.062	0.062	0.075	0.074	0.078	0.057	0.056	0.053	0.053	0.056	0.052
250	0.297	0.075	0.077	0.074	0.077	0.086	0.079	0.056	0.070	0.057	0.057	0.065	0.057
500	0.297	0.071	0.071	0.071	0.076	0.076	0.073	0.056	0.066	0.055	0.058	0.063	0.055
1000	0.297	0.066	0.066	0.067	0.073	0.073	0.073	0.057	0.059	0.052	0.053	0.059	0.053
250	0.497	0.083	0.081	0.082	0.077	0.087	0.078	0.064	0.075	0.065	0.065	0.071	0.063
500	0.497	0.075	0.074	0.078	0.078	0.077	0.075	0.063	0.070	0.064	0.063	0.070	0.064
1000	0.497	0.074	0.074	0.076	0.073	0.075	0.077	0.062	0.065	0.059	0.061	0.066	0.059
250	0.697	0.085	0.086	0.085	0.078	0.088	0.076	0.075	0.076	0.075	0.075	0.081	0.074
500	0.697	0.078	0.077	0.077	0.074	0.078	0.072	0.072	0.075	0.070	0.071	0.074	0.073
1000	0.697	0.072	0.076	0.069	0.068	0.075	0.069	0.066	0.076	0.064	0.068	0.070	0.069
250	0.897	0.077	0.090	0.081	0.063	0.083	0.066	0.073	0.081	0.076	0.073	0.085	0.075
500	0.897	0.068	0.083	0.069	0.062	0.077	0.060	0.071	0.073	0.069	0.069	0.077	0.070
1000	0.897	0.063	0.082	0.061	0.056	0.073	0.058	0.065	0.073	0.063	0.067	0.072	0.068
250	1.097	0.076	0.092	0.079	0.062	0.083	0.064	0.076	0.085	0.079	0.077	0.088	0.077
500	1.097	0.062	0.085	0.065	0.057	0.079	0.058	0.076	0.078	0.071	0.074	0.080	0.072
1000	1.097	0.056	0.083	0.057	0.054	0.076	0.056	0.068	0.076	0.070	0.071	0.078	0.073

Table 16: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.8, 2.1]$; $f(x) = x$; $d_* = 0.444$; **DGP1**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.156	0.072	0.072	0.071	0.074	0.081	0.081	0.055	0.063	0.055	0.054	0.065	0.054
500	-0.156	0.067	0.066	0.069	0.078	0.078	0.076	0.051	0.058	0.049	0.052	0.057	0.054
1000	-0.156	0.065	0.066	0.065	0.073	0.077	0.072	0.050	0.057	0.048	0.049	0.055	0.047
250	0.044	0.073	0.068	0.072	0.078	0.079	0.079	0.053	0.063	0.055	0.055	0.064	0.054
500	0.044	0.067	0.069	0.070	0.077	0.079	0.075	0.053	0.057	0.047	0.052	0.058	0.053
1000	0.044	0.066	0.066	0.064	0.074	0.078	0.073	0.050	0.057	0.048	0.049	0.053	0.048
250	0.244	0.072	0.072	0.075	0.077	0.079	0.078	0.054	0.064	0.053	0.055	0.065	0.053
500	0.244	0.072	0.071	0.072	0.070	0.076	0.073	0.053	0.059	0.049	0.053	0.059	0.054
1000	0.244	0.068	0.067	0.067	0.071	0.077	0.071	0.050	0.058	0.049	0.050	0.055	0.047
250	0.444	0.084	0.077	0.082	0.075	0.080	0.079	0.059	0.068	0.059	0.058	0.069	0.058
500	0.444	0.079	0.074	0.078	0.075	0.077	0.075	0.057	0.064	0.054	0.057	0.064	0.060
1000	0.444	0.077	0.072	0.076	0.071	0.073	0.071	0.053	0.063	0.052	0.053	0.060	0.052
250	0.644	0.089	0.083	0.088	0.078	0.084	0.078	0.071	0.074	0.065	0.068	0.077	0.068
500	0.644	0.081	0.082	0.080	0.072	0.081	0.074	0.067	0.067	0.064	0.065	0.071	0.070
1000	0.644	0.077	0.075	0.073	0.068	0.074	0.065	0.064	0.068	0.063	0.064	0.068	0.066
250	0.844	0.082	0.086	0.079	0.065	0.084	0.063	0.071	0.075	0.066	0.068	0.079	0.071
500	0.844	0.072	0.082	0.070	0.060	0.080	0.061	0.063	0.071	0.064	0.063	0.073	0.065
1000	0.844	0.066	0.079	0.063	0.052	0.075	0.056	0.064	0.070	0.061	0.059	0.070	0.065
250	1.044	0.074	0.090	0.076	0.058	0.087	0.057	0.074	0.079	0.070	0.071	0.080	0.072
500	1.044	0.066	0.087	0.063	0.056	0.081	0.055	0.066	0.074	0.065	0.064	0.076	0.064
1000	1.044	0.059	0.082	0.057	0.051	0.074	0.051	0.063	0.071	0.063	0.059	0.075	0.067

Table 17: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.6, 2.1]$; $f(x) = x$; $d_* = 0.375$; **DGP1**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.225	0.071	0.069	0.072	0.077	0.074	0.076	0.048	0.060	0.050	0.049	0.057	0.052
500	-0.225	0.067	0.069	0.065	0.071	0.075	0.071	0.049	0.055	0.049	0.047	0.052	0.046
1000	-0.225	0.068	0.062	0.067	0.067	0.071	0.070	0.046	0.048	0.047	0.046	0.052	0.044
250	-0.025	0.067	0.070	0.070	0.078	0.077	0.076	0.050	0.060	0.050	0.048	0.056	0.052
500	-0.025	0.063	0.066	0.065	0.071	0.076	0.072	0.049	0.056	0.049	0.046	0.052	0.047
1000	-0.025	0.067	0.066	0.064	0.070	0.074	0.071	0.048	0.047	0.047	0.046	0.052	0.045
250	0.175	0.070	0.072	0.070	0.075	0.075	0.073	0.050	0.059	0.049	0.049	0.057	0.050
500	0.175	0.069	0.065	0.068	0.069	0.072	0.069	0.047	0.055	0.048	0.046	0.053	0.048
1000	0.175	0.065	0.065	0.068	0.069	0.071	0.070	0.050	0.049	0.046	0.046	0.052	0.045
250	0.375	0.081	0.077	0.082	0.077	0.077	0.073	0.054	0.062	0.051	0.047	0.061	0.052
500	0.375	0.078	0.074	0.076	0.067	0.074	0.070	0.050	0.058	0.051	0.047	0.057	0.048
1000	0.375	0.071	0.071	0.074	0.067	0.071	0.068	0.052	0.053	0.045	0.048	0.056	0.046
250	0.575	0.089	0.082	0.087	0.080	0.077	0.076	0.065	0.068	0.061	0.057	0.069	0.063
500	0.575	0.080	0.076	0.081	0.071	0.077	0.070	0.060	0.065	0.059	0.058	0.065	0.057
1000	0.575	0.080	0.078	0.080	0.068	0.074	0.070	0.061	0.060	0.054	0.055	0.063	0.054
250	0.775	0.084	0.085	0.084	0.068	0.077	0.066	0.067	0.074	0.067	0.063	0.076	0.067
500	0.775	0.071	0.082	0.074	0.059	0.074	0.058	0.063	0.070	0.064	0.062	0.070	0.060
1000	0.775	0.069	0.082	0.072	0.055	0.072	0.056	0.061	0.065	0.059	0.057	0.066	0.058
250	0.975	0.072	0.090	0.076	0.056	0.080	0.055	0.064	0.077	0.064	0.065	0.077	0.065
500	0.975	0.062	0.085	0.062	0.052	0.076	0.050	0.060	0.071	0.061	0.060	0.074	0.059
1000	0.975	0.057	0.084	0.063	0.049	0.073	0.050	0.057	0.066	0.055	0.058	0.068	0.056

Table 18: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.4, 2.1]$; $f(x) = x$; $d_* = 0.286$; **DGP1**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.314	0.068	0.067	0.069	0.065	0.075	0.068	0.050	0.056	0.046	0.050	0.054	0.049
500	-0.314	0.066	0.065	0.066	0.066	0.072	0.063	0.044	0.051	0.046	0.048	0.049	0.044
1000	-0.314	0.067	0.067	0.064	0.061	0.068	0.065	0.045	0.045	0.044	0.039	0.048	0.042
250	-0.114	0.067	0.069	0.068	0.065	0.074	0.065	0.048	0.055	0.046	0.051	0.052	0.049
500	-0.114	0.063	0.065	0.068	0.065	0.071	0.065	0.044	0.049	0.047	0.048	0.048	0.044
1000	-0.114	0.065	0.065	0.064	0.062	0.067	0.065	0.044	0.044	0.044	0.040	0.046	0.042
250	0.086	0.064	0.066	0.065	0.066	0.073	0.064	0.048	0.053	0.048	0.048	0.053	0.050
500	0.086	0.059	0.062	0.062	0.066	0.069	0.064	0.044	0.049	0.047	0.048	0.050	0.043
1000	0.086	0.060	0.063	0.061	0.062	0.065	0.063	0.044	0.045	0.044	0.040	0.046	0.041
250	0.286	0.073	0.077	0.077	0.067	0.072	0.067	0.047	0.056	0.048	0.047	0.053	0.048
500	0.286	0.074	0.071	0.071	0.067	0.067	0.067	0.045	0.052	0.046	0.047	0.050	0.043
1000	0.286	0.068	0.067	0.071	0.063	0.068	0.059	0.044	0.047	0.044	0.040	0.048	0.043
250	0.486	0.083	0.080	0.089	0.073	0.074	0.072	0.053	0.059	0.055	0.051	0.060	0.051
500	0.486	0.082	0.079	0.083	0.068	0.070	0.068	0.049	0.059	0.051	0.052	0.057	0.047
1000	0.486	0.078	0.075	0.078	0.067	0.070	0.061	0.051	0.054	0.049	0.046	0.055	0.046
250	0.686	0.088	0.086	0.088	0.067	0.077	0.069	0.061	0.068	0.064	0.059	0.066	0.060
500	0.686	0.078	0.080	0.080	0.064	0.071	0.064	0.059	0.066	0.059	0.059	0.064	0.057
1000	0.686	0.076	0.081	0.076	0.061	0.070	0.057	0.056	0.063	0.056	0.057	0.063	0.055
250	0.886	0.079	0.094	0.080	0.054	0.077	0.057	0.057	0.073	0.060	0.060	0.070	0.062
500	0.886	0.070	0.089	0.071	0.053	0.069	0.051	0.058	0.068	0.057	0.055	0.067	0.053
1000	0.886	0.062	0.086	0.064	0.048	0.066	0.046	0.050	0.064	0.053	0.054	0.063	0.053

Table 19: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.2, 2.1]$; $f(x) = x$; $d_* = 0.167$; **DGP1**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.433	0.067	0.067	0.066	0.066	0.067	0.061	0.045	0.052	0.044	0.046	0.051	0.044
500	-0.433	0.062	0.068	0.065	0.062	0.068	0.058	0.040	0.047	0.046	0.044	0.046	0.046
1000	-0.433	0.064	0.064	0.064	0.060	0.062	0.058	0.040	0.043	0.040	0.039	0.041	0.045
250	-0.233	0.068	0.070	0.066	0.063	0.067	0.061	0.045	0.051	0.042	0.046	0.049	0.044
500	-0.233	0.063	0.069	0.064	0.061	0.067	0.057	0.041	0.047	0.044	0.045	0.044	0.044
1000	-0.233	0.062	0.065	0.063	0.060	0.060	0.058	0.039	0.042	0.040	0.037	0.040	0.043
250	-0.033	0.067	0.070	0.068	0.060	0.067	0.061	0.045	0.048	0.041	0.047	0.049	0.044
500	-0.033	0.069	0.071	0.063	0.059	0.065	0.057	0.041	0.047	0.044	0.043	0.044	0.044
1000	-0.033	0.070	0.068	0.065	0.061	0.058	0.057	0.038	0.041	0.039	0.038	0.040	0.042
250	0.167	0.066	0.065	0.063	0.062	0.065	0.061	0.045	0.049	0.043	0.047	0.049	0.044
500	0.167	0.067	0.063	0.060	0.059	0.059	0.059	0.040	0.048	0.044	0.044	0.044	0.045
1000	0.167	0.063	0.063	0.062	0.058	0.059	0.058	0.038	0.041	0.040	0.038	0.041	0.043
250	0.367	0.083	0.074	0.079	0.066	0.069	0.065	0.048	0.053	0.045	0.048	0.051	0.045
500	0.367	0.082	0.075	0.081	0.064	0.062	0.062	0.043	0.050	0.045	0.044	0.049	0.045
1000	0.367	0.082	0.072	0.076	0.061	0.064	0.061	0.042	0.046	0.041	0.041	0.046	0.044
250	0.567	0.092	0.082	0.089	0.066	0.074	0.069	0.053	0.058	0.053	0.054	0.058	0.053
500	0.567	0.088	0.083	0.087	0.064	0.067	0.063	0.049	0.057	0.052	0.051	0.056	0.051
1000	0.567	0.084	0.081	0.083	0.064	0.068	0.061	0.050	0.052	0.048	0.048	0.053	0.048
250	0.767	0.087	0.088	0.085	0.057	0.078	0.059	0.056	0.062	0.059	0.056	0.066	0.057
500	0.767	0.082	0.089	0.080	0.051	0.068	0.051	0.052	0.060	0.054	0.053	0.062	0.052
1000	0.767	0.073	0.083	0.074	0.052	0.066	0.050	0.052	0.059	0.051	0.053	0.057	0.051

Table 20: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1, 2.1]$; $f(x) = x$; $d_* = 0$; **DGP1**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.60	0.063	0.066	0.066	0.054	0.063	0.054	0.045	0.048	0.037	0.043	0.045	0.042
500	-0.60	0.063	0.064	0.066	0.051	0.057	0.051	0.040	0.043	0.042	0.043	0.044	0.044
1000	-0.60	0.063	0.062	0.061	0.055	0.056	0.049	0.038	0.044	0.038	0.042	0.042	0.039
250	-0.40	0.063	0.066	0.064	0.052	0.063	0.053	0.043	0.046	0.038	0.043	0.043	0.041
500	-0.40	0.062	0.065	0.065	0.049	0.056	0.050	0.039	0.042	0.042	0.042	0.043	0.042
1000	-0.40	0.062	0.061	0.059	0.055	0.055	0.049	0.038	0.042	0.038	0.041	0.041	0.039
250	-0.20	0.065	0.070	0.070	0.052	0.061	0.052	0.042	0.044	0.040	0.043	0.042	0.041
500	-0.20	0.065	0.066	0.066	0.049	0.057	0.049	0.038	0.040	0.040	0.042	0.042	0.040
1000	-0.20	0.067	0.063	0.062	0.055	0.054	0.049	0.037	0.039	0.039	0.040	0.039	0.037
250	0.00	0.060	0.066	0.059	0.055	0.061	0.053	0.041	0.045	0.042	0.043	0.043	0.042
500	0.00	0.061	0.067	0.059	0.051	0.056	0.048	0.039	0.040	0.039	0.043	0.041	0.040
1000	0.00	0.059	0.063	0.057	0.051	0.052	0.051	0.037	0.038	0.039	0.040	0.039	0.038
250	0.20	0.067	0.063	0.065	0.056	0.061	0.057	0.043	0.046	0.042	0.044	0.044	0.042
500	0.20	0.064	0.062	0.068	0.057	0.057	0.052	0.039	0.041	0.039	0.042	0.042	0.040
1000	0.20	0.065	0.058	0.066	0.054	0.053	0.052	0.038	0.039	0.041	0.042	0.041	0.037
250	0.40	0.084	0.078	0.084	0.060	0.065	0.064	0.043	0.051	0.043	0.043	0.047	0.044
500	0.40	0.081	0.074	0.084	0.058	0.062	0.059	0.041	0.046	0.041	0.043	0.045	0.043
1000	0.40	0.081	0.074	0.078	0.057	0.057	0.056	0.042	0.043	0.043	0.044	0.045	0.041
250	0.60	0.092	0.088	0.094	0.061	0.069	0.066	0.047	0.058	0.051	0.050	0.055	0.052
500	0.60	0.086	0.081	0.089	0.058	0.065	0.060	0.048	0.053	0.048	0.050	0.051	0.049
1000	0.60	0.088	0.081	0.084	0.057	0.062	0.055	0.049	0.052	0.049	0.049	0.053	0.050

Table 21: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [0.8, 2.1]$; $f(x) = x$; $d_* = -0.25$; **DGP1**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.85	0.065	0.066	0.064	0.051	0.056	0.049	0.042	0.046	0.043	0.042	0.044	0.040
500	-0.85	0.062	0.065	0.064	0.048	0.053	0.048	0.041	0.040	0.043	0.041	0.041	0.041
1000	-0.85	0.062	0.062	0.064	0.046	0.047	0.047	0.039	0.040	0.038	0.040	0.039	0.039
250	-0.65	0.059	0.061	0.062	0.051	0.052	0.050	0.042	0.045	0.043	0.041	0.043	0.040
500	-0.65	0.060	0.061	0.059	0.048	0.052	0.048	0.039	0.040	0.043	0.041	0.040	0.041
1000	-0.65	0.057	0.055	0.058	0.046	0.048	0.046	0.038	0.040	0.038	0.039	0.039	0.039
250	-0.45	0.058	0.063	0.061	0.051	0.053	0.048	0.042	0.044	0.043	0.040	0.042	0.039
500	-0.45	0.058	0.059	0.059	0.048	0.052	0.048	0.038	0.040	0.043	0.042	0.040	0.040
1000	-0.45	0.056	0.055	0.055	0.046	0.048	0.047	0.037	0.040	0.038	0.038	0.038	0.037
250	-0.25	0.062	0.069	0.065	0.049	0.052	0.050	0.041	0.042	0.043	0.039	0.041	0.038
500	-0.25	0.064	0.066	0.060	0.047	0.053	0.047	0.037	0.038	0.042	0.040	0.039	0.040
1000	-0.25	0.060	0.060	0.059	0.046	0.046	0.046	0.035	0.039	0.037	0.037	0.038	0.036
250	-0.05	0.077	0.081	0.076	0.048	0.051	0.049	0.041	0.042	0.043	0.040	0.042	0.038
500	-0.05	0.078	0.079	0.075	0.044	0.052	0.045	0.037	0.038	0.040	0.039	0.039	0.040
1000	-0.05	0.073	0.075	0.074	0.042	0.047	0.043	0.035	0.037	0.035	0.037	0.038	0.036
250	0.15	0.059	0.057	0.059	0.051	0.049	0.052	0.039	0.043	0.041	0.040	0.041	0.038
500	0.15	0.061	0.058	0.058	0.046	0.053	0.047	0.037	0.037	0.041	0.039	0.040	0.040
1000	0.15	0.054	0.056	0.057	0.045	0.051	0.044	0.035	0.037	0.037	0.037	0.038	0.037
250	0.35	0.075	0.071	0.079	0.056	0.056	0.055	0.041	0.046	0.044	0.043	0.043	0.040
500	0.35	0.075	0.073	0.072	0.051	0.056	0.051	0.040	0.042	0.044	0.041	0.044	0.042
1000	0.35	0.073	0.072	0.074	0.051	0.055	0.050	0.038	0.039	0.038	0.040	0.041	0.040

Table 22: Empirical size (nominal size 5%): **DGP1**; $\alpha = 2.0$; $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$; $d_* = 0.5$

n	d	W-IVX					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.30	0.081	0.079	0.082	0.082	0.082	0.081	0.080	0.083	0.083	0.083
500	-0.30	0.078	0.077	0.078	0.076	0.075	0.079	0.078	0.078	0.077	0.075
1000	-0.30	0.075	0.075	0.076	0.076	0.076	0.075	0.075	0.076	0.076	0.076
250	-0.10	0.080	0.080	0.084	0.085	0.084	0.082	0.081	0.084	0.087	0.085
500	-0.10	0.081	0.077	0.080	0.078	0.075	0.081	0.077	0.079	0.077	0.075
1000	-0.10	0.076	0.074	0.075	0.078	0.077	0.076	0.074	0.075	0.079	0.078
250	0.10	0.079	0.082	0.082	0.084	0.083	0.080	0.085	0.084	0.087	0.085
500	0.10	0.082	0.077	0.082	0.078	0.076	0.083	0.078	0.083	0.079	0.076
1000	0.10	0.075	0.073	0.075	0.080	0.078	0.076	0.075	0.075	0.080	0.078
250	0.30	0.079	0.083	0.086	0.084	0.082	0.092	0.093	0.091	0.093	0.094
500	0.30	0.079	0.076	0.080	0.077	0.074	0.089	0.084	0.084	0.085	0.082
1000	0.30	0.071	0.071	0.073	0.074	0.073	0.082	0.077	0.075	0.080	0.082
250	0.50	0.080	0.086	0.089	0.087	0.083	0.126	0.113	0.101	0.114	0.128
500	0.50	0.078	0.077	0.082	0.079	0.076	0.118	0.104	0.092	0.105	0.116
1000	0.50	0.072	0.071	0.079	0.073	0.073	0.108	0.094	0.087	0.097	0.108
250	0.70	0.077	0.082	0.089	0.083	0.078	0.190	0.155	0.110	0.155	0.197
500	0.70	0.074	0.077	0.080	0.073	0.071	0.188	0.149	0.103	0.148	0.185
1000	0.70	0.066	0.069	0.076	0.070	0.066	0.178	0.137	0.095	0.139	0.179
250	0.90	0.066	0.075	0.086	0.074	0.066	0.281	0.214	0.121	0.207	0.281
500	0.90	0.065	0.068	0.082	0.067	0.063	0.279	0.205	0.114	0.202	0.273
1000	0.90	0.056	0.061	0.077	0.061	0.056	0.269	0.193	0.108	0.193	0.269
250	1.10	0.063	0.072	0.092	0.070	0.066	0.346	0.251	0.132	0.243	0.345
500	1.10	0.060	0.067	0.083	0.065	0.060	0.341	0.239	0.125	0.239	0.338
1000	1.10	0.055	0.058	0.080	0.057	0.056	0.338	0.231	0.116	0.228	0.330
250	1.30	0.073	0.078	0.098	0.075	0.070	0.366	0.260	0.136	0.251	0.363
500	1.30	0.067	0.070	0.090	0.072	0.069	0.355	0.247	0.127	0.249	0.357
1000	1.30	0.063	0.067	0.081	0.061	0.065	0.351	0.239	0.115	0.234	0.352

Table 23: Empirical size (nominal size 5%): **DGP1**; $\alpha = 1.99$; $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$; $d_* = 0.497$

n	d	W-IVX					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.303	0.081	0.083	0.084	0.081	0.079	0.082	0.084	0.084	0.081	0.079
500	-0.303	0.080	0.078	0.080	0.077	0.076	0.080	0.079	0.080	0.078	0.076
1000	-0.303	0.075	0.075	0.076	0.074	0.072	0.075	0.075	0.076	0.075	0.072
250	-0.103	0.082	0.083	0.084	0.082	0.080	0.082	0.083	0.084	0.082	0.081
500	-0.103	0.079	0.081	0.080	0.079	0.078	0.080	0.080	0.080	0.078	0.079
1000	-0.103	0.075	0.074	0.076	0.074	0.073	0.076	0.074	0.076	0.075	0.074
250	0.097	0.081	0.081	0.082	0.081	0.079	0.084	0.083	0.083	0.082	0.082
500	0.097	0.078	0.082	0.081	0.081	0.079	0.078	0.083	0.083	0.082	0.080
1000	0.097	0.075	0.074	0.075	0.073	0.071	0.076	0.075	0.075	0.074	0.072
250	0.297	0.081	0.079	0.083	0.083	0.079	0.090	0.092	0.086	0.092	0.092
500	0.297	0.074	0.080	0.081	0.078	0.074	0.084	0.086	0.084	0.086	0.084
1000	0.297	0.075	0.074	0.076	0.073	0.070	0.083	0.079	0.079	0.080	0.077
250	0.497	0.082	0.082	0.084	0.083	0.082	0.127	0.112	0.096	0.111	0.127
500	0.497	0.075	0.080	0.079	0.080	0.074	0.114	0.105	0.092	0.105	0.113
1000	0.497	0.077	0.078	0.077	0.073	0.073	0.114	0.101	0.084	0.098	0.107
250	0.697	0.080	0.081	0.086	0.080	0.078	0.195	0.155	0.106	0.153	0.199
500	0.697	0.067	0.074	0.081	0.075	0.071	0.182	0.143	0.102	0.150	0.181
1000	0.697	0.070	0.074	0.077	0.072	0.065	0.184	0.139	0.094	0.140	0.175
250	0.897	0.068	0.076	0.086	0.068	0.069	0.287	0.213	0.117	0.207	0.281
500	0.897	0.059	0.065	0.084	0.067	0.063	0.273	0.198	0.115	0.203	0.276
1000	0.897	0.057	0.064	0.077	0.060	0.053	0.277	0.194	0.104	0.194	0.272
250	1.097	0.066	0.073	0.087	0.067	0.063	0.347	0.252	0.127	0.241	0.346
500	1.097	0.057	0.063	0.084	0.064	0.064	0.340	0.234	0.123	0.240	0.340
1000	1.097	0.055	0.061	0.075	0.056	0.053	0.338	0.229	0.111	0.230	0.335
250	1.297	0.071	0.077	0.095	0.075	0.070	0.364	0.260	0.132	0.247	0.363
500	1.297	0.065	0.069	0.091	0.070	0.070	0.355	0.248	0.127	0.248	0.360
1000	1.297	0.062	0.068	0.082	0.065	0.061	0.348	0.240	0.115	0.237	0.353

Table 24: Empirical size (nominal size 5%): **DGP1**; $\alpha = 1.80$; $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$; $d_* = 0.444$

n	d	W-IVX					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.356	0.076	0.078	0.083	0.074	0.074	0.076	0.079	0.083	0.075	0.076
500	-0.356	0.075	0.078	0.074	0.078	0.074	0.075	0.079	0.074	0.078	0.074
1000	-0.356	0.077	0.074	0.076	0.073	0.072	0.077	0.073	0.076	0.073	0.072
250	-0.156	0.077	0.078	0.084	0.076	0.076	0.078	0.079	0.084	0.077	0.078
500	-0.156	0.075	0.077	0.075	0.078	0.075	0.077	0.078	0.075	0.078	0.076
1000	-0.156	0.079	0.073	0.076	0.073	0.072	0.080	0.073	0.076	0.073	0.072
250	0.044	0.077	0.077	0.084	0.077	0.079	0.079	0.078	0.085	0.079	0.081
500	0.044	0.077	0.076	0.075	0.078	0.076	0.078	0.078	0.076	0.079	0.077
1000	0.044	0.077	0.073	0.075	0.072	0.072	0.078	0.074	0.076	0.072	0.073
250	0.244	0.076	0.079	0.083	0.075	0.076	0.089	0.086	0.090	0.083	0.086
500	0.244	0.072	0.075	0.077	0.075	0.073	0.082	0.081	0.080	0.081	0.081
1000	0.244	0.071	0.073	0.074	0.071	0.069	0.078	0.079	0.078	0.075	0.076
250	0.444	0.080	0.080	0.083	0.078	0.078	0.118	0.106	0.095	0.104	0.114
500	0.444	0.074	0.077	0.077	0.073	0.073	0.108	0.099	0.089	0.099	0.108
1000	0.444	0.069	0.073	0.073	0.071	0.070	0.103	0.094	0.084	0.091	0.101
250	0.644	0.082	0.081	0.083	0.078	0.077	0.180	0.145	0.107	0.139	0.173
500	0.644	0.069	0.074	0.079	0.074	0.074	0.167	0.136	0.099	0.138	0.166
1000	0.644	0.069	0.073	0.077	0.073	0.067	0.163	0.134	0.093	0.129	0.159
250	0.844	0.068	0.073	0.084	0.070	0.066	0.262	0.194	0.119	0.192	0.254
500	0.844	0.059	0.067	0.077	0.066	0.058	0.252	0.188	0.112	0.187	0.246
1000	0.844	0.054	0.062	0.075	0.062	0.056	0.250	0.183	0.107	0.183	0.246
250	1.044	0.062	0.065	0.085	0.066	0.060	0.330	0.236	0.128	0.231	0.324
500	1.044	0.054	0.061	0.078	0.061	0.054	0.321	0.227	0.121	0.232	0.314
1000	1.044	0.054	0.060	0.075	0.058	0.052	0.324	0.225	0.115	0.222	0.318
250	1.244	0.066	0.068	0.088	0.070	0.066	0.356	0.249	0.132	0.250	0.351
500	1.244	0.059	0.066	0.084	0.064	0.058	0.348	0.249	0.123	0.247	0.340
1000	1.244	0.059	0.060	0.079	0.063	0.056	0.351	0.239	0.117	0.237	0.343

Table 25: Empirical size (nominal size 5%): **DGP1**; $\alpha = 1.60$; $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$; $d_* = 0.375$

n	d	W-IVX					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.425	0.070	0.075	0.080	0.070	0.074	0.071	0.075	0.080	0.071	0.075
500	-0.425	0.072	0.070	0.074	0.073	0.071	0.072	0.071	0.075	0.073	0.071
1000	-0.425	0.067	0.073	0.071	0.069	0.073	0.067	0.073	0.071	0.068	0.073
250	-0.225	0.070	0.074	0.078	0.072	0.073	0.071	0.076	0.078	0.074	0.076
500	-0.225	0.071	0.071	0.073	0.073	0.070	0.072	0.072	0.073	0.073	0.071
1000	-0.225	0.070	0.072	0.073	0.068	0.072	0.070	0.073	0.074	0.068	0.073
250	-0.025	0.070	0.077	0.080	0.074	0.074	0.073	0.078	0.082	0.075	0.077
500	-0.025	0.073	0.073	0.073	0.074	0.072	0.074	0.075	0.074	0.076	0.074
1000	-0.025	0.070	0.073	0.074	0.072	0.070	0.071	0.074	0.076	0.073	0.072
250	0.175	0.069	0.075	0.083	0.074	0.074	0.078	0.082	0.087	0.081	0.085
500	0.175	0.069	0.073	0.071	0.071	0.070	0.077	0.077	0.075	0.077	0.077
1000	0.175	0.067	0.069	0.072	0.068	0.069	0.073	0.073	0.075	0.074	0.075
250	0.375	0.071	0.075	0.083	0.074	0.074	0.101	0.095	0.095	0.096	0.107
500	0.375	0.070	0.073	0.075	0.073	0.071	0.098	0.092	0.084	0.092	0.098
1000	0.375	0.067	0.068	0.070	0.071	0.066	0.090	0.085	0.081	0.090	0.090
250	0.575	0.075	0.075	0.084	0.076	0.076	0.152	0.128	0.107	0.128	0.157
500	0.575	0.068	0.071	0.074	0.074	0.070	0.143	0.120	0.094	0.120	0.145
1000	0.575	0.065	0.070	0.070	0.074	0.066	0.131	0.118	0.093	0.121	0.135
250	0.775	0.071	0.069	0.080	0.071	0.066	0.227	0.169	0.120	0.173	0.226
500	0.775	0.057	0.066	0.075	0.063	0.058	0.215	0.168	0.109	0.162	0.221
1000	0.775	0.054	0.062	0.070	0.066	0.057	0.205	0.162	0.104	0.168	0.214
250	0.975	0.060	0.062	0.081	0.065	0.059	0.297	0.212	0.134	0.219	0.297
500	0.975	0.051	0.059	0.075	0.057	0.053	0.293	0.210	0.123	0.207	0.289
1000	0.975	0.049	0.054	0.069	0.057	0.049	0.281	0.207	0.116	0.213	0.286
250	1.175	0.058	0.065	0.084	0.068	0.059	0.342	0.238	0.139	0.246	0.334
500	1.175	0.050	0.061	0.077	0.057	0.053	0.333	0.236	0.128	0.233	0.325
1000	1.175	0.052	0.055	0.072	0.057	0.050	0.320	0.230	0.121	0.237	0.324

Table 26: Empirical size (nominal size 5%): **DGP1**; $\alpha = 1.40$; $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$; $d_* = 0.286$

n	d	W-IVX					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.514	0.067	0.070	0.072	0.069	0.069	0.068	0.071	0.073	0.070	0.069
500	-0.514	0.067	0.066	0.072	0.067	0.065	0.067	0.066	0.072	0.067	0.066
1000	-0.514	0.068	0.067	0.068	0.070	0.064	0.068	0.067	0.068	0.070	0.064
250	-0.314	0.067	0.070	0.072	0.067	0.067	0.069	0.071	0.072	0.069	0.069
500	-0.314	0.066	0.066	0.070	0.066	0.065	0.068	0.067	0.072	0.067	0.066
1000	-0.314	0.066	0.066	0.067	0.069	0.063	0.067	0.067	0.067	0.070	0.064
250	-0.114	0.069	0.069	0.073	0.068	0.068	0.071	0.072	0.075	0.070	0.071
500	-0.114	0.065	0.065	0.072	0.066	0.064	0.067	0.066	0.073	0.067	0.067
1000	-0.114	0.066	0.065	0.066	0.068	0.061	0.068	0.067	0.067	0.069	0.063
250	0.086	0.066	0.073	0.074	0.068	0.069	0.074	0.078	0.078	0.072	0.077
500	0.086	0.064	0.063	0.068	0.066	0.061	0.071	0.068	0.071	0.069	0.067
1000	0.086	0.064	0.065	0.066	0.068	0.062	0.070	0.067	0.068	0.071	0.068
250	0.286	0.069	0.071	0.074	0.069	0.071	0.092	0.089	0.085	0.087	0.095
500	0.286	0.066	0.064	0.068	0.065	0.059	0.087	0.083	0.081	0.082	0.079
1000	0.286	0.065	0.064	0.067	0.066	0.063	0.083	0.079	0.076	0.082	0.082
250	0.486	0.072	0.077	0.076	0.070	0.073	0.129	0.116	0.099	0.113	0.131
500	0.486	0.068	0.069	0.071	0.069	0.064	0.123	0.107	0.093	0.106	0.115
1000	0.486	0.063	0.068	0.067	0.067	0.063	0.111	0.104	0.085	0.105	0.114
250	0.686	0.066	0.076	0.079	0.070	0.068	0.186	0.154	0.116	0.153	0.191
500	0.686	0.061	0.066	0.074	0.064	0.058	0.180	0.147	0.111	0.145	0.174
1000	0.686	0.056	0.066	0.069	0.063	0.057	0.167	0.140	0.100	0.141	0.172
250	0.886	0.056	0.067	0.079	0.064	0.060	0.254	0.196	0.131	0.195	0.262
500	0.886	0.051	0.061	0.075	0.057	0.050	0.246	0.192	0.127	0.188	0.244
1000	0.886	0.045	0.055	0.067	0.053	0.048	0.235	0.181	0.117	0.185	0.240
250	1.086	0.052	0.063	0.080	0.061	0.053	0.299	0.226	0.141	0.225	0.312
500	1.086	0.047	0.059	0.079	0.057	0.048	0.296	0.224	0.138	0.218	0.295
1000	1.086	0.046	0.053	0.071	0.052	0.047	0.281	0.211	0.127	0.210	0.284

Table 27: Empirical size (nominal size 5%): **DGP1**; $\alpha = 1.20$; $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$; $d_* = 0.167$

n	d	W-IVX					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.633	0.066	0.067	0.069	0.064	0.065	0.066	0.067	0.068	0.064	0.066
500	-0.633	0.060	0.065	0.067	0.063	0.058	0.060	0.065	0.067	0.063	0.058
1000	-0.633	0.059	0.061	0.065	0.065	0.062	0.059	0.061	0.065	0.065	0.062
250	-0.433	0.064	0.067	0.069	0.064	0.064	0.066	0.068	0.070	0.065	0.066
500	-0.433	0.061	0.064	0.067	0.063	0.057	0.062	0.065	0.068	0.064	0.059
1000	-0.433	0.059	0.061	0.064	0.064	0.060	0.060	0.062	0.064	0.064	0.061
250	-0.233	0.064	0.067	0.068	0.063	0.063	0.068	0.069	0.071	0.065	0.067
500	-0.233	0.060	0.063	0.066	0.062	0.056	0.063	0.065	0.068	0.064	0.060
1000	-0.233	0.058	0.060	0.064	0.062	0.058	0.061	0.062	0.065	0.064	0.060
250	-0.033	0.063	0.064	0.068	0.063	0.061	0.066	0.066	0.070	0.064	0.064
500	-0.033	0.059	0.061	0.065	0.062	0.059	0.061	0.064	0.067	0.063	0.062
1000	-0.033	0.058	0.059	0.064	0.060	0.056	0.062	0.061	0.064	0.064	0.059
250	0.167	0.061	0.064	0.066	0.062	0.063	0.076	0.076	0.075	0.076	0.078
500	0.167	0.059	0.061	0.063	0.062	0.057	0.072	0.072	0.072	0.075	0.071
1000	0.167	0.055	0.057	0.066	0.060	0.055	0.069	0.070	0.071	0.071	0.068
250	0.367	0.064	0.064	0.068	0.064	0.064	0.104	0.096	0.090	0.096	0.104
500	0.367	0.061	0.065	0.064	0.064	0.061	0.097	0.093	0.083	0.091	0.098
1000	0.367	0.059	0.062	0.064	0.061	0.057	0.090	0.087	0.080	0.090	0.092
250	0.567	0.065	0.070	0.070	0.068	0.066	0.151	0.131	0.107	0.128	0.145
500	0.567	0.062	0.068	0.064	0.064	0.061	0.140	0.125	0.099	0.123	0.136
1000	0.567	0.059	0.063	0.063	0.063	0.060	0.131	0.115	0.093	0.117	0.132
250	0.767	0.059	0.065	0.074	0.063	0.057	0.214	0.169	0.126	0.171	0.208
500	0.767	0.054	0.062	0.067	0.058	0.049	0.198	0.163	0.119	0.159	0.193
1000	0.767	0.049	0.055	0.065	0.057	0.049	0.186	0.152	0.111	0.155	0.186
250	0.967	0.051	0.061	0.076	0.059	0.051	0.267	0.207	0.142	0.201	0.259
500	0.967	0.045	0.058	0.072	0.055	0.043	0.251	0.195	0.134	0.193	0.247
1000	0.967	0.043	0.050	0.066	0.052	0.043	0.240	0.184	0.126	0.187	0.233

Table 28: Empirical size (nominal size 5%): **DGP1**; $\alpha = 1.0$; $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$; $d_* = 0$

n	d	W-IVX					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-0.80	0.056	0.058	0.059	0.058	0.052	0.056	0.058	0.059	0.058	0.053
500	-0.80	0.053	0.057	0.059	0.055	0.053	0.053	0.058	0.059	0.055	0.053
1000	-0.80	0.051	0.054	0.055	0.054	0.052	0.051	0.054	0.055	0.054	0.052
250	-0.60	0.055	0.057	0.059	0.058	0.053	0.056	0.058	0.059	0.059	0.054
500	-0.60	0.054	0.057	0.060	0.054	0.054	0.055	0.058	0.060	0.054	0.055
1000	-0.60	0.050	0.054	0.055	0.055	0.053	0.050	0.054	0.055	0.055	0.053
250	-0.40	0.055	0.057	0.059	0.058	0.054	0.057	0.058	0.061	0.060	0.057
500	-0.40	0.053	0.057	0.059	0.054	0.054	0.055	0.058	0.061	0.055	0.055
1000	-0.40	0.049	0.053	0.054	0.055	0.052	0.050	0.054	0.055	0.056	0.054
250	-0.20	0.054	0.056	0.059	0.057	0.054	0.058	0.060	0.061	0.061	0.059
500	-0.20	0.053	0.055	0.059	0.052	0.053	0.057	0.057	0.061	0.055	0.056
1000	-0.20	0.050	0.052	0.053	0.054	0.052	0.052	0.055	0.056	0.056	0.055
250	0.00	0.057	0.058	0.057	0.059	0.054	0.060	0.060	0.060	0.059	0.056
500	0.00	0.053	0.054	0.057	0.052	0.052	0.055	0.056	0.058	0.054	0.055
1000	0.00	0.050	0.054	0.054	0.054	0.052	0.052	0.054	0.054	0.055	0.053
250	0.20	0.056	0.059	0.062	0.059	0.055	0.079	0.077	0.080	0.078	0.079
500	0.20	0.051	0.056	0.054	0.056	0.055	0.074	0.072	0.070	0.074	0.076
1000	0.20	0.052	0.054	0.052	0.054	0.052	0.070	0.071	0.065	0.070	0.070
250	0.40	0.061	0.062	0.068	0.063	0.059	0.107	0.100	0.097	0.104	0.110
500	0.40	0.054	0.060	0.058	0.059	0.059	0.099	0.096	0.086	0.095	0.106
1000	0.40	0.055	0.055	0.057	0.058	0.056	0.095	0.089	0.084	0.089	0.096
250	0.60	0.061	0.064	0.070	0.067	0.061	0.148	0.133	0.114	0.135	0.150
500	0.60	0.055	0.061	0.064	0.059	0.055	0.139	0.125	0.109	0.124	0.145
1000	0.60	0.056	0.055	0.061	0.058	0.055	0.134	0.115	0.102	0.118	0.136
250	0.80	0.051	0.060	0.072	0.063	0.052	0.196	0.169	0.135	0.170	0.198
500	0.80	0.045	0.054	0.064	0.053	0.046	0.189	0.160	0.127	0.160	0.195
1000	0.80	0.045	0.048	0.062	0.051	0.044	0.181	0.147	0.122	0.150	0.183

Table 29: Empirical size (nominal size 5%): **DGP1**; $\alpha = 0.80$; $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$; $d_* = -0.25$

n	d	W-IVX					OLS				
		$\delta=-0.99$	-0.75	0.00	0.75	0.99	$\delta=-0.99$	-0.75	0.00	0.75	0.99
250	-1.05	0.051	0.050	0.054	0.052	0.049	0.051	0.051	0.054	0.052	0.049
500	-1.05	0.050	0.049	0.049	0.052	0.051	0.050	0.049	0.049	0.052	0.051
1000	-1.05	0.045	0.048	0.049	0.049	0.044	0.044	0.048	0.049	0.049	0.044
250	-0.85	0.050	0.051	0.052	0.051	0.048	0.051	0.051	0.052	0.051	0.048
500	-0.85	0.050	0.048	0.050	0.052	0.051	0.050	0.048	0.050	0.052	0.051
1000	-0.85	0.045	0.048	0.049	0.048	0.046	0.045	0.048	0.050	0.048	0.045
250	-0.65	0.048	0.051	0.052	0.051	0.048	0.049	0.052	0.052	0.051	0.048
500	-0.65	0.049	0.049	0.050	0.051	0.050	0.050	0.049	0.051	0.052	0.051
1000	-0.65	0.046	0.046	0.050	0.046	0.045	0.046	0.046	0.050	0.047	0.045
250	-0.45	0.048	0.052	0.053	0.051	0.048	0.050	0.053	0.054	0.053	0.049
500	-0.45	0.050	0.048	0.052	0.051	0.051	0.051	0.050	0.053	0.052	0.052
1000	-0.45	0.045	0.047	0.050	0.047	0.046	0.046	0.048	0.051	0.048	0.047
250	-0.25	0.048	0.052	0.053	0.050	0.048	0.051	0.055	0.056	0.054	0.051
500	-0.25	0.048	0.047	0.052	0.049	0.049	0.051	0.050	0.054	0.052	0.051
1000	-0.25	0.046	0.046	0.050	0.048	0.046	0.048	0.049	0.051	0.050	0.049
250	-0.05	0.046	0.050	0.051	0.050	0.046	0.049	0.052	0.053	0.052	0.048
500	-0.05	0.046	0.046	0.049	0.047	0.046	0.049	0.051	0.050	0.050	0.049
1000	-0.05	0.043	0.045	0.049	0.045	0.043	0.046	0.048	0.050	0.048	0.044
250	0.15	0.050	0.049	0.054	0.052	0.050	0.069	0.069	0.068	0.070	0.064
500	0.15	0.047	0.048	0.050	0.050	0.051	0.063	0.066	0.065	0.065	0.068
1000	0.15	0.045	0.045	0.049	0.049	0.044	0.058	0.059	0.064	0.062	0.056
250	0.35	0.056	0.057	0.059	0.057	0.053	0.098	0.094	0.089	0.092	0.093
500	0.35	0.053	0.053	0.055	0.054	0.055	0.090	0.088	0.085	0.088	0.092
1000	0.35	0.050	0.050	0.055	0.054	0.050	0.086	0.082	0.083	0.084	0.084
250	0.55	0.060	0.060	0.062	0.058	0.056	0.133	0.121	0.109	0.125	0.129
500	0.55	0.057	0.058	0.061	0.057	0.056	0.127	0.118	0.103	0.115	0.127
1000	0.55	0.055	0.054	0.059	0.055	0.052	0.122	0.108	0.102	0.111	0.118

Table 30: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [2, 2.1]$; $f(x) = x$; $d_* = 0.5$; **DGP2**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.10	0.052	0.049	0.049	0.052	0.052	0.052	0.050	0.053	0.050	0.052	0.051	0.049
500	-0.10	0.048	0.052	0.054	0.050	0.050	0.052	0.052	0.050	0.052	0.052	0.050	0.051
1000	-0.10	0.050	0.053	0.051	0.047	0.050	0.051	0.054	0.050	0.050	0.051	0.050	0.051
250	0.10	0.052	0.052	0.050	0.051	0.052	0.052	0.050	0.052	0.050	0.050	0.051	0.049
500	0.10	0.052	0.053	0.051	0.050	0.049	0.050	0.051	0.050	0.052	0.051	0.051	0.051
1000	0.10	0.050	0.053	0.049	0.049	0.051	0.052	0.054	0.049	0.048	0.051	0.051	0.052
250	0.30	0.051	0.050	0.049	0.052	0.052	0.053	0.049	0.053	0.053	0.050	0.052	0.049
500	0.30	0.054	0.050	0.052	0.048	0.049	0.048	0.051	0.051	0.050	0.053	0.051	0.051
1000	0.30	0.049	0.054	0.049	0.050	0.050	0.051	0.053	0.048	0.048	0.051	0.052	0.051
250	0.50	0.050	0.051	0.048	0.055	0.052	0.056	0.052	0.054	0.058	0.058	0.054	0.053
500	0.50	0.050	0.051	0.050	0.051	0.050	0.050	0.053	0.051	0.052	0.055	0.050	0.054
1000	0.50	0.047	0.053	0.045	0.052	0.050	0.052	0.052	0.050	0.051	0.051	0.049	0.055
250	0.70	0.048	0.051	0.046	0.053	0.050	0.056	0.054	0.052	0.057	0.060	0.051	0.062
500	0.70	0.044	0.052	0.042	0.050	0.050	0.051	0.054	0.051	0.053	0.060	0.051	0.058
1000	0.70	0.043	0.051	0.044	0.050	0.050	0.052	0.052	0.052	0.053	0.058	0.052	0.058
250	0.90	0.041	0.049	0.041	0.047	0.052	0.047	0.051	0.051	0.053	0.059	0.050	0.060
500	0.90	0.039	0.051	0.034	0.046	0.050	0.045	0.052	0.049	0.048	0.057	0.051	0.054
1000	0.90	0.033	0.048	0.034	0.043	0.051	0.044	0.049	0.052	0.046	0.056	0.053	0.055
250	1.10	0.037	0.049	0.037	0.047	0.050	0.044	0.053	0.049	0.053	0.065	0.048	0.064
500	1.10	0.037	0.046	0.034	0.046	0.052	0.045	0.052	0.048	0.051	0.058	0.052	0.056
1000	1.10	0.033	0.043	0.032	0.043	0.052	0.043	0.049	0.050	0.048	0.059	0.052	0.057

Table 31: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.99, 2.1]$; $f(x) = x$; $d_* = 0.497$; **DGP2**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.103	0.050	0.054	0.050	0.051	0.051	0.051	0.051	0.051	0.052	0.051	0.055	0.055
500	-0.103	0.054	0.054	0.050	0.051	0.053	0.050	0.049	0.056	0.051	0.054	0.053	0.049
1000	-0.103	0.050	0.051	0.051	0.049	0.051	0.051	0.050	0.051	0.051	0.048	0.049	0.052
250	0.097	0.050	0.054	0.049	0.052	0.052	0.050	0.051	0.051	0.052	0.051	0.054	0.054
500	0.097	0.051	0.052	0.053	0.051	0.053	0.048	0.050	0.056	0.050	0.051	0.052	0.048
1000	0.097	0.051	0.052	0.049	0.050	0.051	0.049	0.048	0.050	0.051	0.049	0.048	0.050
250	0.297	0.048	0.054	0.050	0.051	0.051	0.050	0.048	0.050	0.051	0.051	0.055	0.053
500	0.297	0.052	0.050	0.048	0.052	0.051	0.048	0.050	0.054	0.049	0.052	0.054	0.047
1000	0.297	0.049	0.052	0.051	0.049	0.049	0.047	0.050	0.049	0.052	0.049	0.048	0.050
250	0.497	0.050	0.051	0.051	0.053	0.050	0.052	0.053	0.052	0.053	0.054	0.052	0.055
500	0.497	0.054	0.050	0.047	0.052	0.052	0.050	0.051	0.050	0.052	0.055	0.054	0.051
1000	0.497	0.049	0.053	0.048	0.050	0.049	0.049	0.051	0.051	0.051	0.053	0.051	0.052
250	0.697	0.045	0.051	0.045	0.053	0.051	0.050	0.058	0.052	0.055	0.060	0.052	0.062
500	0.697	0.046	0.051	0.046	0.051	0.053	0.050	0.054	0.051	0.053	0.061	0.052	0.056
1000	0.697	0.045	0.051	0.045	0.049	0.048	0.049	0.053	0.053	0.052	0.060	0.049	0.056
250	0.897	0.041	0.048	0.040	0.046	0.051	0.045	0.056	0.051	0.051	0.057	0.052	0.058
500	0.897	0.037	0.050	0.038	0.045	0.051	0.044	0.051	0.050	0.048	0.057	0.051	0.059
1000	0.897	0.036	0.050	0.039	0.043	0.049	0.045	0.048	0.051	0.047	0.055	0.051	0.054
250	1.097	0.035	0.045	0.037	0.042	0.047	0.044	0.052	0.048	0.053	0.061	0.053	0.060
500	1.097	0.035	0.045	0.035	0.044	0.050	0.046	0.052	0.049	0.049	0.059	0.048	0.062
1000	1.097	0.032	0.046	0.035	0.044	0.048	0.045	0.049	0.050	0.050	0.057	0.051	0.056

Table 32: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.8, 2.1]$; $f(x) = x$; $d_* = 0.444$; **DGP2**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.156	0.051	0.050	0.050	0.050	0.052	0.051	0.051	0.050	0.051	0.055	0.053	0.049
500	-0.156	0.050	0.052	0.052	0.050	0.051	0.053	0.047	0.051	0.051	0.053	0.047	0.053
1000	-0.156	0.048	0.049	0.049	0.053	0.050	0.050	0.051	0.051	0.053	0.050	0.049	0.050
250	0.044	0.053	0.052	0.048	0.051	0.053	0.050	0.051	0.050	0.049	0.054	0.054	0.050
500	0.044	0.053	0.050	0.053	0.051	0.049	0.052	0.046	0.051	0.049	0.050	0.049	0.052
1000	0.044	0.051	0.049	0.051	0.052	0.051	0.049	0.051	0.050	0.052	0.052	0.050	0.050
250	0.244	0.050	0.053	0.047	0.052	0.053	0.049	0.050	0.052	0.048	0.051	0.052	0.049
500	0.244	0.051	0.051	0.049	0.049	0.051	0.053	0.047	0.053	0.047	0.049	0.050	0.051
1000	0.244	0.049	0.051	0.049	0.050	0.053	0.051	0.050	0.051	0.051	0.052	0.049	0.051
250	0.444	0.053	0.051	0.048	0.053	0.053	0.051	0.053	0.052	0.049	0.054	0.051	0.055
500	0.444	0.049	0.048	0.052	0.050	0.049	0.053	0.050	0.051	0.050	0.051	0.051	0.054
1000	0.444	0.050	0.050	0.048	0.050	0.053	0.051	0.050	0.050	0.048	0.051	0.050	0.050
250	0.644	0.050	0.049	0.044	0.053	0.054	0.055	0.058	0.050	0.056	0.064	0.053	0.061
500	0.644	0.048	0.049	0.048	0.051	0.051	0.051	0.053	0.051	0.055	0.058	0.050	0.059
1000	0.644	0.046	0.053	0.046	0.049	0.052	0.052	0.053	0.048	0.053	0.056	0.048	0.057
250	0.844	0.042	0.047	0.039	0.047	0.053	0.049	0.054	0.049	0.051	0.062	0.049	0.059
500	0.844	0.038	0.048	0.040	0.043	0.050	0.048	0.048	0.049	0.050	0.056	0.049	0.060
1000	0.844	0.039	0.050	0.038	0.042	0.051	0.047	0.050	0.050	0.051	0.053	0.048	0.054
250	1.044	0.036	0.043	0.037	0.045	0.051	0.044	0.052	0.049	0.051	0.062	0.051	0.058
500	1.044	0.035	0.045	0.034	0.040	0.052	0.046	0.047	0.051	0.050	0.057	0.049	0.058
1000	1.044	0.033	0.046	0.033	0.042	0.050	0.044	0.051	0.050	0.052	0.054	0.050	0.056

Table 33: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.6, 2.1]$; $f(x) = x$; $d_* = 0.375$; **DGP2**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.225	0.051	0.050	0.050	0.052	0.051	0.051	0.050	0.050	0.049	0.051	0.053	0.051
500	-0.225	0.048	0.052	0.052	0.048	0.050	0.048	0.051	0.052	0.050	0.050	0.052	0.049
1000	-0.225	0.049	0.051	0.049	0.050	0.050	0.051	0.051	0.052	0.051	0.049	0.050	0.051
250	-0.025	0.053	0.053	0.053	0.051	0.051	0.049	0.051	0.051	0.050	0.049	0.052	0.052
500	-0.025	0.047	0.052	0.051	0.050	0.052	0.050	0.048	0.052	0.051	0.049	0.052	0.050
1000	-0.025	0.048	0.049	0.048	0.051	0.051	0.052	0.051	0.052	0.050	0.048	0.049	0.049
250	0.175	0.052	0.050	0.053	0.052	0.051	0.048	0.050	0.052	0.050	0.048	0.051	0.053
500	0.175	0.052	0.051	0.050	0.049	0.051	0.051	0.049	0.051	0.051	0.050	0.050	0.050
1000	0.175	0.050	0.051	0.048	0.050	0.051	0.051	0.052	0.052	0.049	0.049	0.049	0.049
250	0.375	0.053	0.053	0.053	0.053	0.052	0.052	0.051	0.054	0.050	0.049	0.051	0.054
500	0.375	0.051	0.052	0.049	0.049	0.051	0.052	0.051	0.051	0.052	0.052	0.051	0.051
1000	0.375	0.048	0.050	0.047	0.051	0.049	0.052	0.051	0.050	0.050	0.049	0.048	0.049
250	0.575	0.050	0.051	0.054	0.053	0.054	0.054	0.057	0.053	0.056	0.057	0.051	0.059
500	0.575	0.050	0.054	0.047	0.050	0.050	0.055	0.053	0.052	0.055	0.055	0.050	0.056
1000	0.575	0.049	0.046	0.050	0.053	0.051	0.052	0.053	0.051	0.053	0.055	0.051	0.054
250	0.775	0.046	0.050	0.042	0.051	0.055	0.051	0.054	0.051	0.056	0.063	0.051	0.061
500	0.775	0.042	0.052	0.043	0.046	0.049	0.052	0.052	0.052	0.053	0.058	0.049	0.056
1000	0.775	0.041	0.047	0.042	0.044	0.049	0.049	0.052	0.049	0.050	0.055	0.051	0.056
250	0.975	0.037	0.047	0.036	0.046	0.051	0.047	0.051	0.050	0.055	0.061	0.051	0.059
500	0.975	0.033	0.051	0.036	0.044	0.049	0.046	0.048	0.050	0.052	0.058	0.048	0.054
1000	0.975	0.033	0.048	0.034	0.043	0.050	0.044	0.049	0.050	0.048	0.055	0.050	0.056

Table 34: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.4, 2.1]$; $f(x) = x$; $d_* = 0.286$; **DGP2**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.314	0.049	0.050	0.052	0.052	0.056	0.051	0.052	0.050	0.049	0.048	0.050	0.052
500	-0.314	0.050	0.052	0.050	0.051	0.052	0.048	0.046	0.048	0.050	0.050	0.050	0.051
1000	-0.314	0.048	0.050	0.050	0.052	0.049	0.050	0.049	0.049	0.049	0.053	0.051	0.051
250	-0.114	0.049	0.050	0.051	0.051	0.055	0.052	0.052	0.049	0.049	0.048	0.049	0.051
500	-0.114	0.050	0.052	0.051	0.051	0.053	0.048	0.048	0.048	0.051	0.050	0.051	0.050
1000	-0.114	0.049	0.051	0.050	0.052	0.050	0.050	0.050	0.049	0.050	0.052	0.049	0.052
250	0.086	0.052	0.051	0.050	0.051	0.053	0.052	0.050	0.052	0.050	0.048	0.049	0.052
500	0.086	0.051	0.051	0.052	0.050	0.051	0.048	0.048	0.049	0.050	0.049	0.051	0.049
1000	0.086	0.050	0.052	0.050	0.053	0.050	0.047	0.050	0.048	0.049	0.051	0.049	0.050
250	0.286	0.053	0.049	0.050	0.051	0.051	0.051	0.048	0.052	0.049	0.049	0.050	0.053
500	0.286	0.050	0.052	0.055	0.050	0.051	0.049	0.049	0.050	0.049	0.049	0.049	0.050
1000	0.286	0.049	0.048	0.052	0.050	0.048	0.046	0.049	0.050	0.052	0.051	0.050	0.048
250	0.486	0.050	0.051	0.051	0.052	0.055	0.054	0.052	0.053	0.052	0.054	0.050	0.056
500	0.486	0.050	0.050	0.052	0.051	0.050	0.052	0.050	0.051	0.051	0.053	0.050	0.056
1000	0.486	0.049	0.050	0.049	0.049	0.050	0.046	0.052	0.050	0.052	0.054	0.051	0.053
250	0.686	0.050	0.050	0.048	0.050	0.053	0.054	0.054	0.053	0.056	0.059	0.051	0.063
500	0.686	0.046	0.050	0.045	0.050	0.051	0.051	0.051	0.052	0.053	0.058	0.050	0.062
1000	0.686	0.045	0.046	0.043	0.047	0.050	0.047	0.054	0.050	0.053	0.057	0.051	0.059
250	0.886	0.040	0.046	0.039	0.045	0.052	0.047	0.052	0.050	0.053	0.056	0.049	0.059
500	0.886	0.036	0.046	0.038	0.046	0.050	0.043	0.046	0.053	0.048	0.055	0.050	0.056
1000	0.886	0.036	0.048	0.035	0.042	0.049	0.042	0.052	0.051	0.047	0.053	0.050	0.056

Table 35: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1.2, 2.1]$; $f(x) = x$; $d_* = 0.167$; **DGP2**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.433	0.049	0.052	0.051	0.049	0.050	0.052	0.049	0.055	0.051	0.047	0.051	0.051
500	-0.433	0.052	0.054	0.051	0.054	0.050	0.049	0.050	0.048	0.048	0.052	0.049	0.049
1000	-0.433	0.051	0.051	0.050	0.051	0.049	0.053	0.048	0.052	0.050	0.051	0.050	0.050
250	-0.233	0.051	0.051	0.052	0.050	0.050	0.051	0.049	0.053	0.051	0.046	0.052	0.050
500	-0.233	0.049	0.052	0.049	0.052	0.051	0.048	0.050	0.048	0.047	0.051	0.050	0.049
1000	-0.233	0.049	0.050	0.051	0.052	0.049	0.052	0.049	0.052	0.050	0.051	0.050	0.049
250	-0.033	0.050	0.053	0.051	0.051	0.051	0.051	0.050	0.053	0.051	0.048	0.054	0.049
500	-0.033	0.052	0.052	0.050	0.052	0.051	0.051	0.050	0.048	0.048	0.051	0.048	0.048
1000	-0.033	0.048	0.049	0.051	0.051	0.048	0.051	0.050	0.051	0.051	0.050	0.050	0.051
250	0.167	0.048	0.050	0.049	0.051	0.050	0.052	0.049	0.055	0.052	0.048	0.053	0.050
500	0.167	0.052	0.053	0.051	0.052	0.051	0.049	0.049	0.051	0.048	0.052	0.048	0.048
1000	0.167	0.049	0.048	0.051	0.051	0.048	0.048	0.049	0.052	0.051	0.049	0.050	0.050
250	0.367	0.050	0.049	0.051	0.051	0.050	0.053	0.051	0.053	0.053	0.051	0.053	0.049
500	0.367	0.052	0.052	0.049	0.051	0.051	0.050	0.049	0.050	0.048	0.052	0.047	0.049
1000	0.367	0.049	0.049	0.049	0.051	0.051	0.049	0.049	0.052	0.051	0.048	0.049	0.049
250	0.567	0.051	0.049	0.049	0.054	0.051	0.052	0.055	0.052	0.055	0.056	0.054	0.056
500	0.567	0.051	0.049	0.049	0.051	0.050	0.052	0.052	0.049	0.052	0.057	0.049	0.058
1000	0.567	0.050	0.049	0.049	0.052	0.049	0.052	0.053	0.054	0.053	0.055	0.049	0.055
250	0.767	0.045	0.051	0.047	0.048	0.054	0.047	0.055	0.052	0.053	0.059	0.050	0.059
500	0.767	0.044	0.047	0.041	0.049	0.050	0.045	0.053	0.049	0.051	0.059	0.051	0.059
1000	0.767	0.044	0.049	0.040	0.049	0.050	0.048	0.049	0.052	0.051	0.056	0.050	0.056

Table 36: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [1, 2.1]$; $f(x) = x$; $d_* = 0$; **DGP2**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.60	0.051	0.054	0.050	0.051	0.051	0.051	0.046	0.053	0.048	0.046	0.052	0.048
500	-0.60	0.052	0.053	0.050	0.051	0.050	0.050	0.051	0.051	0.051	0.049	0.048	0.046
1000	-0.60	0.049	0.049	0.051	0.050	0.053	0.049	0.049	0.051	0.051	0.047	0.049	0.045
250	-0.40	0.049	0.053	0.049	0.052	0.052	0.052	0.047	0.052	0.047	0.046	0.049	0.047
500	-0.40	0.050	0.051	0.049	0.050	0.051	0.050	0.051	0.051	0.050	0.050	0.047	0.046
1000	-0.40	0.050	0.051	0.050	0.050	0.053	0.049	0.050	0.053	0.050	0.048	0.047	0.045
250	-0.20	0.051	0.054	0.051	0.051	0.052	0.052	0.045	0.051	0.048	0.045	0.048	0.046
500	-0.20	0.053	0.050	0.050	0.051	0.050	0.049	0.051	0.052	0.050	0.049	0.048	0.045
1000	-0.20	0.050	0.051	0.052	0.049	0.050	0.049	0.050	0.053	0.049	0.046	0.047	0.045
250	0.00	0.050	0.052	0.050	0.051	0.052	0.051	0.046	0.051	0.050	0.043	0.047	0.045
500	0.00	0.052	0.053	0.051	0.051	0.049	0.049	0.050	0.052	0.051	0.049	0.048	0.046
1000	0.00	0.051	0.050	0.051	0.049	0.049	0.051	0.048	0.051	0.048	0.047	0.048	0.046
250	0.20	0.052	0.051	0.049	0.052	0.051	0.051	0.047	0.051	0.049	0.044	0.048	0.045
500	0.20	0.052	0.052	0.053	0.051	0.049	0.050	0.048	0.051	0.051	0.048	0.047	0.045
1000	0.20	0.050	0.052	0.051	0.050	0.050	0.049	0.048	0.051	0.049	0.049	0.047	0.046
250	0.40	0.051	0.053	0.053	0.052	0.052	0.051	0.050	0.050	0.052	0.047	0.048	0.047
500	0.40	0.051	0.052	0.052	0.052	0.050	0.050	0.050	0.053	0.052	0.049	0.049	0.045
1000	0.40	0.049	0.052	0.052	0.051	0.051	0.050	0.048	0.051	0.049	0.050	0.049	0.049
250	0.60	0.055	0.052	0.050	0.054	0.052	0.057	0.056	0.051	0.056	0.054	0.050	0.054
500	0.60	0.051	0.052	0.051	0.055	0.051	0.052	0.053	0.054	0.055	0.055	0.050	0.052
1000	0.60	0.051	0.053	0.051	0.052	0.051	0.053	0.051	0.049	0.052	0.053	0.052	0.054

Table 37: Empirical size (nominal size 5%): W-IVX; $[\alpha, \nu] = [0.8, 2.1]$; $f(x) = x$; $d_* = -0.25$; **DGP2**

$w(x)$		$x^{(0)}$			$x^{(0.5)}$			$x^{(0.75)}$			$x^{(1)}$		
n	d	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99	$\delta=-0.99$	0.00	0.99
250	-0.85	0.053	0.053	0.054	0.052	0.053	0.050	0.047	0.050	0.043	0.045	0.045	0.043
500	-0.85	0.053	0.051	0.051	0.050	0.051	0.050	0.047	0.047	0.047	0.045	0.045	0.044
1000	-0.85	0.050	0.051	0.052	0.049	0.051	0.051	0.047	0.048	0.047	0.044	0.047	0.044
250	-0.65	0.052	0.052	0.052	0.051	0.052	0.050	0.047	0.050	0.043	0.046	0.045	0.043
500	-0.65	0.052	0.052	0.051	0.050	0.052	0.049	0.047	0.048	0.048	0.045	0.044	0.043
1000	-0.65	0.049	0.053	0.050	0.048	0.051	0.051	0.047	0.049	0.048	0.045	0.047	0.044
250	-0.45	0.051	0.051	0.050	0.051	0.052	0.049	0.048	0.051	0.044	0.045	0.045	0.043
500	-0.45	0.050	0.053	0.052	0.050	0.053	0.049	0.048	0.046	0.046	0.043	0.043	0.041
1000	-0.45	0.050	0.053	0.049	0.049	0.049	0.052	0.048	0.047	0.047	0.044	0.046	0.043
250	-0.25	0.051	0.051	0.056	0.052	0.051	0.048	0.047	0.050	0.043	0.044	0.046	0.041
500	-0.25	0.050	0.051	0.049	0.048	0.052	0.047	0.049	0.047	0.045	0.043	0.042	0.040
1000	-0.25	0.048	0.053	0.050	0.048	0.049	0.050	0.048	0.047	0.046	0.043	0.045	0.043
250	-0.05	0.051	0.054	0.056	0.052	0.053	0.048	0.047	0.048	0.044	0.045	0.046	0.040
500	-0.05	0.052	0.053	0.051	0.050	0.054	0.047	0.046	0.048	0.046	0.043	0.042	0.040
1000	-0.05	0.048	0.053	0.050	0.045	0.050	0.052	0.047	0.047	0.047	0.042	0.046	0.041
250	0.15	0.052	0.050	0.052	0.051	0.053	0.048	0.046	0.048	0.047	0.045	0.045	0.041
500	0.15	0.050	0.052	0.053	0.051	0.051	0.049	0.044	0.048	0.048	0.044	0.042	0.041
1000	0.15	0.049	0.052	0.051	0.047	0.050	0.050	0.046	0.050	0.043	0.043	0.047	0.040
250	0.35	0.052	0.057	0.054	0.050	0.052	0.049	0.049	0.049	0.048	0.045	0.048	0.043
500	0.35	0.052	0.050	0.054	0.052	0.052	0.048	0.046	0.050	0.048	0.046	0.045	0.043
1000	0.35	0.051	0.051	0.054	0.050	0.048	0.050	0.049	0.054	0.045	0.044	0.050	0.042

Empirical Power

Figure 1: Asymptotic Power (nominal size 5%): $[f(x), w(x)] = [x, x^{(0)}]$, $\delta = -0.95$, $n = 1000$

(a) $\alpha = 2$, $d_* = 0.5$

(b) $\alpha = 1.8$, $d_* = 0.444$.

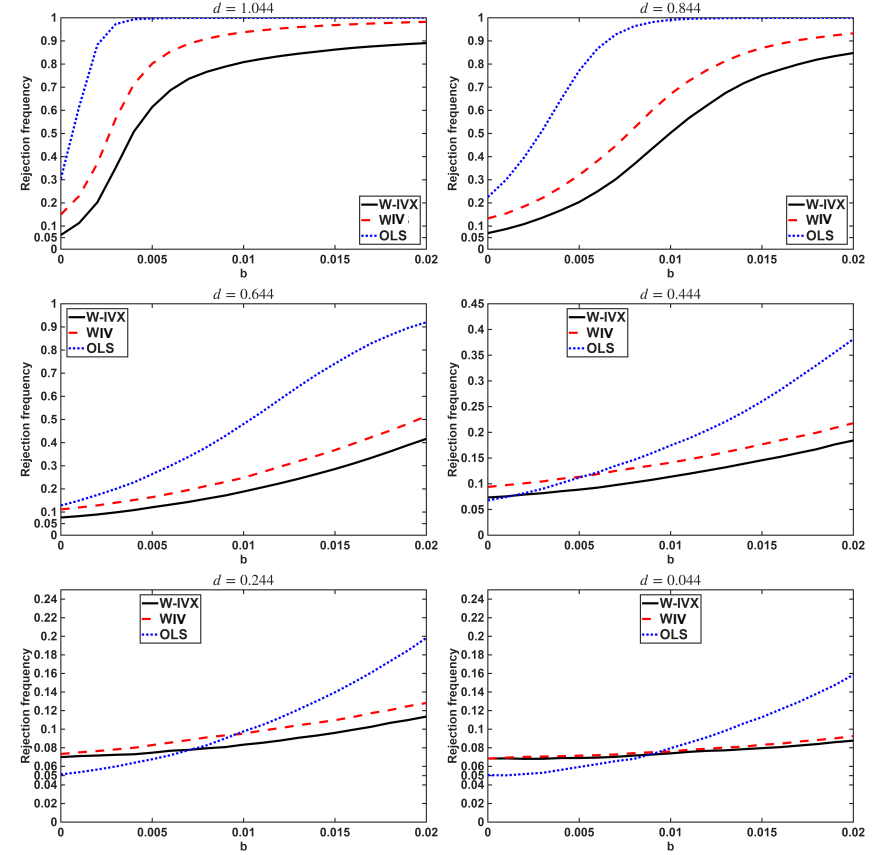
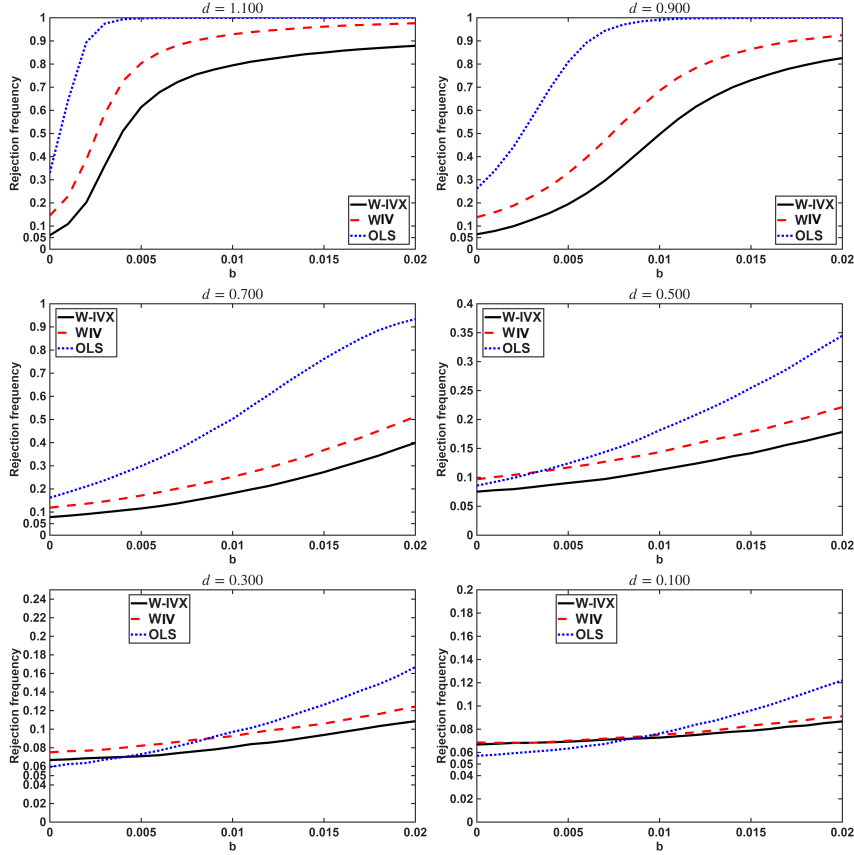


Figure 2: Asymptotic Power (nominal size 5%): $[f(x), w(x)] = [x, x^{(0.5)}]$, $\delta = -0.95$, $n = 1000$

(a) $\alpha = 2$, $d_* = 0.5$

(b) $\alpha = 1.8$, $d_* = 0.444$.

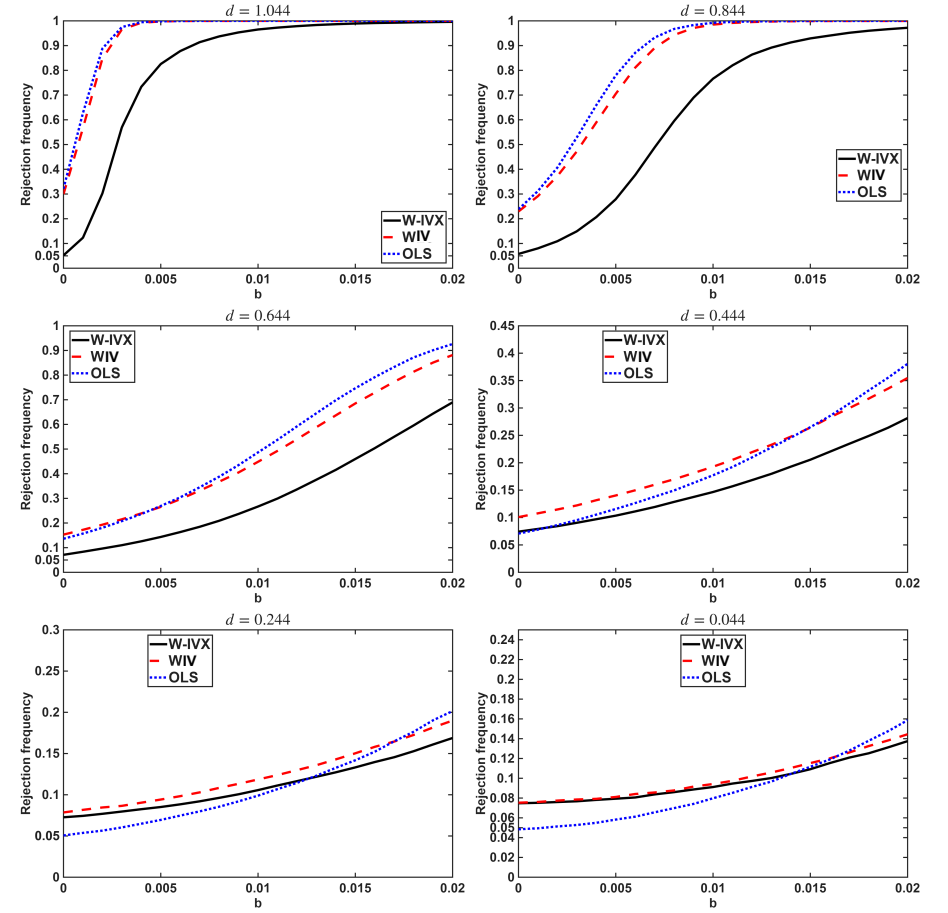
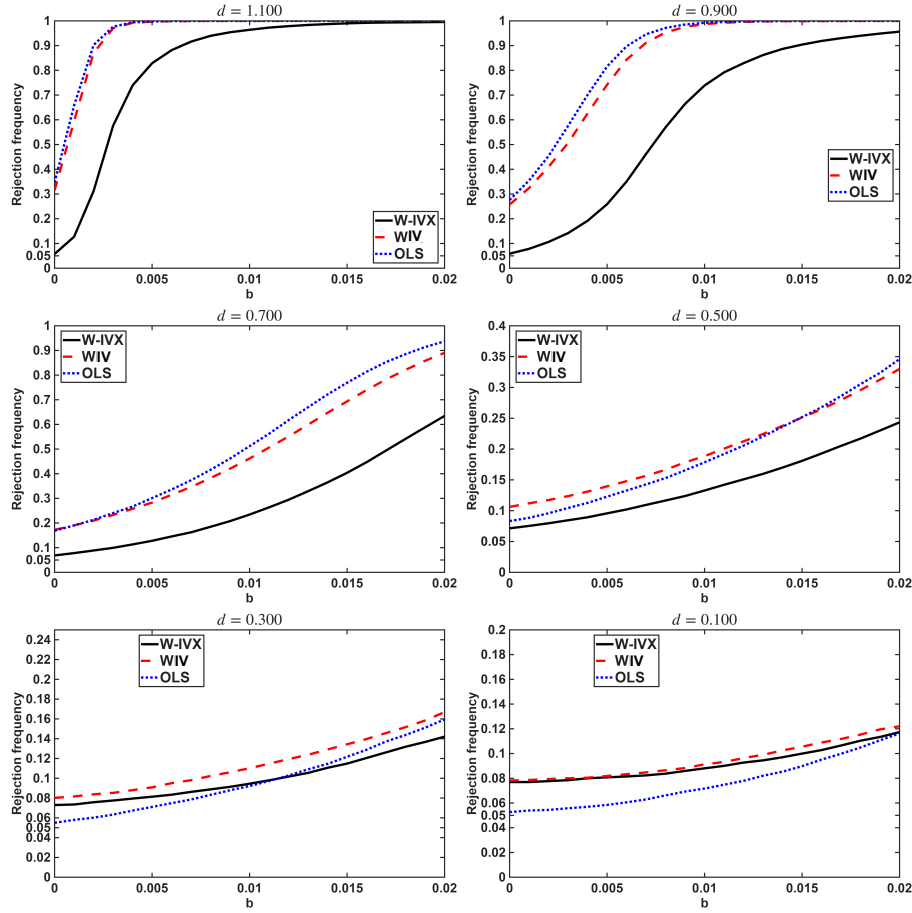


Figure 3: Asymptotic Power (nominal size 5%): $[f(x), w(x)] = [x, x^{(0.75)}]$, $\delta = -0.95$, $n = 1000$

(a) $\alpha = 2$, $d_* = 0.5$

(b) $\alpha = 1.8$, $d_* = 0.444$.

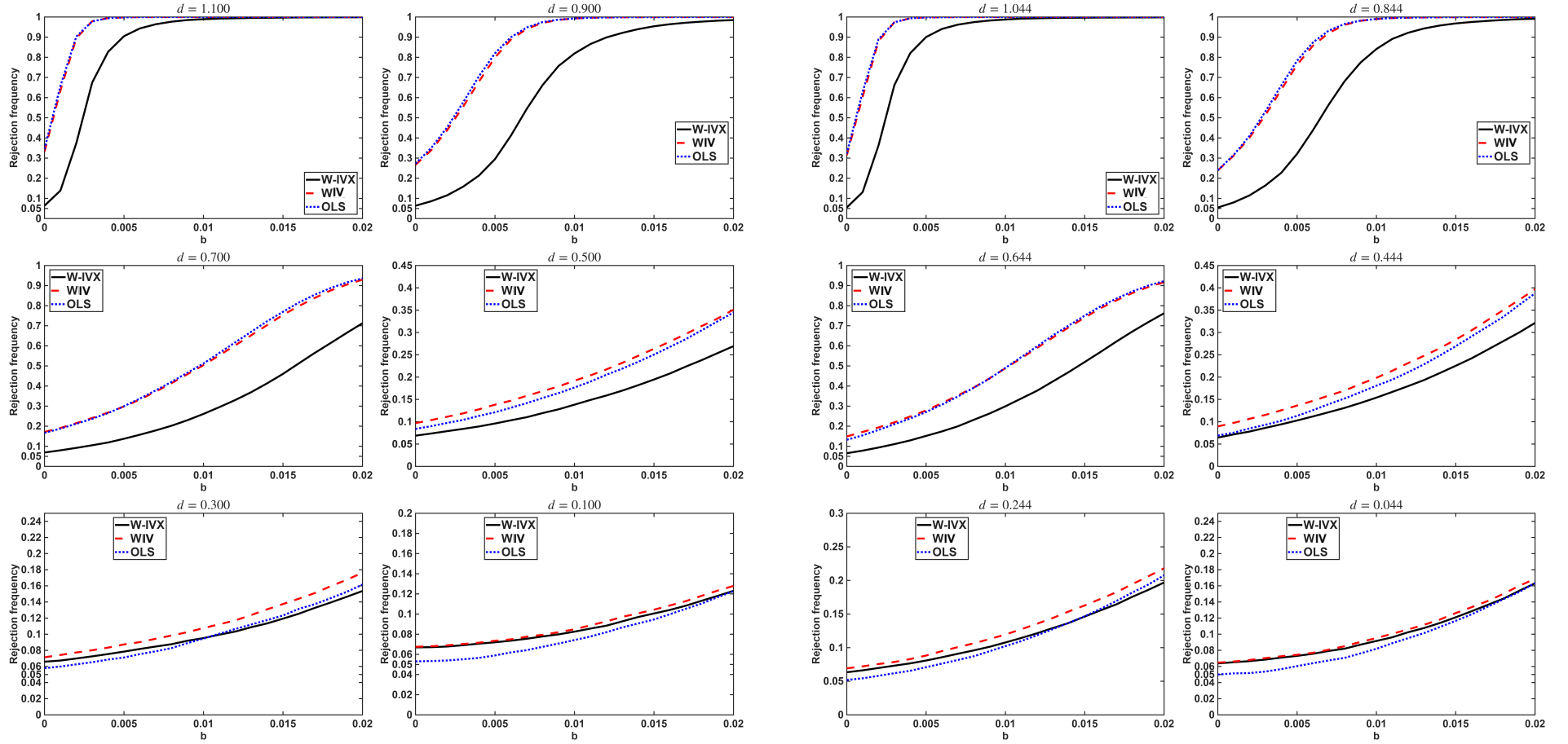


Figure 4: Asymptotic Power (nominal size 5%): $[f(x), w(x)] = [x, x]$, $\delta = -0.95$, $n = 1000$

(a) $\alpha = 2$, $d_* = 0.5$

(b) $\alpha = 1.8$, $d_* = 0.444$.

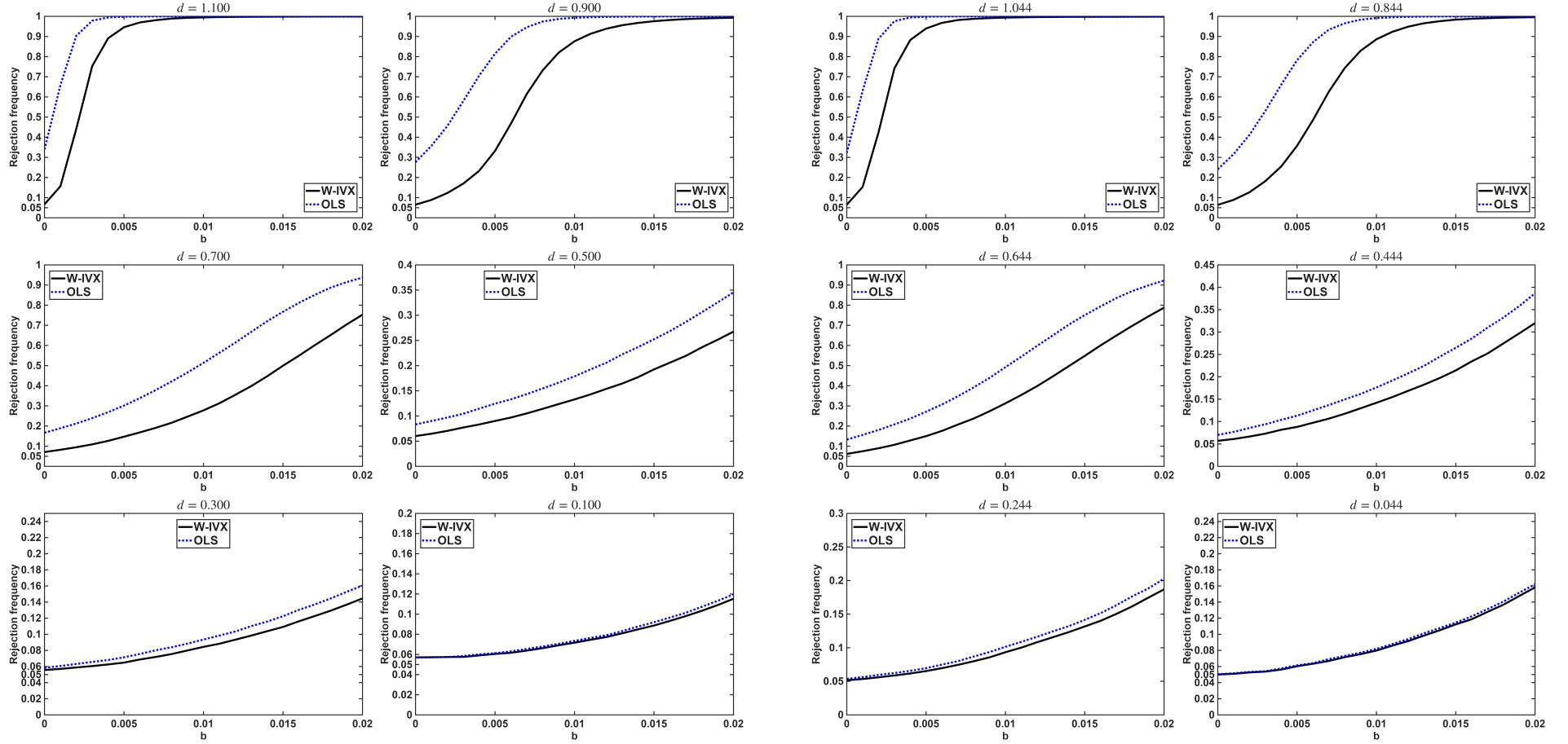


Figure 5: Size Adjusted Power (nominal size 5%): **DGP1**, $[f(x), w(x)] = [x, x^{(0)}]$, $\delta = -0.95$, $n = 1000$

(a) $\alpha = 2$, $d_* = 0.5$

(b) $\alpha = 1.8$, $d_* = 0.444$

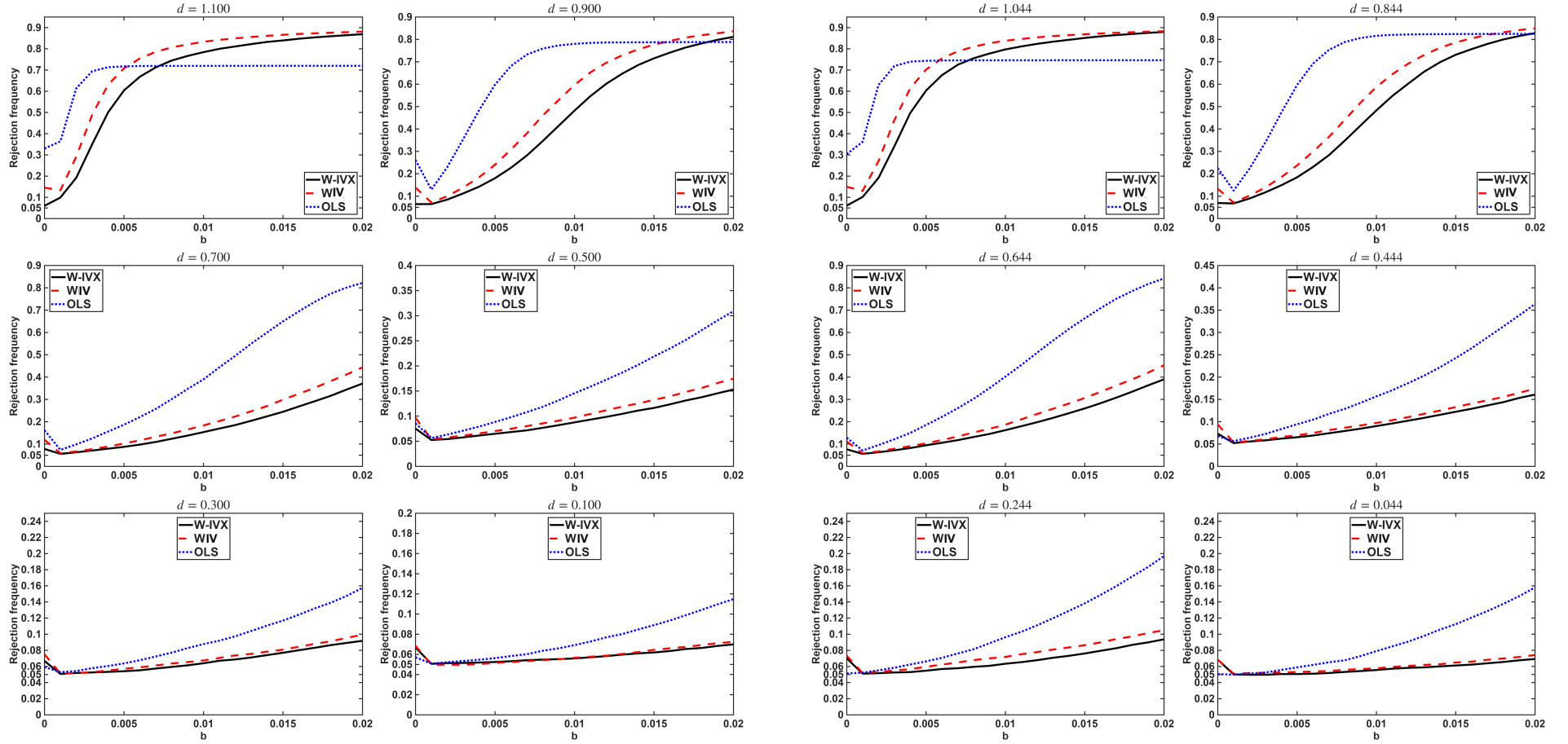


Figure 5: Size Adjusted Power (nominal size 5%): $[f(x), w(x)] = [x, x^{(0)}]$, $\delta = -0.95$, $n = 1000$ (continued)

(c) $\alpha = 1.6$, $d_* = 0.375$

(d) $\alpha = 1.4$, $d_* = 0.286$

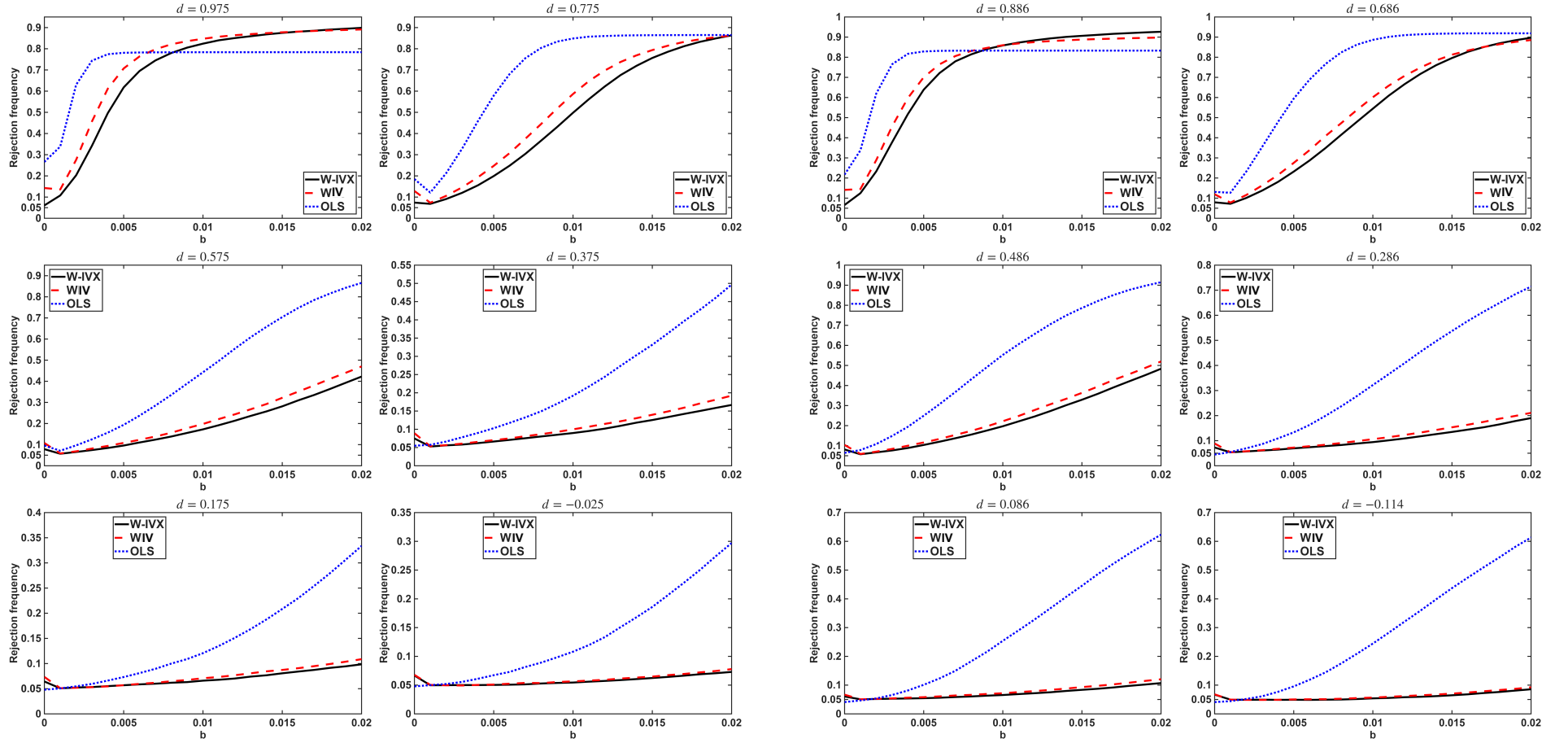


Figure 6: Size Adjusted Power (nominal size 5%): **DGP1**, $[f(x), w(x)] = [x, x^{(0.5)}]$, $\delta = -0.95$, $n = 1000$

(a) $\alpha = 2$, $d_* = 0.5$

(b) $\alpha = 1.8$, $d_* = 0.444$

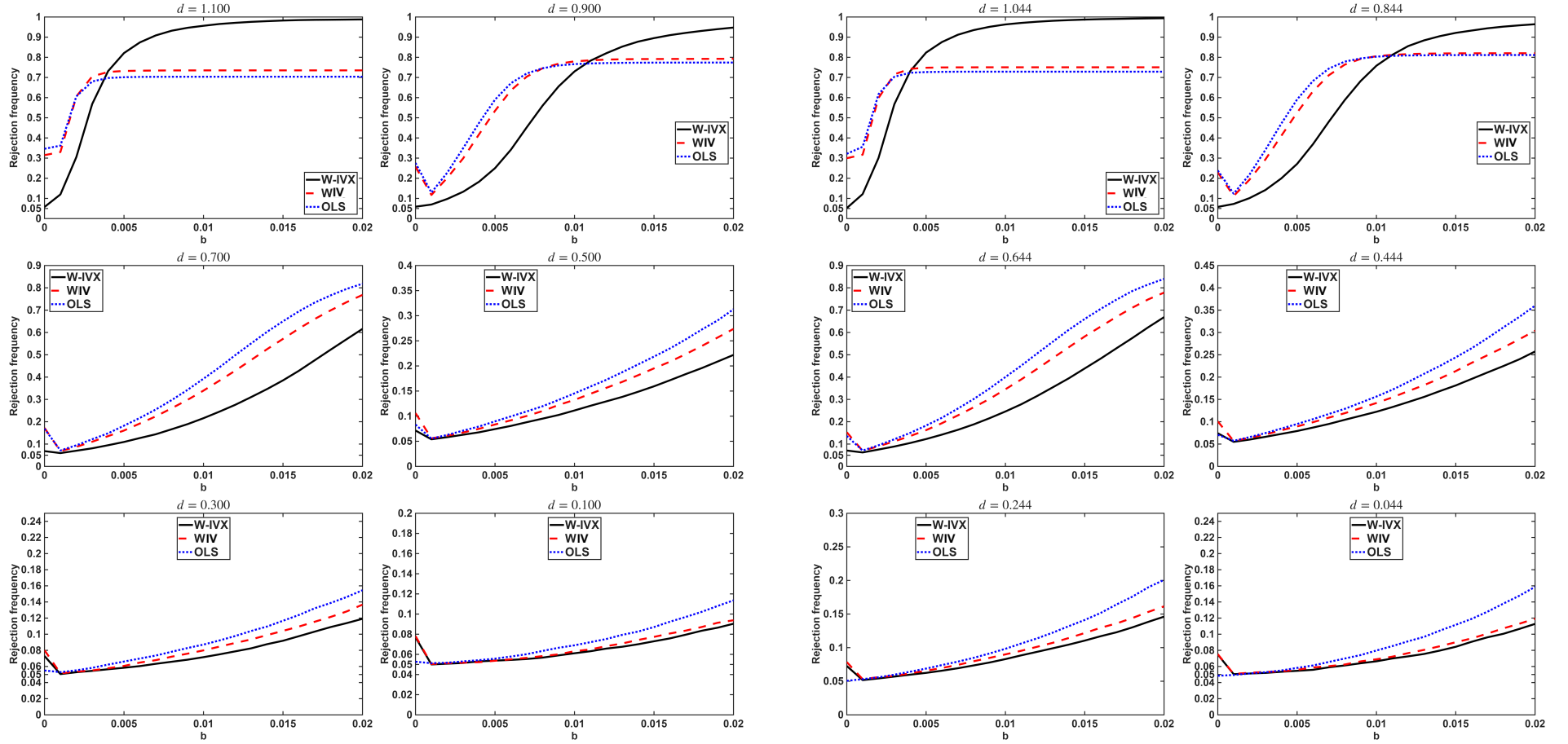


Figure 6: Size Adjusted Power (nominal size 5%): $[f(x), w(x)] = [x, x^{(0.5)}]$, $\delta = -0.95$, $n = 1000$ (continued)

(c) $\alpha = 1.6$, $d_* = 0.5$

(d) $\alpha = 1.4$, $d_* = 0.444$

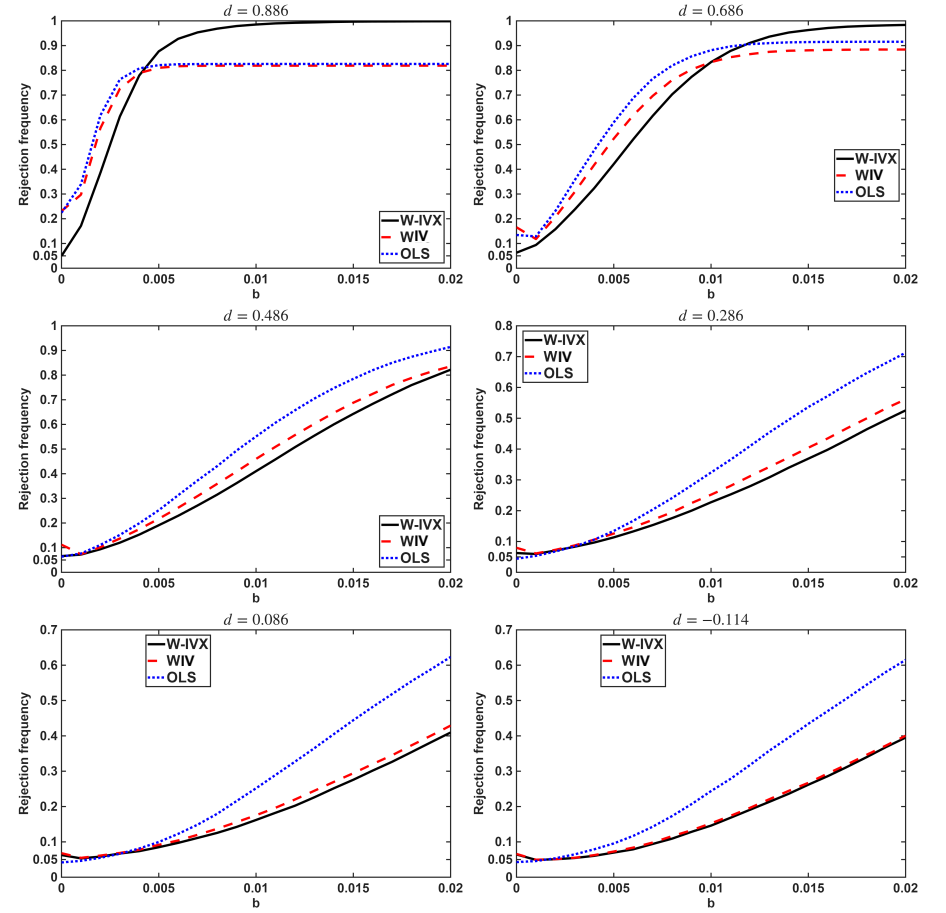
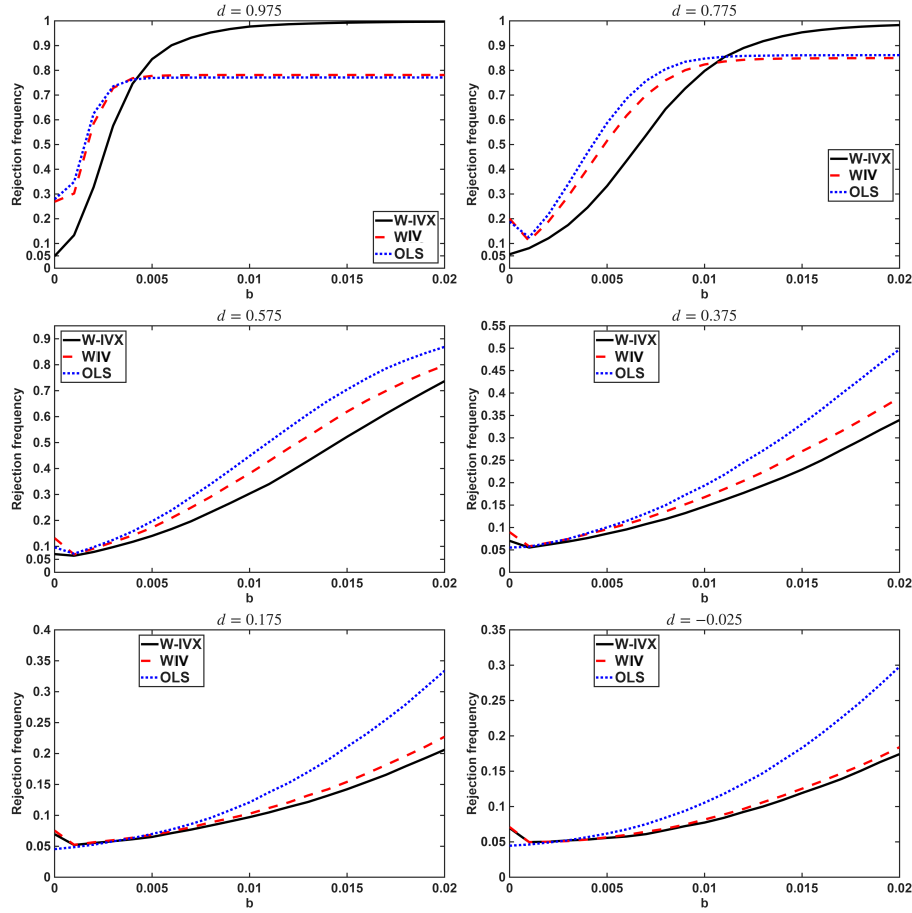


Figure 7: Size Adjusted Power (nominal size 5%): **DGP1**, $[f(x), w(x)] = [x, x^{(0.75)}]$, $\delta = -0.95$, $n = 1000$

(a) $\alpha = 2$, $d_* = 0.5$

(b) $\alpha = 1.8$, $d_* = 0.444$

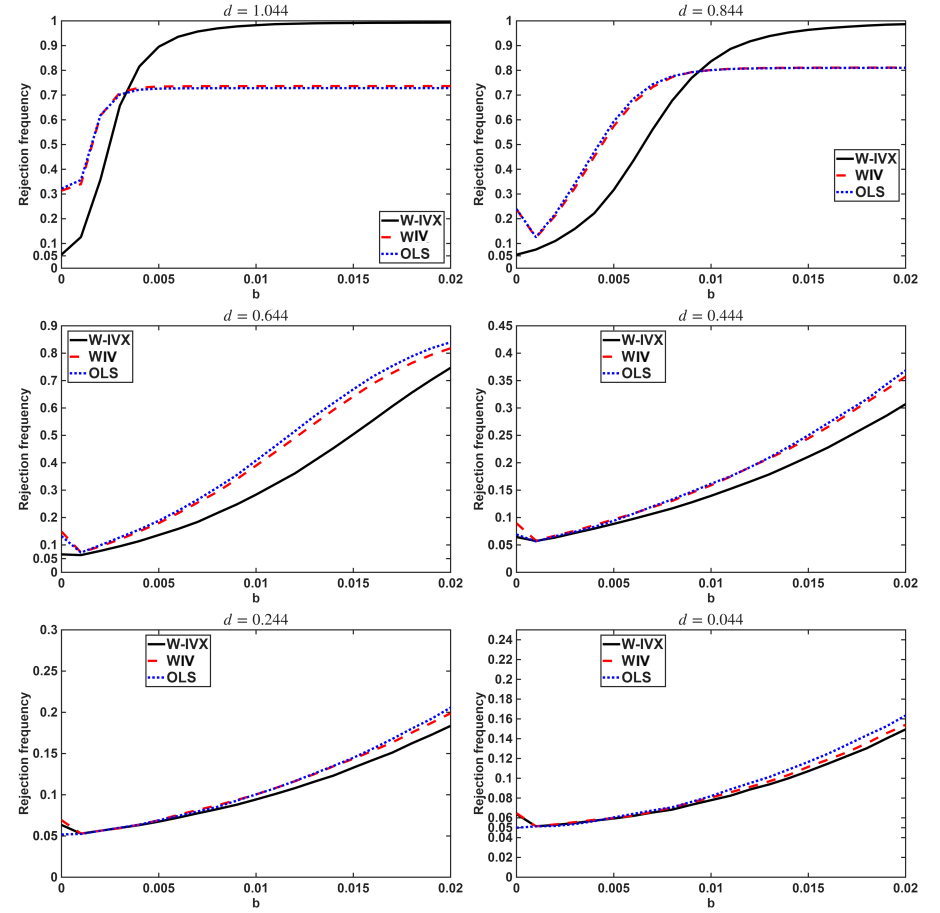
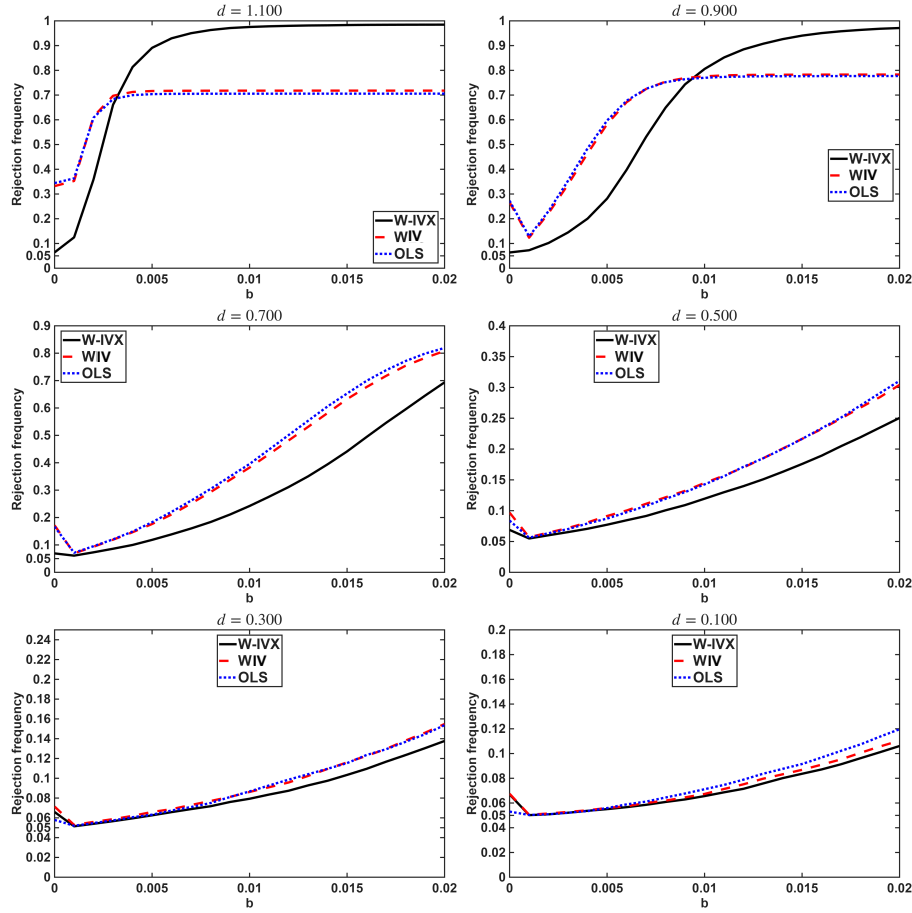


Figure 7: Size Adjusted Power (nominal size 5%): $[f(x), w(x)] = [x, x^{(0.75)}]$, $\delta = -0.95$, $n = 1000$ (continued)

(c) $\alpha = 1.6$, $d_* = 0.375$

(d) $\alpha = 1.4$, $d_* = 0.286$

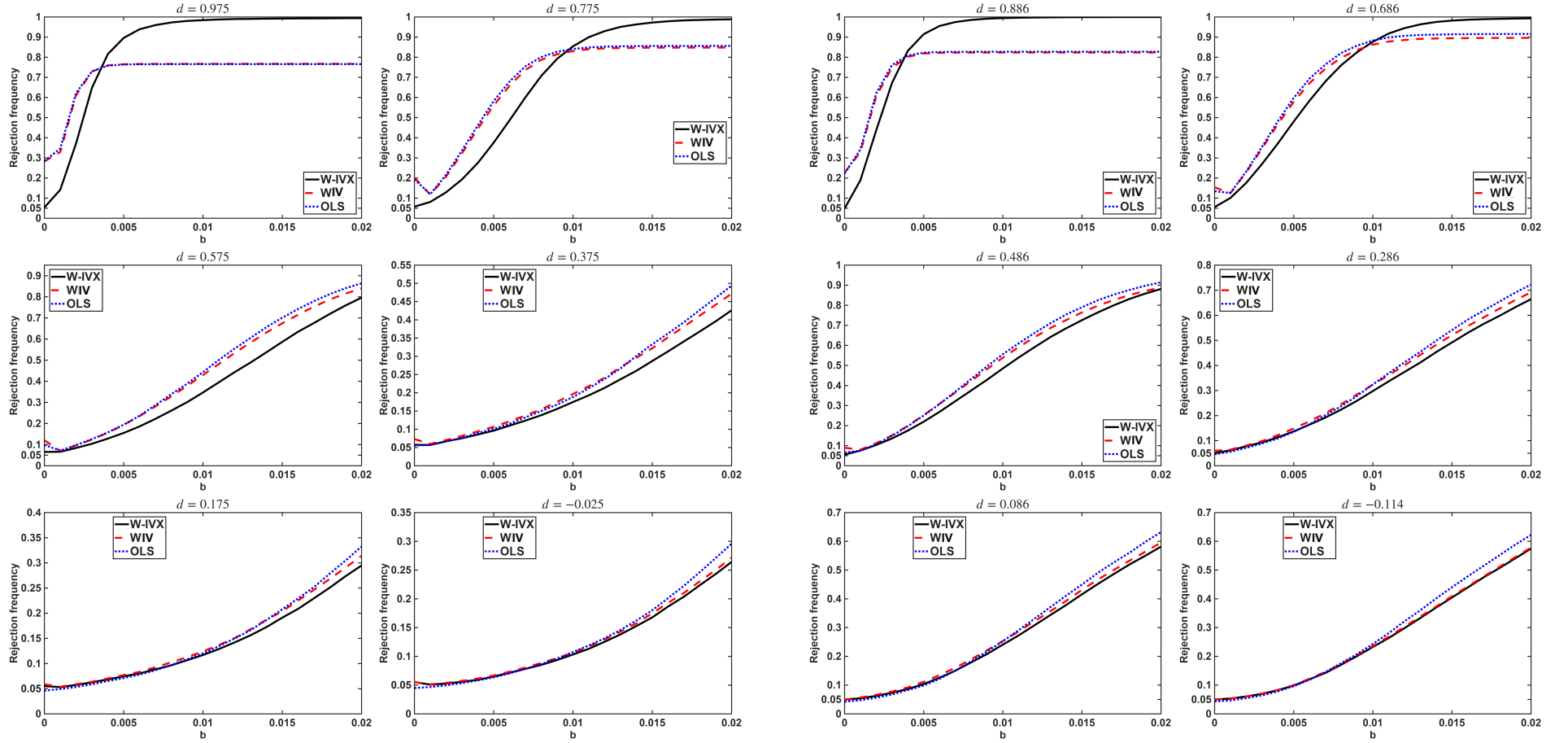


Figure 8: Size Adjusted Power (nominal size 5%): **DGP1**, $[f(x), w(x)] = [x, x]$, $\delta = -0.95$, $n = 1000$

(a) $\alpha = 2$, $d_* = 0.5$

(b) $\alpha = 1.8$, $d_* = 0.444$

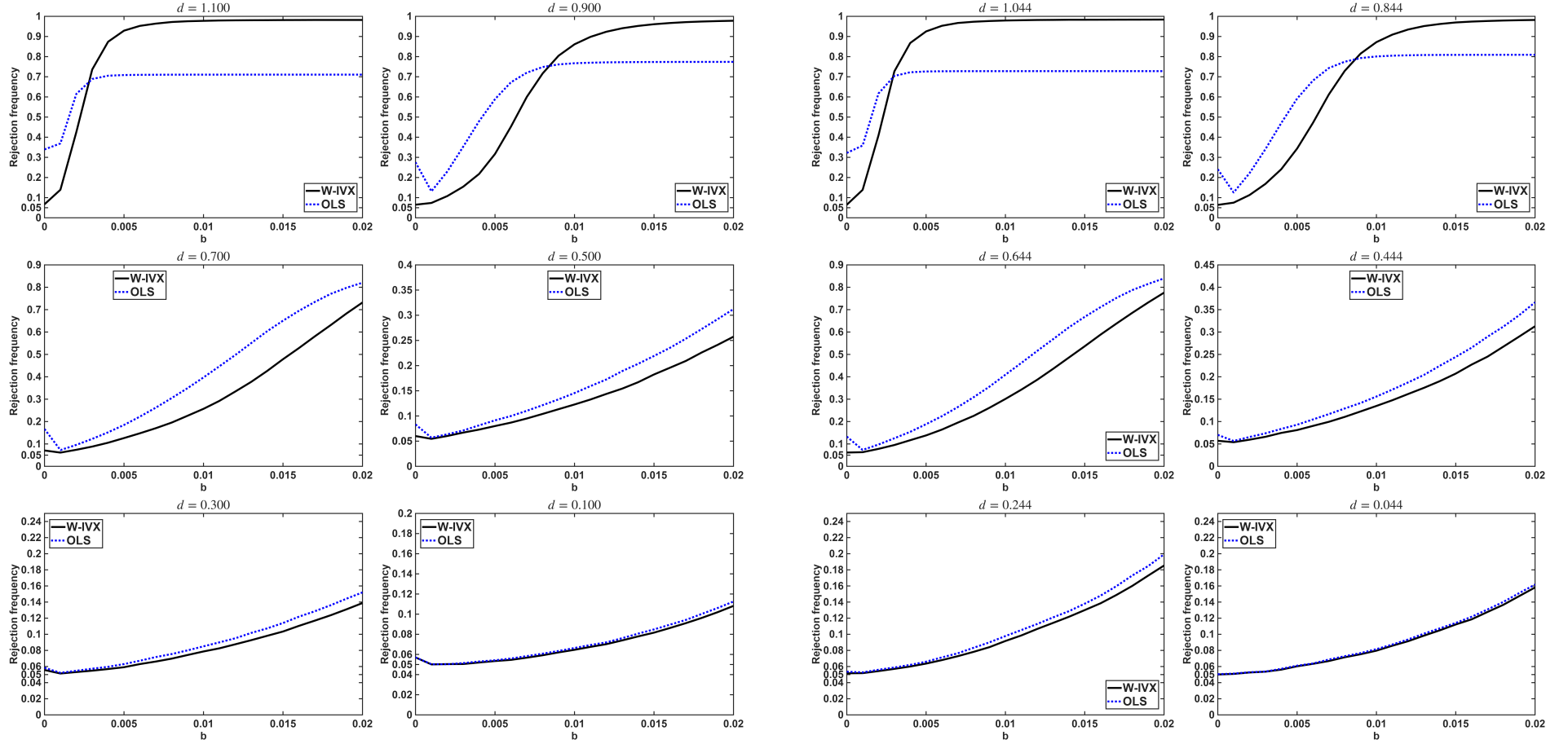


Figure 8: Size Adjusted Power (nominal size 5%): $[f(x), w(x)] = [x, x]$, $\delta = -0.95$, $n = 1000$ (continued)

(c) $\alpha = 1.6$, $d_* = 0.375$

(d) $\alpha = 1.4$, $d_* = 0.286$

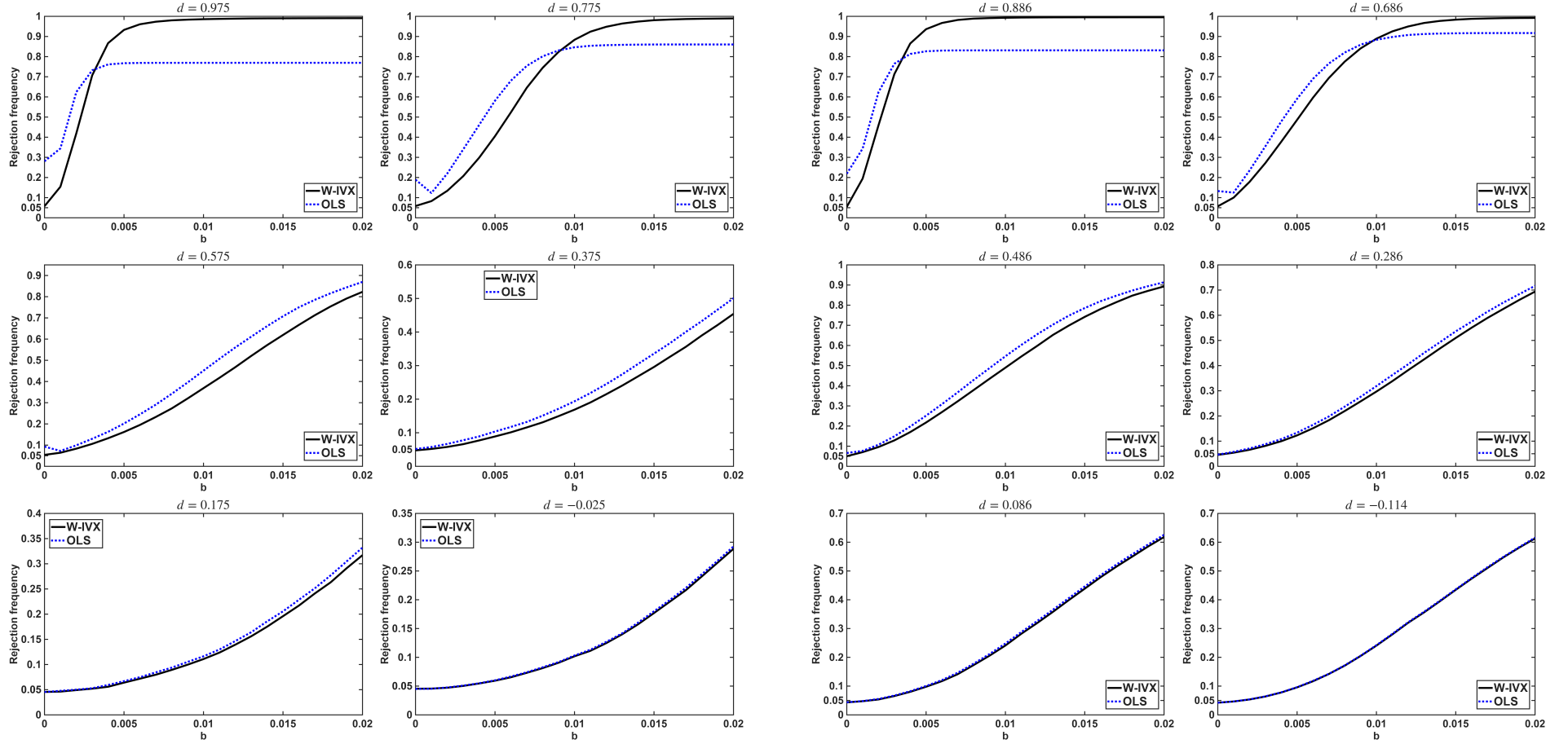


Figure 9: Size Adjusted Power (nominal size 5%): **DGP1**, $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$, $\delta = -0.95$, $n = 1000$

(a) $\alpha = 2$, $d_* = 0.5$

(b) $\alpha = 1.8$, $d_* = 0.444$

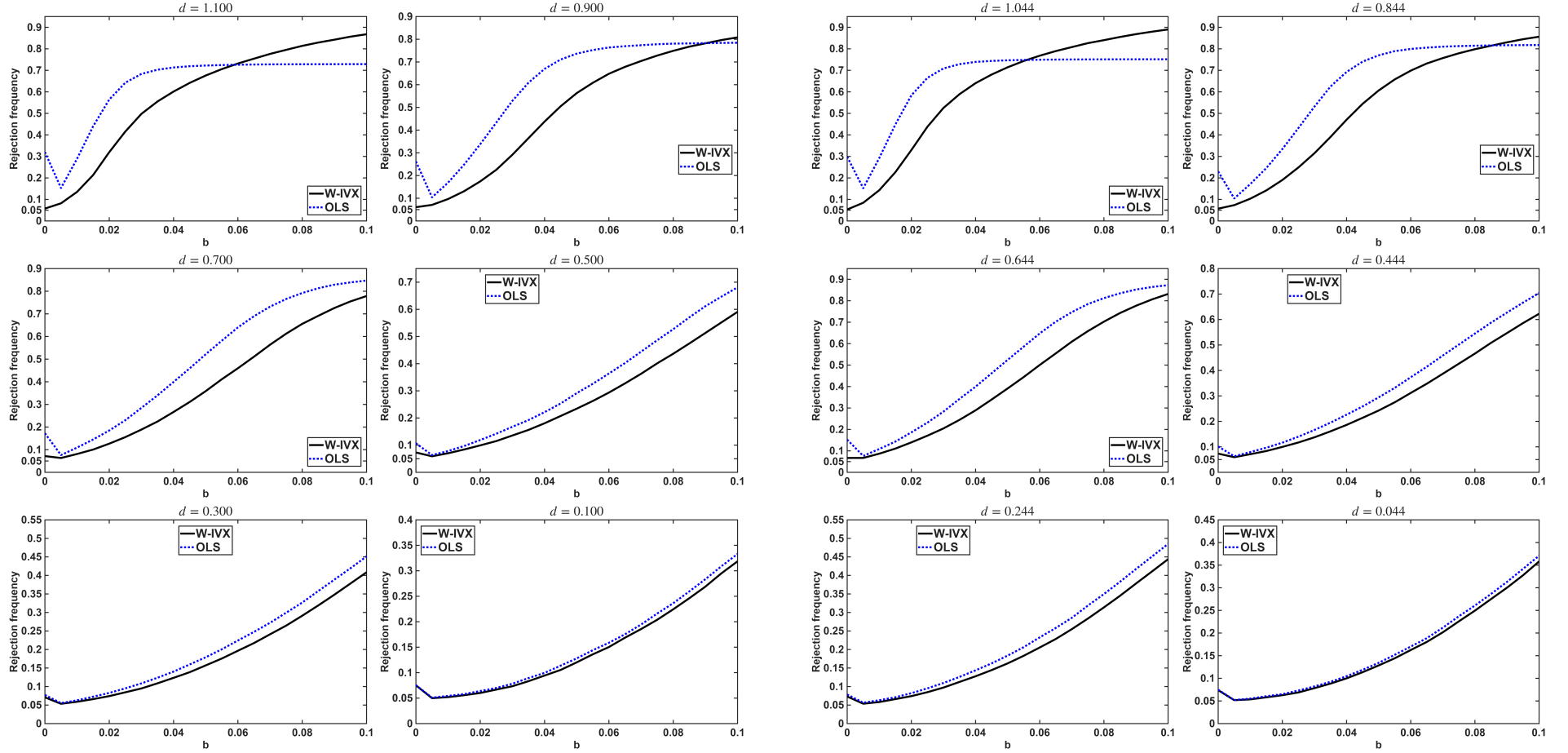
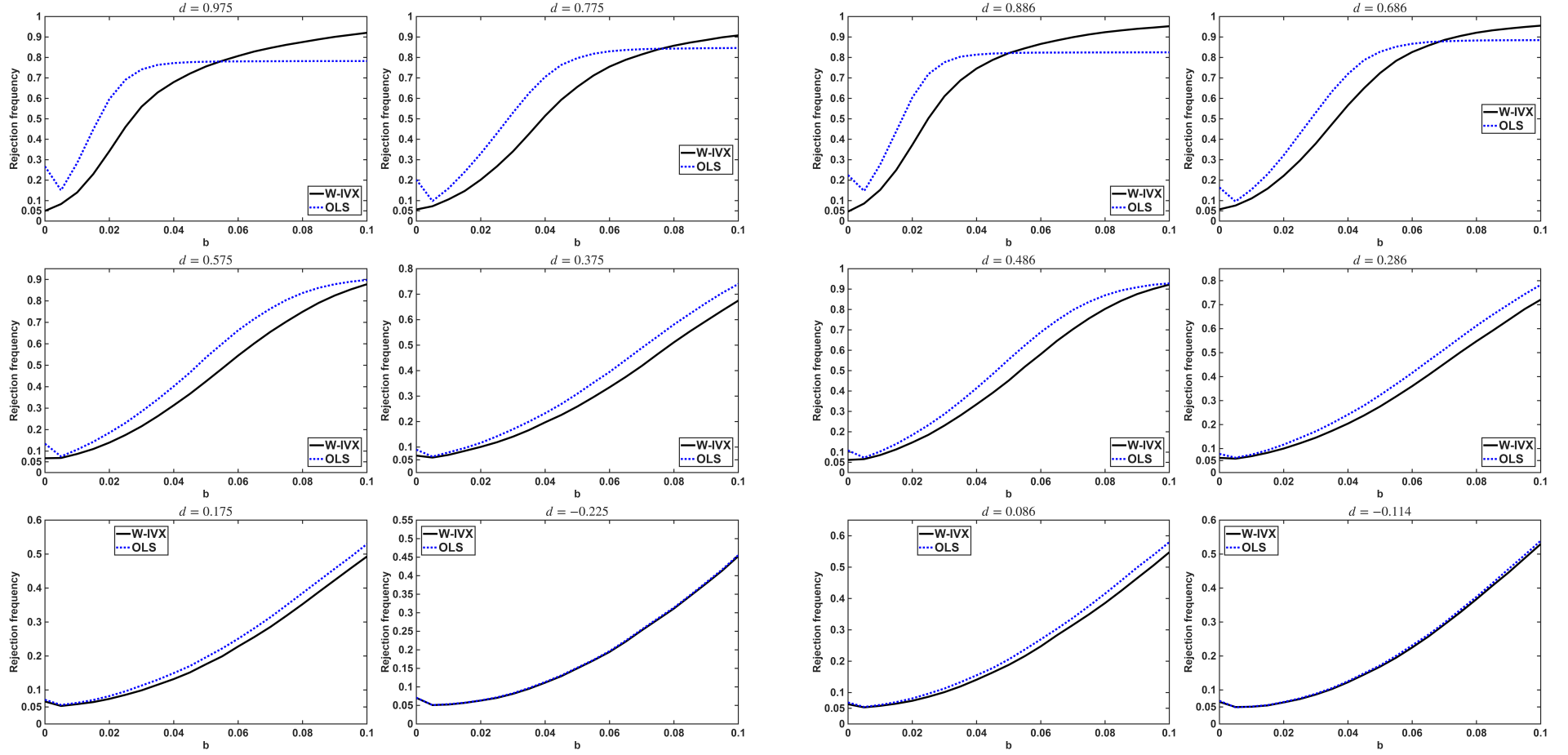


Figure 9: Size Adjusted Power (nominal size 5%): $[f(x), w(x)] = [x^{(0.5)}, x^{(0.5)}]$, $\delta = -0.95$, $n = 1000$ (continued)

(c) $\alpha = 1.6$, $d_* = 0.375$

(d) $\alpha = 1.4$, $d_* = 0.286$



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