University of Cyprus
Department of Economics

Working Paper 03-2017

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Dimitrios Xefteris and Nicholas Ziros

Department of Economics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus
Tel.: +357-22893700, Fax: +357-22895028, Web site: http://www.ucy.ac.cy/econ/en

Strategic Vote Trading under Complete Information

Dimitrios Xefteris*   Nicholas Ziros†

June, 2017

Abstract

We study two-party elections considering that: a) prior to the voting stage voters are free to trade votes for money according to the rules of the Shapley-Shubik strategic market games; and b) voters' preferences –both ordinal rankings and cardinal intensities– are public information. While under plurality rule no trade occurs, under a power-sharing system (voters' utilities are proportionally increasing in the vote share of their favorite party) *full trade is always an equilibrium* (two voters –the strongest supporter of each party– buy the votes of all others). Notably, this equilibrium implements *proportional justice* with respect to the two buyers: the ratio of the parties' vote shares is equal to the ratio of the preference intensities of the two most opposing voters.

JEL: C72, D72, P16

Keywords: Vote trading; Complete information; Strategic market games; Power sharing; Proportional justice.

*Department of Economics, University of Cyprus, P.O. Box 20537, Nicosia 1678, Cyprus (e-mail: xefteris.dimitrios@ucy.ac.cy)

†Department of Economics, University of Cyprus, P.O. Box 20537, Nicosia 1678, Cyprus (e-mail: n.ziros@ucy.ac.cy).

1 Introduction

Vote trading is a common practice in bodies of collective decision making because it allows voters to express their preference intensities over alternatives and can be seen in various forms, such as exchange of supports for different proposals (logrolling) or exchange of ballots for money or other commodities. The theoretical investigation of vote markets has attracted the interest of scholars from many disciplines not only for their frequent use but also for their distinct attributes. However, the literature has not provided yet definite answers to many central questions; and hence there are no widely accepted conclusions regarding the properties of vote trading.[1]

The quality of information that individuals hold about the policy preferences of the other individuals, is considered particularly relevant in the debate. Indeed, as Piketty (1994) argues, there can be no price-taking equilibrium with active vote trading under complete information, as incentives for trading are hard to be aligned when individual (and, thus, most probably conflicting) preferences are publicly known. For this reason, most works in the literature feature incomplete information and, in specific, consider either that there is uncertainty about both the ordinal and cardinal preferences of other voters (e.g., Casella et al., 2012; Xefteris and Ziros, 2017), or that ordinal preferences are publicly known but the intensities of these preferences are private information (e.g., Casella et al., 2014; Casella and Turban, 2014).

In this paper we present a simple vote-trading model that differs from earlier approaches in various ways, so as to offer new insights on the effects of information on vote trading. Our main goal is to try to understand the consequences of vote trading in the least explored informational environment: that of complete information. Undeniably, incomplete information is a reasonable assumption in many environments, but a proper understanding of vote markets –as it is the case with markets for standard goods– may be achieved only if we have a good idea of what to expect in a complete information setting as well. Apart from this, our approach is further motivated by the fact that in most legislatures or committees it is not realistic to assume uncertainty about the preferred

[1] See Philipson and Snyder (1996); Casella et al. (2012) for a detailed exposition of the issue.

policy alternatives of their members. In addition, in most cases not only their preferred alternatives but also the intensities of these preferences are publicly known, which also determine the vote buying or selling incentives when vote trading is allowed. Hence, we believe that our complete information approach is relevant in many bodies of group decision making.

Our approach employs a strategic rather than a price-taking exchange framework, as vote trading is conducted via the mechanism of strategic market games (introduced in Shubik, 1973; Shapley and Shubik, 1977), which maps agents' actions to prices and allocations. We study a non-cooperative game in strategic form which allows us: a) to use Nash equilibrium as a solution concept and b) not to impose any price-taking hypothesis as the standard approaches on vote markets (e.g., Philipson and Snyder, 1996; Casella et al., 2012); and hence to effectively deal with the conceptual and practical problems of competitive equilibrium analysis in markets with externalities (see, for instance, del Mercato, 2006). In particular, we study a two-party election in which prior to the voting stage individuals are free to trade votes for money, if they find it profitable to do so. That is, an individual can offer her vote in exchange for money or can place a monetary bid in exchange for votes. In this setup, the price of a vote is endogenously determined by the actions of vote traders, while the distribution mechanism allocates the supplied votes to vote buyers in proportion to their bids and accordingly distributes monetary bids to those who chose to sell their votes.

In this framework we study the consequences of vote trading with perfect information under different electoral systems. Initially we argue that under the simple plurality rule the unique equilibrium involves all players abstaining from vote trading. Then, we move on to examine whether an alternative electoral system, which avoids some deficiencies associated with plurality rule (e.g., severe discontinuities in the outcome function), can guarantee a generic existence of an equilibrium with vote trading. To this end, we consider a power-sharing system, in which the decision-making power is distributed between the two competing parties in proportion to their vote shares. Similar frameworks have been extensively employed in the political economics literature[2], but, to the best of our

[2]See, for instance, Lijphart (1984), Austen-Smith and Banks (1988), Ortuño-Ortín (1997), Alesina and Rosenthal (2000), Llavador (2006), Sahuguet and Persico (2006), Herrera et al. (2014), Iaryczower

knowledge, only Xefteris and Ziros (2017) have studied vote trading in such systems.[3] In such a setup the whole distribution of votes is crucial for the determination of policies and a voter's utility is proportionally increasing in the vote share of her favorite party.

We provide a full characterization of all Nash equilibria under the power-sharing electoral rule. Apart from the no-trade equilibrium we show that, for every generic preference profile, there exists a unique full-trade equilibrium. In this equilibrium only two players, the strongest supporter of each party, are buying votes whereas all the other players prefer to sell their votes. Moreover, we show that partial-trade equilibria might exist, but only for specific classes of preference profiles. That is, depending on the precise preference profile we might additionally have equilibria in which trade occurs, but not among all players. In these equilibria, again, only the strongest supporter of each party buys votes, some players sell their votes while the rest –with preference intensities within a party-specific interval– prefer to refrain from vote trading and simply vote for their preferred party during the elections. Hence, in all equilibria with active trading the competition between two vote buyers determines in a large degree the final vote shares of the two parties. It should be noted that similar results with respect to the number of vote traders have been obtained by the means of alternative equilibrium concepts and institutional settings in Casella et al. (2012), Casella et al. (2014) and Casella and Turban (2014), where the two voters with the highest valuations buy votes and all other voters sell their votes.[4]

Concerning the welfare properties of vote trading, the earlier literature has produced both positive (for example, Buchanan and Tullock, 1962) and negative (for example, Riker and Brams, 1973) results about the superiority of vote trading over the no-trade option, focusing on Benthamite/utilitarian criteria. More recently, Casella et al. (2012), Casella et al. (2014), and Casella and Turban (2014) showed that vote trading is welfare

and Mattozzi (2013), Saporiti (2014), Matakos et al. (2016) among others.

[3]That paper considered incomplete information regarding voters' preferences –which, as explained above, enhances the prospects of equilibrium existence–, symmetric uncertainty –in the sense that no party is expected to be supported by a more voters than the other– and restricted strategy spaces –in the sense that vote buyers were not allowed to bid any arbitrary monetary amount. In this paper, we consider instead perfect information, arbitrary voters' preferences and unrestricted strategy spaces: each voter is free to bid any monetary amount she deems best.

[4]In Casella et al. (2014) and Casella and Turban (2014) these two vote buyers are necessarily one from each party, whereas in Casella et al. (2012) there is no such restriction.

decreasing when compared to plurality rule without vote trading, in the sense that it implements less frequently the alternative that maximizes the sum of individual cardinal utilities. On the other hand, Xefteris and Ziros (2017), in an incomplete information variant of the current framework, proved that vote trading is welfare improving because when vote trading is allowed all players' expected utility is larger compared to the case where vote trading is prohibited. The welfare analysis of vote trading under complete information is orthogonal to all these results, since in certain cases, vote trading leads to a larger social utility compared to simple voting, and in some others not.

In fact, vote trading under complete information in power-sharing systems is found to implement a different welfare optimum: it achieves *proportional justice in policy* with respect to the two buyers. That is, the ratio of the parties' vote shares is equal to the ratio of the preference intensities of their strongest supporters. The origins of proportional justice with respect to a distributional problem involving two individuals may be traced back to Aristotle and it has been recently studied by Broome (1984, 1991) and Segal (2006), in a more standard economics' context.[5] This result is arguably of independent interest as, to the best of our knowledge, this is the first known mechanism that takes into account only each party's stronger supporter and is "fairly biased" –in the context of Segal (2006)– towards the one with the most intense preferences. Of course this welfare analysis holds specifically for our unique full-trade equilibrium, and does not extend to other outcomes possibilities. But since in our complete information environment, the full-trade equilibrium is the unique one that exists for every generic preference profile, it is the only reasonable candidate for a comprehensive welfare analysis: other equilibria might only deliver insights for merely a fraction of possible preference distributions.

Overall, our analysis underlines the importance of three interacting aspects of the vote-trading environment. Namely: a) the information that voters' hold about their fellow citizens' preferences, b) the voting rule and c) the vote-trading mechanism in operation; with a particular emphasis on the latter. It is shown that under strategic vote trading institutions –as opposed to price-taking settings– equilibria exist under both plurality

[5] We should note that we consider implementation in the limit: we demonstrate that when the number of voters becomes arbitrarily large, the full-trade equilibrium is such that the ratio of the vote shares of the two parties converges to the ratio of the preference intensities of the two buyers.

and power-sharing rules, even under the demanding assumption of perfect information. Indeed, in the first case (plurality rule) the only equilibrium outcome is that no trade takes place and in the second case (power sharing), in the most robust equilibrium of the game, all voters engage in vote trading. But in both cases equilibrium behavior is well-defined and possible to be fully characterized. In a way, an important message of this paper is that the study of vote trading can prove more fruitful in the future if it is conducted in the strategic setting rather than in the competitive one, since the strategic framework of interactions is designed to nest externalities, while a competitive one cannot do that without additional and arguably less widely accepted assumptions.

The remainder of the paper is organized as follows. In Section 2 we develop the model, in Section 3 we present the main results and in Section 4 we discuss the welfare properties of the full-trade equilibrium. Some concluding remarks follow in Section 5. The discussion about partial-trade equilibria can be found in the Appendix.

2 The Model

We consider a committee of $n > 2$ voters and two parties (or policy alternatives), L and R. Voters fall into two types depending on their ordinal preferences, $t_i \in \{L, R\}$, where $t_i = L$ if $L \succ R$ and $t_i = R$ if $R \succ L$ for voter i. Hence we have two sets of voters with cardinality $n_L \geq 1$ and $n_R \geq 1$ respectively, where $n_L + n_R = n$. Each voter i is also characterized by her distinct intensity parameter $w_i > 0$ and let us denote with $\bar{w}^L$, $\bar{w}^R$ the valuations of the each party's strongest supporter. All voters have one vote each and concerning their monetary endowments we assume that they are significantly large (i.e., no individual faces liquidity constraints).[6]

The timing of the game is as follows: initially vote trading takes place; next players cast the amount of votes they have after the vote-trading stage in order to maximize their utilities; finally the payoffs of all players are computed.

Vote trading takes place in a trading post where each voter chooses whether to offer her

[6] This is a standard assumption in the vote-trading (e.g., Casella et al., 2014; Casella and Turban, 2014) and the vote-buying literature (e.g., Lalley and Weyl, 2016; Goeree and Zhang, 2016).

whole vote for sale, $q_i \in \{0, 1\}$, or whether to place a monetary bid, $b_i \geq 0$, for purchase of votes, with the restriction that a voter is not allowed to be active in both sides of the market. Hence, the strategy set of voter i is $S_i = \{(b_i, q_i) : b_i \geq 0, q_i \in \{0, 1\},\ b_i q_i = 0\}$.

Given a strategy profile $(b, q) \in \prod_{i \in I} S_i$ let B, Q denote aggregate bids and offers of all voters and B^T, Q^T denote aggregate bids and offers of all voters of type $T \in \{L, R\}$. In addition, for each i define B_{-i}, Q_{-i} as aggregate bids and offers of all voters other than voter i and $B^{t_i}_{-i}$, $Q^{t_i}_{-i}$ as aggregate bids and offers of all voters of type t_i other than voter i.

For a strategy profile with $BQ \neq 0$ the price of a vote is given by the fraction

$$p = \frac{B}{Q}.$$

The amount of votes, x_i, that a voter ends up with after the vote-trading stage and her net monetary transfers, m_i, are

$$(x_i, m_i) = \begin{cases} (1 + b_i/p, -b_i) & \text{if she is a vote buyer,} \\ (0, p) & \text{if she is a vote seller,} \\ (1, 0) & \text{if she chooses not to trade.} \end{cases}$$

The interpretation of this allocation rule is that the amount of votes offered for sale is distributed among vote buyers in proportion to their bids, whereas the monetary bids are equally distributed among vote sellers. In our framework, votes are perfectly divisible and, hence, a trader might end up having a non-integer number of votes. This is perfectly legitimate in our framework as, both under plurality rule and in a power-sharing system, all that matters is the share and not the actual number of votes that each alternative receives.

The vote shares $v_L \in [0, 1]$ and $v_R = 1 - v_L$ of the two parties after vote trading are given by

$$v_L = \frac{1}{n}\left(n_L - Q^L + B^L/p\right)$$

and

$$v_R = \frac{1}{n}\left(n_L - Q^R + B^R/p\right).$$

Under plurality rule the utility of voter i after the election is given by

$$u_i = \theta \times w_i + m_i, \text{ where } \theta = \begin{cases} 1 & \text{if } v_{t_i} > 1/2, \\ \frac{1}{2} & \text{if } v_{t_i} = 1/2, \\ 0 & \text{if } v_{t_i} < 1/2. \end{cases}$$

In case of a power-sharing system the utility of voter i after the election is given by

$$u_i = v_{t_i} \times w_i + m_i.^7$$

Given that the behavior of players in the voting stage is completely unambiguous (i.e., casting all votes that one has to her preferred party is her dominant strategy), we essentially have an one-shot game and an equilibrium is defined as a profile of pure strategies $(b, q) \in \prod_{i \in I} S_i$ that forms a Nash equilibrium.

3 Main Results

Let us first discuss the difficulties that arise in our vote-trading model under plurality rule. Proposition 1 highlights the problem of nonexistence of equilibrium with active trading in a majoritarian environment.

Proposition 1 *No trade is the only equilibrium under plurality rule.*

Proof. We can straightforwardly claim that no trade is always an equilibrium, as abstaining from trade is the best response of an individual when all other voters choose not to trade.

[7] This formulation of voters' preferences is perfectly compatible with other papers studying power-sharing systems (see, for example, Herrera et al., 2014; Iaryczower and Mattozzi, 2013).

Now let us show that any equilibrium with vote trading must involve vote buyers of both types. Suppose, on the contrary, that there are only type L vote buyers. In such an eventuality a vote buyer of type L will always deviate by reducing her bid, hence no equilibrium involves vote buyers of only one type.

With vote trading two cases arise as possible outcomes; in Case 1 there is a tie between the two alternatives, that is both parties have the same number of votes, whereas in Case 2 one of the two alternatives wins. Case 1 cannot be an equilibrium outcome, as in such an eventuality a vote buyer will be willing to slightly increase her bid, which in turn will result in her favorite alternative acquiring a majority position and thus to a substantial increase in her payoff.

In Case 2, suppose that party L is the plurality winner after the vote-trading stage. Then a vote buyer of type R will deviate to selling her vote. In other words, there will be no vote buyers of type R, but, on the contrary, all type R individuals will be willing to sell their votes. Similarly, if party R is the plurality winner after the vote-trading stage a type L vote buyer will deviate to selling her vote. Hence, there cannot be an equilibrium with vote trading resulting in a party being the plurality winner.

Hence, under plurality rule no equilibrium involves vote trading. ■

We now turn our attention to the power-sharing systems where, apart from no trade, we show that equilibria involving trade always exist. We start by characterizing the behavior of players in such equilibria (Lemma 1 and Lemma 2) and then we establish that they generically exist (Proposition 1).

Lemma 1 *In a power-sharing system, an equilibrium with vote trading is such that exactly two voters –the strongest supporter of each party– buy votes.*

Proof. Notice that if in an equilibrium we have vote trading, it must be the case that at least a player sells her vote and at least one player bids a positive monetary amount: i.e., $BQ > 0$. First, we show that in an equilibrium (b, q) with vote trading only one voter from each party buys votes. Consider a voter i of type L with valuation w_i who buys

$b_i > 0$ votes in some equilibrium (b, q). This individual faces the following –constrained– problem

$$\max_{b_i \geq 0} u_i = \frac{1}{n}\left(n_L - Q^L_{-i} + (b_i + B^L_{-i})\frac{Q^L_{-i} + Q^R}{b_i + B^L_{-i} + B^R}\right) w_i - b_i,$$

which is well-behaved in $b_i \in [0, +\infty)$.[8] By solving this problem we get that b_i must be such that $b_i = -B^L_{-i} - B^R + \left(\frac{w_i Q B^R}{n}\right)^{1/2}$. Hence, for the equilibrium profile (b, q), the total bids of type L individuals can be written as

$$B^L = -B^R + \left(\frac{w_i Q B^R}{n}\right)^{1/2}. \tag{1}$$

Now assume that another voter j of type L, with valuation $w_j \neq w_i$, also buys votes $b_j > 0$ in equilibrium (b, q). Solving the maximization problem for this voter one can derive that b_j is such that $b_j = -B^L_{-j} - B^R + \left(\frac{w_j Q B^R}{n}\right)^{1/2}$ and hence the corresponding total bids of type L individuals are

$$B^L = -B^R + \left(\frac{w_j Q B^R}{n}\right)^{1/2}. \tag{2}$$

However in order for the expressions (1) and (2) to hold at the same time we need $w_j = w_i$, which contradicts our assumption that each voter of type L is characterized by a distinct preference intensity. Hence, there is no equilibrium (b, q) with two or more buyers of type L. Similarly, in equilibrium (b, q), only one individual of type R may be buying votes.

Next we argue that there can be no equilibrium with active trading if there is only one buyer. In other words, in an equilibrium (b, q) with vote trading there must be one buyer of each type. Indeed, consider an equilibrium (b, q) such that there is only one buyer of type L and no buyer of type R. Then, from the above expressions we get that the type L buyer is submitting a zero bid, which leads to a contradiction.

[8] We use the term constrained since the presented maximization does not consider the possibility of selling one's vote, and we note that it is well behaved in b_i in the sense that, as long as at least one other player sells her vote, it is well defined, differentiable, and strictly concave in $[0, +\infty)$.

Now we proceed to show that in an equilibrium (b, q) with vote trading, the two vote buyers must be the strongest supporters of each party. Suppose, on the contrary, that a voter i with valuation $w_i < \bar{w}^L$ is the only vote buyer of type L. In an equilibrium (b, q) with vote trading in which i is a vote buyer, it must be true that $b_i = -B^R + \left(\frac{w_i Q B^R}{n}\right)^{1/2}$. But given this profile of strategies (b, q), the best response bid of the strongest supporter of type L will be $\bar{b}^L = -b_i - B^R + \left(\frac{\bar{w}^L Q B^R}{n}\right)^{1/2} = \left(\frac{\bar{w}^L Q B^R}{n}\right)^{1/2} - \left(\frac{w_i Q B^R}{n}\right)^{1/2} > 0$, that is, she also submits a positive bid for purchase of votes, which is a contradiction to the fact that there can be no equilibrium with two buyers of the same type. Hence, in an equilibrium (b, q) with vote trading only the individual of type L with the highest valuation buys votes. Similarly, only the strongest supporter of type R buys votes.

So if an equilibrium (b, q) with vote trading exists it must be such that exactly two voters –the strongest supporter of each party– buy votes. ■

Let us note that in an equilibrium (b, q) with vote trading, it must be true for the equilibrium bids of the two vote buyers that $\frac{\bar{b}^L}{\bar{b}^R} = \frac{\bar{w}^L}{\bar{w}^R}$. Indeed, the equilibrium bids of type L and type R are $\bar{b}^L = \frac{Q(\bar{w}^L)^2 \bar{w}^R}{n(\bar{w}^L + \bar{w}^R)^2}$, $\bar{b}^R = \frac{Q\bar{w}^L (\bar{w}^R)^2}{n(\bar{w}^L + \bar{w}^R)^2}$ and aggregate bids are $B = \frac{Q\bar{w}^L \bar{w}^R}{n(\bar{w}^L + \bar{w}^R)}$. The corresponding equilibrium price of a vote is $p = \frac{\bar{w}^L \bar{w}^R}{n(\bar{w}^L + \bar{w}^R)}$ and clearly depends on the valuations of the two vote buyers, on the size of the electorate but not on the number of vote sellers. The logic behind this result is that, for a given number of voters, a change in aggregate offers is offset, in equilibrium, by a change in aggregate bids and hence the price remains constant.

Now we turn our attention to the other side of the market and we examine the behavior of vote sellers in a full-trade equilibrium. That is, in an equilibrium in which every player either sells her vote or places monetary bids to acquire more votes.

Lemma 2 *In a full-trade equilibrium of a power-sharing system, an individual i of type T chooses to sell her vote if and only if $w_i < \bar{w}^{t_i}$.*

Proof. This lemma is in fact a trivial corollary of Lemma 1. If individual i of type L is characterized by an intensity parameter $w_i < \bar{w}^L$ then, given the arguments in the proof of Lemma 1, she is not placing monetary bids to acquire more votes. If we are in a full-trade equilibrium and i is not buying votes then she must be selling her vote. This establishes the "if" direction. To establish also the "only if" direction, notice that if individual i of type L, sells her vote in a full-trade equilibrium, she must be characterized by an intensity parameter $w_i < \bar{w}^L$, as, by Lemma 1, we know that the the strongest supporter of type L never sells votes in any equilibrium with vote trading, including the full-trade one. ■

The next Proposition proves the existence of a unique full-trade equilibrium for every possible parameter values. In such an equilibrium only the two preference intensities of the players who decided to buy votes actually shape the voting outcome.

Proposition 2 *In a power-sharing system, for any distribution of intensity parameters, there exists a unique full-trade equilibrium and it is such that exactly two voters –the strongest supporter of each party– buy votes by bidding* $\bar{b}^L = \frac{(n-2)}{n}\frac{(\bar{w}^L)^2\bar{w}^R}{(\bar{w}^L+\bar{w}^R)^2}$, $\bar{b}^R = \frac{(n-2)}{n}\frac{\bar{w}^L(\bar{w}^R)^2}{(\bar{w}^L+\bar{w}^R)^2}$ *and all other players offer their vote for sale.*

Proof. From Lemma 1 we have established that in an equilibrium (b, q) with vote trading only the two players with the most intense preferences, one from each party, buy votes. In a full-trade equilibrium (where $Q = n-2$) the equilibrium bids of the two vote buyers are $\bar{b}^L = \frac{(n-2)}{n}\frac{(\bar{w}^L)^2\bar{w}^R}{(\bar{w}^L+\bar{w}^R)^2}$, $\bar{b}^R = \frac{(n-2)}{n}\frac{\bar{w}^L(\bar{w}^R)^2}{(\bar{w}^L+\bar{w}^R)^2}$ yielding $B = \frac{(n-2)}{n}\frac{\bar{w}^L\bar{w}^R}{(\bar{w}^L+\bar{w}^R)}$.

The type L vote buyer is willing to deviate from this strategy to selling her vote if

$$\left(\frac{(n-1)\bar{w}^L+\bar{w}^R}{n(\bar{w}^L+\bar{w}^R)}\right)\bar{w}^L - \frac{(n-2)}{n}\frac{(\bar{w}^L)^2\bar{w}^R}{(\bar{w}^L+\bar{w}^R)^2} < \frac{(n-2)}{n(n-1)}\frac{\bar{w}^L(\bar{w}^R)^2}{(\bar{w}^L+\bar{w}^R)^2}$$

resulting in $(n-1)\left(\bar{w}^R+(n-1)\bar{w}^L\right)^2 < 0$, which is impossible. Hence, the vote buyer of type L is not willing to deviate to selling her vote. Similarly, the vote buyer of type R is also not willing to deviate to selling her vote. Straightforwardly, none of them wishes to deviate to any other bidding amount in $[0, +\infty)$ given the concavity of the maximization problem $\max_{b_i \geq 0} u_i$, and hence the posited strategies are their unique best responses.

Let us now turn our attention to vote sellers. In a full-trade equilibrium an individual i of type L with intensity parameter $w_i < \bar{w}^L$ sells her vote if

$$\frac{1}{n}\left(n_L - (Q^L_{-i}+1) + \frac{B^L}{B}(Q_{-i}+1)\right)w_i + \frac{B}{Q_{-i}+1} > \frac{1}{n}(n_L - Q^L_{-i} + \frac{B^L}{B}Q_{-i})w_i,$$

which reduces to $w_i < \dfrac{nB^2}{B^R(Q_{-i}+1)}$. Substituting for the best response bids, we can calculate that in a full-trade equilibrium ($Q = Q_{-i}+1 = n-2$) individual i sells her vote if $w_i < \bar{w}^L$, which is always the case. That is, the best response of such an individual is to sell her vote. Similarly, we get that an individual i of type R sells her vote if $w_i < \bar{w}^R$, which is again always the case. So we established that all individuals, other than the two individuals with highest valuations for each party, are selling their votes. Hence, all voters participate in vote trading. ■

The logic of the full-trade equilibrium is quite clear. Positive bidding for votes comes only from the strongest supporter of each party, whose objective is to secure as many votes as possible, since she benefits the most from an increase in the vote share of her preferred alternative. On the other hand, all other individuals find it profitable to sell their votes and to abstain from voting. Given the complete information setup, an individual with valuation ($w_i < \bar{w}^{t_i}$) knows that there is always demand for her vote by the high-value supporters and moreover, since no individual is liquidity constrained, that her sale will lead to an increase of their monetary bids. Additionally, given the anonymity of our trading scenario, as buying and selling orders from supporters of both parties are treated equally, each vote seller knows that there is always some intra-group trading and a fraction of her vote will certainly end up to her favorite party, as the high-value supporter of her party preempts the sale of her whole vote to the buyer from the other party.

As far as the comparative statics of the equilibria of the game are concerned we notice that the bids of vote buyers are increasing in the size of the electorate and in their valuations $\bar{w}^L$,$\bar{w}^R$, with one's own valuation having the greater effect. It should be stressed here that only the total number of voters, and not the relative sizes of the two opposing groups (n_L, n_R), affects equilibrium bids and hence whether a vote buyer belongs to the minority or not does not affect her behavior. This observation is in contrast with the results in Casella and Turban (2014) where of the two individuals who buy votes, the one belonging to the minority can be the most aggressive buyer even when she has lower valuation. Finally, the price of a vote is decreasing with the size of the electorate, as a greater number of votes makes less the influence of a single vote on the voting outcome.

In the Appendix we characterize all non-generic equilibria of our game and discuss their properties.[9]

4 Welfare Analysis

"The just [...] involves at least four terms; for the persons for whom it is in fact just are two, and the things in which it is manifested, the objects distributed, are two. [...] and the ratio between one pair is the same as that between the other pair; for there is a similar distinction between the persons and between the things." Aristotle (Nicomachean Ethics, V.3)

Concerning social welfare, the adoption of the standard utilitarian approach of maximizing total welfare does not yield definite results as to whether our vote-trading approach produces equilibrium outcomes that are welfare increasing when compared to simple voting in power-sharing system. It is clear that voting alone yields positive vote shares for both parties, whereas the Benthamite/utilitarian optimum dictates having one party (the party with the greater sum of supporters' valuations) getting all votes. In order for the full-trade equilibrium outcome to be closer to the utilitarian optimum than the simple voting outcome, a necessary condition is that the voter with the highest overall valuation is a voter of the party with the greater sum of individual valuations. Consequently, both welfare improving and decreasing results are likely to be produced by our vote-trading framework.

However, one can discuss the properties of this full-trade equilibrium in terms of other –non-utilitarian– welfare optima. To this end, based on Aristotle's notion of proportional justice, we identify an alternative welfare criterion that suggests an outcome that takes

[9] These equilibria exhibit partial vote trading –the strongest supporter of each party buys votes, some players sell their votes and others prefer to refrain from trading and just vote– and exist only for particular preference profiles.

into account only the valuations of the two most opposing voters (i.e., the strongest supporter of each party).

Definition 1 *An outcome satisfies proportional justice in policy with respect to the preferences of the two most opposing voters, if the ratio of vote shares of the two parties is equal to the ratio of the preference intensities of the two most opposing voters.*

The following result exhibits that as the population becomes arbitrarily large the full-trade equilibrium yields vote shares for the two alternatives that satisfy proportional justice.

Lemma 3 *The full-trade equilibrium implements proportional justice in policy with respect to the preferences of the two most opposing voters, when $n \to \infty$.*

Proof. The vote share of alternative L in the full-trade equilibrium is $v_L = \dfrac{(n-1)\bar{w}^L + \bar{w}^R}{n(\bar{w}^L + \bar{w}^R)} = \dfrac{\bar{w}^L}{(\bar{w}^L + \bar{w}^R)} + \dfrac{\bar{w}^R - \bar{w}^L}{n(\bar{w}^L + \bar{w}^R)}$ and hence $\lim_{n\to\infty} v_L = \dfrac{\bar{w}^L}{(\bar{w}^L + \bar{w}^R)}$ and $\lim_{n\to\infty} v_R = 1 - \dfrac{\bar{w}^L}{(\bar{w}^L + \bar{w}^R)} = \dfrac{\bar{w}^R}{(\bar{w}^L + \bar{w}^R)}$. Therefore, for $n \to \infty$ we have $\dfrac{v_L}{v_R} = \dfrac{\bar{w}^L}{\bar{w}^R}$. ■

The possibility of implementation of intuitive welfare optima via simple mechanisms is an important open issue in economics, since most general results in implementation literature (one is referred to Maskin, 1999 for an excellent initial reference) employ complicated mechanisms –like integer games– that cannot be applied to real life decision making. Several simple mechanisms have been proposed that deliver utilitarian efficiency or implementation of general families of rules in certain contexts[10] but, to our knowledge, this is the first mechanism that implements proportional justice in policy with respect to the preferences of the two most opposing voters. Now, is this a relevant welfare criterion? In many cases it certainly is: historically, disagreements and disputes between two groups of individuals have been settled by a verbal –or even physical– duel between a representative of each group. So arriving to an outcome that reflects the characteristics

[10] For example, Yamamura and Kawasaki (2013), Gershkov et al. (2016) and Núñez and Xefteris (2017) propose simple voting mechanisms that implement Moulin's (1980) generalized median rules in the domain of single-peaked preferences.

of only two individuals –the "champion" of each group– is something that in many contexts was and is still considered desirable. Understandably, one could disagree with the principles behind this welfare optimum. We note though that our goal here is barely to argue that proportional justice is neither an abstract construction, nor unheard of in the dispute-settling history; and not to argue in favor or against any particular welfare goal.

5 Concluding Remarks

The intention in this paper is to establish an appropriate model of active vote trading for committees in which there is complete information about the preferences, both cardinal and ordinal, of all their members. It is a fair argument that standard models of vote trading exhibit many difficulties with respect to the existence of equilibrium, however this does not mean that these are solely due to complete information. It's the combination of market mechanisms, electoral systems and levels of information that creates such obstacles. For this purpose, we present an appropriate version of a simple vote market mechanism with strategic players and we provide clear-cut results about the existence of equilibrium involving trade in a power-sharing system, as opposed to the simple plurality rule where no trade occurs. Moreover, under a power-sharing system *full trade* is always an equilibrium; a result of particular interest as it establishes the willingness of all voters to participate in decentralized market institutions that complement elections. Notably, this full-trade equilibrium implements proportional justice in policy and hence our setup serves as an example of how this alternative notion of welfare optimality can be applied in issues of political economics.

6 Appendix

In this part of the paper we focus on partial-trade equilibria, which are non-generic as they exist only for particular preference profiles. In other words, for some preference profiles, apart from the full-trade equilibrium, there are also other equilibria where the strongest supporter of each party buys votes, some players sell their votes and others prefer to vote without participating in vote trading. The next Proposition exhibits that in a partial-trade equilibrium a player who chooses to abstain from vote trading must have a valuation within a certain party-specific interval. That is, in an equilibrium with $m < n-2$ sellers, the valuation of a player i who chooses not to trade should be in the interval $(\frac{m}{m+1}\bar{w}^{t_i}, \bar{w}^{t_i})$. This also implies that if there are no players with valuations in the interval $(\frac{m}{m+1}\bar{w}^{t_i}, \bar{w}^{t_i})$, then there are no partial-trade equilibria.

It should also be stressed that we don't claim that in a partial-trade equilibrium all players with valuations $w_i \in (\frac{m}{m+1}\bar{w}^{t_i}, \bar{w}^{t_i})$ refrain from vote trading. Indeed, there might be partial-trade equilibria where some players with valuations $w_i \in (\frac{m}{m+1}\bar{w}^{t_i}, \bar{w}^{t_i})$ trade and others don't. What we essentially prove is that a player who is not expected to sell her vote has incentives to deviate to selling her vote only if her valuation parameter is $w_i < \frac{m}{m+1}\bar{w}^{t_i}$; whereas she has no incentives to deviate to selling her vote if $w_i \in (\frac{m}{m+1}\bar{w}^{t_i}, \bar{w}^{t_i})$.

Proposition 3 *In a power-sharing system a strategy profile* (b, q) *is an equilibrium with* $m < n-2$ *vote sellers if and only if: the two players with the most intense preferences* $\bar{w}^L, \bar{w}^R$ *play* $\bar{b}^L = \dfrac{m(\bar{w}^L)^2\bar{w}^R}{n(\bar{w}^L+\bar{w}^R)^2}$, $\bar{b}^R = \dfrac{m\bar{w}^L(\bar{w}^R)^2}{n(\bar{w}^L+\bar{w}^R)^2}$*; m players with valuations* $w_i < \bar{w}^{t_i}$ *sell their votes; and* $n-2-m$ *players with valuations* $w_i \in (\frac{m}{m+1}\bar{w}^{t_i}, \bar{w}^{t_i})$ *abstain from vote trading.*

Proof. From Lemma 1 we have established that in any equilibrium (b, q) with vote trading only the two players with the most intense preferences, one from each party, buy votes (each one plays $(q_i, b_i) = (0, \bar{b}^{t_i})$). For the case of $m < n-2$ vote sellers, their best response bids are $\bar{b}^L = \dfrac{m(\bar{w}^L)^2\bar{w}^R}{n(\bar{w}^L+\bar{w}^R)^2}$, $\bar{b}^R = \dfrac{m\bar{w}^L(\bar{w}^R)^2}{n(\bar{w}^L+\bar{w}^R)^2}$ yielding $B = \dfrac{m}{n}\dfrac{\bar{w}^L\bar{w}^R}{(\bar{w}^L+\bar{w}^R)}$.

The type L vote buyer is willing to deviate from this strategy to selling her vote if

$$\left(\frac{(n-1)\bar{w}^L+\bar{w}^R}{n(\bar{w}^L+\bar{w}^R)}\right)\bar{w}^L-\frac{m}{n}\frac{(\bar{w}^L)^2\bar{w}^R}{(\bar{w}^L+\bar{w}^R)^2}<\frac{m}{n(n-1)}\frac{\bar{w}^L(\bar{w}^R)^2}{(\bar{w}^L+\bar{w}^R)^2}$$

resulting in $(n-1)\bar{w}^L+(n-1-m)\bar{w}^R<0$, which is impossible. Hence, the vote buyer of type L is not willing to deviate to selling her vote. Similarly, the vote buyer of type R is also not willing to deviate to selling her vote. Straightforwardly, none of them wishes to deviate to any other bidding amount in $[0,+\infty)$ –which also excludes the possibility that they have incentives to abstain from trading– given the strict concavity of the maximization problem $\underset{b_i\geq 0}{max} u_i$, and hence the posited strategies are their unique best responses.

Let us now turn our attention to vote sellers. Consider an individual i of type L with intensity parameter $w_i<\bar{w}^L$ who sells her vote (plays $(b_i,q_i)=(0,1)$) in a partial-trade profile of strategies and all other players expect it (that is, $Q=Q_{-i}+1=m$). From Lemma 1 this type L individual will never deviate to buying votes. Moreover, she is willing to deviate to abstaining from vote trading only if

$$\frac{1}{n}\left(n_L-(Q^L_{-i}+1)+\frac{B^L}{B}(Q_{-i}+1)\right)w_i+\frac{B}{Q_{-i}+1}<\frac{1}{n}(n_L-Q^L_{-i}+\frac{B^L}{B}Q_{-i})w_i,$$

which reduces to $w_i>\dfrac{nB^2}{B^R(Q_{-i}+1)}$. Substituting for the best response bids, we can calculate that individual i deviates to abstaining from vote trading if $w_i>\bar{w}^L$, which is a contradiction to our assumption about her preferences. That is, individual i never deviates from her strategy. Similarly, we get that a type R vote seller will never deviate.

Now let us now consider an individual i of type L with valuation $w_i\in(\frac{m}{m+1}\bar{w}^L,\bar{w}^L)$ who does not sell her vote (plays $(b_i,q_i)=(0,0)$) in a partial-trade profile of strategies and all other players expect it (that is, $Q=Q_{-i}=m$). In the proof of Lemma 1 we have established that this type L individual will never deviate to buying votes. Moreover, she prefers to deviate from this strategy to selling her vote if

$$\frac{1}{n}\left(n_L-(Q^L_{-i}+1)+\frac{B^L}{B}(Q_{-i}+1)\right)w_i+\frac{B}{Q_{-i}+1}>\frac{1}{n}(n_L-Q^L_{-i}+\frac{B^L}{B}Q_{-i})w_i,$$

which reduces to $w_i<\dfrac{nB^2}{B^R(Q_{-i}+1)}$. Substituting for the best response bids of the two buyers we derive that for this voter it must be the case that $w_i<\frac{m}{m+1}\bar{w}^L$, which is a contradiction to our assumption about her preferences. That is, an individual i of type L with valuation $w_i \in (\frac{m}{m+1}\bar{w}^L, \bar{w}^L)$ who does not trade in partial-trade profile of strategies will never deviate to selling her vote. Similarly, an individual i of type R with valuation $w_i \in (\frac{m}{m+1}\bar{w}^R, \bar{w}^R)$ who does not trade in partial-trade profile of strategies will never deviate to selling her vote.

Hence, in an equilibrium (b, q) with $m < n - 2$ vote sellers, the two players with the most intense preferences $\bar{w}^L, \bar{w}^R$ play $\bar{b}^L = \dfrac{m(\bar{w}^L)^2\bar{w}^R}{n(\bar{w}^L + \bar{w}^R)^2}$, $\bar{b}^R = \dfrac{m\bar{w}^L(\bar{w}^R)^2}{n(\bar{w}^L + \bar{w}^R)^2}$; m players with valuations $w_i < \bar{w}^{t_i}$ sell their vote; and $n - 2 - m$ players with valuations $w_i \in (\frac{m}{m+1}\bar{w}^{t_i}, \bar{w}^{t_i})$ abstain from vote trading. ■

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