University of Cyprus
Department of Economics

Working Paper 03-2015

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A Backward-Bending and Forward-Falling Semi-log Model of Labour Supply

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Abstract

This paper proposes a labour supply function that allows not only for backward-bending behaviour at high but also for forward-falling behaviour at low wage rates. The proposed model adheres to the fundamentals of consumer theory and encompasses all well-known and widely used semi-log labour supply models in the literature. It is applied to UK data to investigate female labour supply and, in particular, to demonstrate the importance of including a forward-falling segment in the empirical specification for accurate estimation of labour supply behaviour at the low end of the wage distribution. The policy implications of our empirical findings are considered in the context of a hypothesised minimum wage reform.

Keywords: female labour supply, low-wage work behaviour, S-shaped labour supply, minimum wage reform.

JEL Classification: J22, J16, J38

February 2015

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We are grateful for helpful comments from seminar participants at the University of Cyprus. We are solely responsible for errors and omissions.

1. Introduction

The behaviour of individuals in the labour market is among the most active areas of economic research. This is not surprising given the importance of employment for promoting economic growth, and for combating poverty and social exclusion. Furthermore, knowing the labour supply behaviour of individuals can be a key factor to the success of a wide variety of public policies, ranging from tax and welfare programs to the alteration of institutional features of labour supply.

The foundations of the modern approach to labour supply analysis were laid initially by Mincer (1962), Becker (1965) and Cain (1966), who formalised the separation and measurement of the income and substitution effects; and were developed further by, among others, Ashenfelter and Heckman (1974), Gronau (1974), Heckman (1974a; 1974b) and Burtless and Hausman (1978). Nowadays, the labour supply literature provides enough tools to handle labour supply issues arising, for instance, from unobserved heterogeneity (Blundell et al., 2007a; Chiappori, 1992), nonparticipation (Blundell and MaCurdy, 1999; Brewer et al., 2006; Heckman, 1979; MaCurdy et al., 1990), non-linear budget constraints (Blomquist and Newey, 2002; Moffitt, 1990), life-cycle decision making (Attanasio et al., 2008; Heckman and MaCurdy, 1980; MaCurdy, 1981) and discrete or/and restricted working hour choices (Aaberge et al., 2009; Beffy et al., 2014).

The above literature developments, have led to a labour supply framework equipped with an array of alternative empirical specifications, each offering advantages in terms of highlighting the importance of certain concerns, but also limited by information requirements and by the range of issues that can be analysed. The concern in this paper is limitations placed on the empirical specification by the functional form of the utility function. Early empirical investigation based on the linear (Hausman, 1980), log-linear (Burtless and Hausman, 1978) and semi-log (Heckman, 1974b) specifications restrict labour supply to be monotonically either increasing or decreasing with the wage level. This limitation was remedied by Blundell, Duncan and Meghir (1992) – thereafter BDM - through a generalization of the semi-log model allowing backward-bending behaviour at high wages. This BDM model has since been extensively used for the empirical analysis of labour supply, among others by Duncan (1993), Blundell (1994; 1995), Blundell et al. (2007a) and Frederiksen et al. (2008).

The BDM model has been a major break-through in the analysis of labour supply, given the strong and long recognised theoretical and empirical foundations of the backward-bending labour supply. This paper extends the BDM model to also capture forward-falling labour supply behaviour, i.e. the negative slope of the labour supply at low wages; while adhering to the fundamentals of consumer theory, which are required for meaningful behavioural and welfare interpretation of empirical results. The backward-bending and forward-falling (thereafter BB-FF) model proposed here is argued to be important when investigating the labour supply of low-paid workers. In particular, our contention is that failing to allow for this type of labour supply behaviour in empirical analysis can give rise to misleading conclusions when assessing the employment impact of policies targeting low-paid individuals, such as the reform of guaranteed minimum income and minimum wage schemes.

The existence of subsistence income is crucial for the labour supply function to exhibit forward-falling behaviour. For instance, this behaviour may not be relevant when non-labour income covers the subsistence needs of individuals; in which case extending the BDM model to capture forward-falling labour supply should be redundant. Nevertheless, subsistence consumption can exist even in countries with well-developed social protection systems, due to imperfections in the design (e.g. errors in reference budgets) and/or the implementation (e.g. non-take up of benefits) of these systems. In this sense forward-falling labour supply can be seen as evidence of households falling through the social safety net. According to the Low Pay Commission (2010) in April 2009 the wage for around 242,000 jobs (about 1% of UK employees) was below the national minimum. Also, le Roux et al. (2013), using data from to the Labour Force Survey, find that the extent of non-compliance with the national minimum wage increased from 0.4% in the second quarter of 2000 to 1.7% in the second quarter of 2011.

By accommodating all possible shapes of the labour supply curve (monotonically rising, monotonically falling, backward-bending and forward-falling), the BB-FF labour supply model proposed in this paper encompasses all known semi-log specifications considered in the empirical literature. Yet, it can be estimated by relatively simple econometric methods and used for simulating the impact of tax-benefit reforms on employment without encumbering computation time. The empirical analysis in the paper, based on UK data drawn from the 2011 European Union Statistics on Income and Living Conditions (EU-SILC) database, demonstrates the empirical improvement from using the BB-FF labour supply specification and illustrates its practical usefulness through simulating the employment impact of a hypothetical minimum wage reform.

The outline of the paper is as follows. Section 2 describes the proposed BB-FF labour supply model. Section 3 reports and compares estimates obtained from the application of the BB-FF and BDM models to UK data. Section 4 discusses the shape of female labour supply implied by the empirical findings and reports simulation results obtained from a hypothetical minimum wage reform. Section 5 concludes the paper.

2. A BB-FF labour supply model

In this section we propose an extension of the classic backward-bending semi-log labour supply BDM model to also incorporate a forward-falling section, so as to allow for the possibility of individuals increasing their working hours with wage decreases in order to maintain a minimum (subsistence) consumption level. This extension of the BDM model can be particularly useful for the investigation of the labour supply behaviour of low-paid groups of strong policy interest such as mothers, social assistance recipients and workers at minimum wage. For brevity, here we describe only the basic features of the proposed BB-FF labour supply model. More details, including the standard theoretical properties of the expenditure function the model is derived from, are given in Appendix A.

The theoretical basis of our analysis stems from the standard assumption that individual i maximises a quasi-concave utility function $U(C_i, L_i; z_i)$ subject to the budget constraint $pC_i + w_i L_i = y_i + w_i T_i = M_i$, where C_i is consumption, L_i leisure time, z_i a vector with individual characteristics, w_i hourly wage rate, p is the price of consumption, y_i non-labour income, T_i total time available and M_i the 'full income'. The first order conditions yield the Marshallian demand for consumption $C_i = C(w_i, p, M_i; z_i)$ and leisure $L_i = L(w_i, p, M_i; z_i)$. Then, using the time constraint $L_i + h_i = T_i$ and defining M_i in terms of y_i, we obtain the labour supply equation $h_i^m = h^m(w_i, p, y_i; z_i)$. This function is integrable (its parameters can recover the utility function); and, thus, appropriate for the empirical analysis of both behavioural and welfare aspects of labour supply.

Here we show that a labour supply function satisfying the above properties can be obtained from the expenditure function

$$e(w_i, p, U_i; z_i) = U_i p \left(\frac{w_i}{p}\right)^{-\beta(z_i)} - \frac{w_i}{\beta(z_i)+1}\left[\alpha(z_i) log \left(\frac{w_i}{p}\right) + \gamma(z_i) - \frac{\alpha(z_i)}{\beta(z_i)+1}\right] + w_i T_i - \frac{\delta(z_i)}{\beta(z_i)\frac{1}{p}}, \quad (2.1)$$

where: U_i is the utility level; and $\alpha(z_i)$, $\beta(z_i)$, $\gamma(z_i)$ and $\delta(z_i)$ parameters that depend on the characteristics of the individual. The term $-\delta(z_i)/\beta(z_i)\frac{1}{p}$ is the 'subsistence' cost of individual i, i.e. the income needed for maintaining an absolute minimum standard of living.

As shown in Appendix A, (2.1) is homogeneous of degree 1 in p and w_i, non-decreasing in p and $\mathrm{w_i}$ and concave in p and w_i. Therefore, applying Shephard's lemma we can derive the Hicksian labour supply function

$$h^h(w_i, p, U_i; z_i) = \frac{\beta(z_i)}{\left(\frac{w_i}{p}\right)^{\beta(z_i)+1}} U_i + \frac{1}{\beta(z_i)+1}\left[\alpha(z_i) log\left(\frac{w_i}{p}\right) + \gamma(z_i) + \frac{\alpha(z_i)\beta(z_i)}{\beta(z_i)+1}\right]. \tag{2.2}$$

Then, using $e(w_i, p, u(w_i, p, M_i; z_i); z_i) = M_i$, i.e. the minimum expenditure necessary for the individual *i* to reach utility $U_i = u(w_i, p, M_i; z_i)$, we obtain the indirect utility function

$$u(w_i, p, y_i; z_i) = \frac{\left(\frac{w_i}{p}\right)^{\beta(z_i)+1}}{\beta(z_i)+1}\left[\frac{y_i}{w_i}(\beta(z_i)+1) + \alpha(z_i) log\left(\frac{w_i}{p}\right) + \gamma(z_i) - \frac{\alpha(z_i)}{\beta(z_i)+1} + \frac{\delta(z_i)(\beta(z_i)+1)}{\beta(z_i)\frac{w_i}{p}}\right], \tag{2.3}$$

where y_i is the non-labour income of i^{th} individual.

Substituting (2.3) in (2.2) we obtain the Marshallian labour supply

$$h^m(w_i, p, y_i; z_i) = \alpha(z_i) log\left(\frac{w_i}{p}\right) + \beta(z_i)\frac{1}{\left(\frac{w_i}{p}\right)}\left(\left(\frac{y_i}{p}\right) - \left(-\frac{\delta(z_i)}{\beta(z_i)}\right)\right) + \gamma(z_i), \tag{2.4}$$

which belongs to the family of semi-logarithmic functions. Assuming that the price of consumption is the same for all individuals only the relative price of leisure (w_i) and relative non-labour income (y_i) to the price of consumption are relevant. Thus, setting $p = 1$, (2.4) simplifies to

$$h^m(w_i, y_i; z_i) = \alpha(z_i) log(w_i) + \beta(z_i)\frac{1}{w_i}\left(y_i - \left(-\frac{\delta(z_i)}{\beta(z_i)}\right)\right) + \gamma(z_i). \tag{2.5}$$

Notably, (2.3) and (2.5) are similar to the BDM labour supply function except for the additional term $\delta(z_i)(\beta(z_i)+1)/\beta(z_i)\frac{w_i}{p}$ in (2.3) and $\delta(z_i)\frac{1}{w_i}$ in (2.5), respectively. The inclusion of the term $\delta(z_i)\frac{1}{w_i}$ in (2.5) serves to capture forward-falling labour supply behaviour at low wage rates, as explained below.

The slope of the Marshallian labour supply is given by the Slutsky equation

$$\frac{\partial h^m(w_i, y_i; z_i)}{\partial w_i} = \frac{\partial h^h(w_i, U; z_i)}{\partial w_i} + \frac{\partial h^m(w_i, y_i; z_i)}{\partial y_i} h^m(w_i, y_i; z_i), \tag{2.6}$$

where the first term on the right hand side is the substitution effect, which is positive; and the second term the income effect, which is negative. For the proposed labour supply function (2.5) the substitution effect is given by

$$\frac{\partial h^h(w_i,u(w_i,p,y_i;z_i);z_i)}{\partial w_i} = -\frac{\beta(z_i)}{w_i}\left(\alpha(z_i)log(w_i) + \beta(z_i)\frac{1}{w_i}\left(y_i - \left(-\frac{\delta(z_i)}{\beta(z_i)}\right)\right) + \gamma(z_i)\right)$$
$$+\frac{1}{w_i}\left(\alpha(z_i) - \beta(z_i)\frac{1}{w_i}\left(y_i - \left(-\frac{\delta(z_i)}{\beta(z_i)}\right)\right)\right); \qquad (2.7)$$

and the income effect by

$$\frac{\partial h^m(w_i,y_i;z_i)}{\partial y_i} h^m(w_i, y_i; z_i) = \frac{\beta(z_i)}{w_i}\left(\alpha(z_i)log(w_i) + \beta(z_i)\frac{1}{w_i}\left(y_i - \left(-\frac{\delta(z_i)}{\beta(z_i)}\right)\right) + \gamma(z_i)\right). \qquad (2.8)$$

Thus, the slope of the Marshallian labour supply is

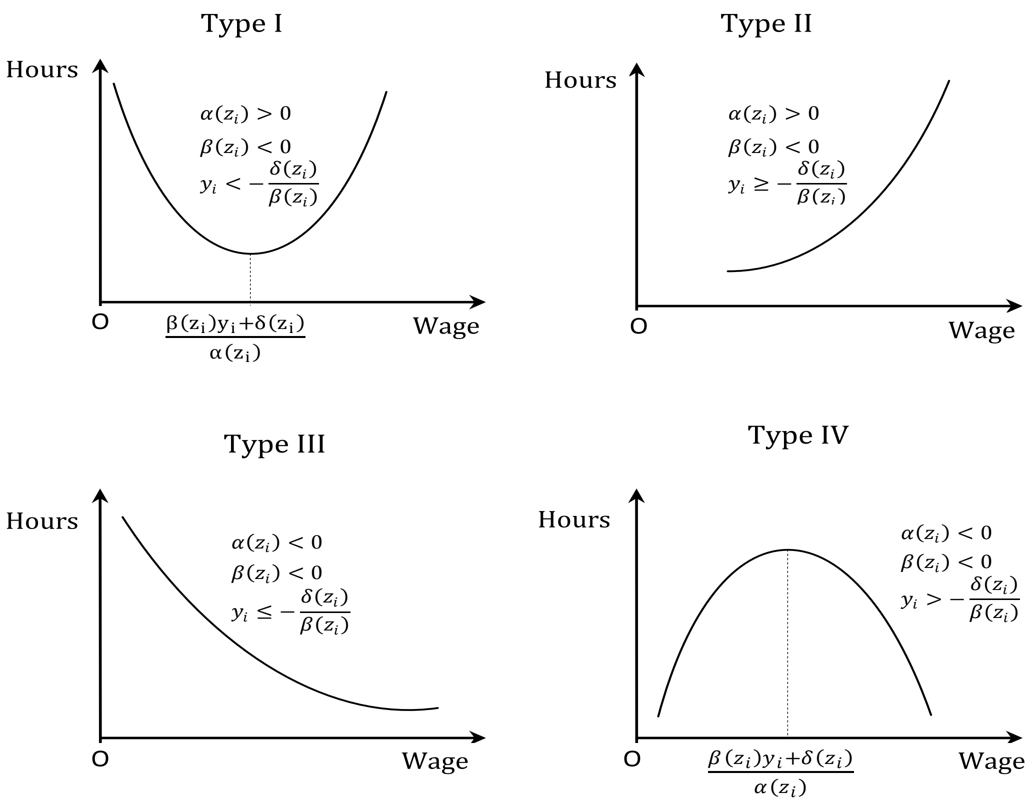
$$\partial h^m \big/ \partial w_i = \frac{1}{w_i}\left(\alpha(z_i) - \beta(z_i)\frac{y_i}{w_i} - \delta(z_i)\frac{1}{w_i}\right) = \frac{1}{w_i}\left(\alpha(z_i) - \beta(z_i)\frac{1}{w_i}\left(y_i - \left(-\frac{\delta(z_i)}{\beta(z_i)}\right)\right)\right). \qquad (2.9)$$

For leisure to be a normal good, $\partial h^m / \partial y_i = \frac{\beta(z_i)}{w_i}$ needs to be negative, thus $\beta(z_i)$ must be negative. Consumer theory does not impose any constraints on the sign of the parameters $\alpha(z_i)$, $\gamma(z_i)$ and $\delta(z_i)$.

Next we take a closer look at the labour supply function (2.5) and, in particular, the term $\beta(z_i)\frac{1}{w_i}\left(y_i + \frac{\delta(z_i)}{\beta(z_i)}\right)$ which differentiates it from other semi-log labour supply functions in the literature. In the context of (2.5) the working hours of an individual depend on the wage rate and the difference between non-labour income y_i and subsistence cost $y_i + \frac{\delta(z_i)}{\beta(z_i)}$. The term $\frac{1}{w_i}\left(y_i + \frac{\delta(z_i)}{\beta(z_i)}\right)$ gives the number of hours individuals work above (if positive) or below (if negative) those corresponding to subsistence. More precisely, given that $\beta(z_i) < 0$, individual i decreases (increases) her/his working hours by $\beta(z_i)\frac{1}{w_i}\left(y_i + \frac{\delta(z_i)}{\beta(z_i)}\right)$ when the non-labour income is above (below) subsistence level.

The term $\beta(z_i)\frac{1}{w_i}\left(y_i + \frac{\delta(z_i)}{\beta(z_i)}\right)$, together with $\alpha(z_i)$, also affect the slope of the labour supply equation (2.9). When the non-labour income is close to subsistence level $\left((y_i + \frac{\delta(z_i)}{\beta(z_i)}) \approx 0\right)$ the slope of the labour supply function depends primarily on $\alpha(z_i)$: if $\alpha(z_i) > 0$ labour supply would be upward and if $\alpha(z_i) < 0$ downward sloping. For non-labour income above or below subsistence level the term $\beta(z_i)\frac{1}{w_i}\left(y_i + \frac{\delta(z_i)}{\beta(z_i)}\right)$ affects the slope of the labour supply mainly at low wage levels, i.e. this term decreases as the wage increases. Consequently, the different sign combinations of $\alpha(z_i)$ and $y_i + \frac{\delta(z_i)}{\beta(z_i)}$ give rise to four different types of labour supply behaviour as shown by the diagrams of Figure 1.

Figure 1: Types of Labour Supply Curves



- Type I labour supply curve has the U-shape form and is obtained when $\beta(z_i) < 0$, $\alpha(z_i) > 0$ and $y_i < -\frac{\delta(z_i)}{\beta(z_i)}$. It has negative slope (the substitution effect is lower than the income effect in absolute values) for $w_i < \frac{\beta(z_i)y_i+\delta(z_i)}{\alpha(z_i)}$; and positive slope (the substitution effect is greater than the income effect in absolute values) for $w_i > \frac{\beta(z_i)y_i+\delta(z_i)}{\alpha(z_i)}$.
- Type II labour supply curve is positively sloped at any wage rate and corresponds to the parameter restrictions $\beta(z_i) < 0, \alpha(z_i) > 0$ and $y_i \geq -\frac{\delta(z_i)}{\beta(z_i)}$.

- Type III labour supply curve is negatively sloped at any wage rate and corresponds to the parameter restrictions $\beta(z_i) < 0, \alpha(z_i) < 0$ and $y_i \leq -\frac{\delta(z_i)}{\beta(z_i)}$. Finally,
- Type IV labour supply curve has the inverse U-shape form and is obtained when $\beta(z_i) < 0$, $\alpha(z_i) < 0$ and $y_i > -\frac{\delta(z_i)}{\beta(z_i)}$. It is positively sloped for wage rates lower than $w_i < \frac{\beta(z_i)y_i+\delta(z_i)}{\alpha(z_i)}$ and negatively sloped for $w_i > \frac{\beta(z_i)y_i+\delta(z_i)}{\alpha(z_i)}$.

It emerges from the exposition above that the BB-FF labour supply specification (2.5), by incorporating the four types of labour supply shown in Figure 1, encompasses backward-bending and forward-looking labour supply behaviour depending on whether: $\alpha(z_i) > 0$ and $y_i < -\left[\frac{\delta(z_i)}{\beta(z_i)}\right]$; or $\alpha(z_i) < 0$ and $y_i > -[\delta(z_i)/\beta(z_i)]$ holds true, respectively. In addition, when $\alpha(z_i) > 0$ and $y_i \geq -[\delta(z_i)/\beta(z_i)]$, then labour supply is monotonically upward and when $\alpha(z_i) < 0$ and $y_i \leq -[\delta(z_i)/\beta(z_i)]$ monotonically downward sloping. Thus the BB-FF function provides a framework general enough to nest the various semi-log labour supply functions used in the empirical literature as special cases; and test them through parametric restrictions. Furthermore, as noted earlier, non-labour income and subsistence cost shape the proposed BB-FF function: $y_i < -[\delta(z_i)/\beta(z_i)]$ is likely to hold for individuals with low non-labour income, resulting in a Type I or III labour supply behaviour; whereas, $y_i > -[\delta(z_i)/\beta(z_i)]$ is likely to hold for individuals with high non-labour income, with their labour supply behaviour assuming a Type II or IV shape.

In the empirical analysis that follows we investigate the above possibilities in the case of female labour supply, using UK data.[1] Females are typically considered to be the secondary earner in a family and their labour supply is often used as a consumption smoothing/self-insurance mechanism against negative wage shocks, i.e. females may adjust their working hours in response to shocks in economic resources to maintain their living standards. This role of female labour supply can be particularly important for families with limited financial assets, as it can help sustain living standards when a shock hits (Blundell et al., 2014; Low, 2005). In particular we focus on whether women increase their working hours with wage reduction because they live at subsistence consumption level and cannot afford an income reduction; and, conversely, work less when the wage increases. For instance,

[1] The labour supply behaviour of males in the UK is also investigated, but no evidence of forward falling behaviour is found (Polycarpou, 2015).

this type of labour supply behaviour can be motivated by mothers whose income from employment is considered as a top-up component of household income. This is not an unreasonable hypothesis given that the same motivation is often found to be behind women's decision to participate in the labour market.

3. Empirical analysis

3.1 Data

The UK data used for the empirical analysis are drawn from the 2011 European Union Statistics on Income and Living Conditions (EU-SILC) database.[2] The data contain information for 9636 female persons. To limit heterogeneity, unnecessary for the purposes of our analysis, the following individuals have been excluded: below 25 or over 65 years old, employers, self-employed, between jobs, having a second job, have changed job during the last 12 months, receiving pension or social benefits, and living in multi-unit households. The resulting subsample consists of 2327 females.

The EU-SILC dataset contains sufficient cross-country comparable information (hours of work, wages and socio-demographic characteristics of each household member) for estimating the parameters of the BB-FF labour supply model.[3] It does not, however, contain information about the after-tax income (net wage) needed for accurate estimation of the labour supply parameters. As no open access microsimulation models using EU-SILC data are available for the UK (e.g. Euromod uses the 2009/10 Family Resources Survey) we proceed with estimation using (i) pre-tax incomes as reported in EU-SILC and (ii) post-tax incomes approximated by an ad hoc microsimulation model based on the basic features of the UK tax-benefit system. The empirical results obtained from the two approaches yield similar conclusions. Here we have chosen to report the results obtained from using the pre-tax data. Those obtained from the simulated post-tax data are available from the authors on request.[4]

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[2] European Commission, Eurostat, cross-sectional EU SILC UDB 2011 - version 2 of August 2013. Eurostat has no responsibility for results obtained and conclusions reached in this paper.

[3] Table B1 in Appendix B shows the descriptive statistics of the variables used in the empirical analysis; and Figure B1 in Appendix B the scatterplot of hours and wages of females in our sample.

[4] The wage rate of non-working females is imputed from the standard two-step Heckman method.

3.2 Semi-parametric analysis

We first use semi-parametric analysis for chequing whether the BB-FF shape of labour supply is supported by the data without any guidance or constraints from theory. For this we employ the procedure based on the nearest neighbour estimator proposed by Estes and Honore (1995) and Yatchew (1997) briefly described as follows.

The working hours are described by the semiparametric regression equation

$$h = f(w) + X\beta + \varepsilon, \tag{3.1}$$

where w is the wage, X the personal socio-economic characteristics, and ε the error term with mean zero and variance σ_ε^2. The term $f(w)$ is the nonparametric and $X\beta$ the parametric part of the regression. The function $f(.)$ is smooth with a bounded first derivative; and $f(w)$ and $X\beta$ are additively separable.

Sorting the data by w such that $w_1 < w_2 \dots < w_N$ and expressing (3.1) in first differences we have

$$h_n - h_{n-1} = f(w_n) - f(w_{n-1}) + (X_n - X_{n-1})\beta + \varepsilon_n - \varepsilon_{n-1}. \tag{3.2}$$

The assumption that $f(.)$ is smooth and has a bounded first derivative implies that $f(w_n) - f(w_{n-1}) \to 0$ as the sample size increases and thus (3.2) is simplified to

$$\Delta h_n = \Delta X_n \beta + u_n, \tag{3.3}$$

where $\Delta h_n = h_n - h_{n-1}$, $\Delta X_n = X_n - X_{n-1}$ and $u_n = \varepsilon_n - \varepsilon_{n-1}$. Regression equation (3.3) can be estimated by OLS and based on Yatchew (1997) the sampling distribution of the estimated parameters, $\hat{\beta}_{diff,}$ can be approximated by

$$\hat{\beta}_{diff} \to N\left(\beta, \frac{1}{N}\frac{1.5\sigma_\varepsilon^2}{\sigma_{x/w}^2}\right),$$

where $\sigma_{x/w}^2$ is the conditional variance of X given w.

After $\hat{\beta}_{diff}$ is estimated, alternative non-parametric techniques can be used to consistently estimate $f(w)$ as if β is known. This can be done by subtracting the estimated parametric part from both sides of (3.1) to obtain

$$h - X\hat{\beta}_{diff} = f(w) + X(\beta - \hat{\beta}_{diff}) + \varepsilon \cong f(w) + \varepsilon \tag{3.4}$$

and apply conventional smoothing methods to (3.4) such as kernel estimation. As shown by Yatchew (2003), $\hat{\beta}_{diff}$ converges sufficiently quickly to β and thus the approximation in the last part of (3.4) leaves asymptotic arguments unaffected.

The parametric part of (3.1) is specified to depend on dummies for the age group, health condition, number of children, age of youngest child, marital status, and spouse working status; and continued variables for the amount of mortgage payments (log) and non-labour income. For the kernel estimation of equation (3.4) we use two alternative bandwidths, 2 and 6: the smaller bandwidth can highlight details in the data that necessitate the use of a complex parametric specification to be captured; whereas, in contrast, the larger bandwidth disregards data details in favour of a more parsimonious parametric model.

Figure 2 reports the plot of the Gaussian kernel estimates of the relationship between working hours and hourly wage rate in our sample.[5] The estimates obtained using the lower bandwidth shows clearly a labour supply curve negatively sloped (forward-falling) at very low wage rates, positively sloped at intermediate wage rates, negatively sloped (backward bending) at high wage rates and positively sloped at very high wage rates. In particular, labour supply has negative slope for wages below 7.95 euro per hour; positive slope for wages between 7.95 euro and 29.30 euro per hour; negative slope for wages between 29.30 euro and 37.50 euro per hour; and positive slope for wages above 37.50 euro per hour. In contrast, the kernel estimation with the higher bandwidth shows a labour supply curve flat at low wages, positively sloped for intermediate wages and slightly negatively sloped at very high wages.

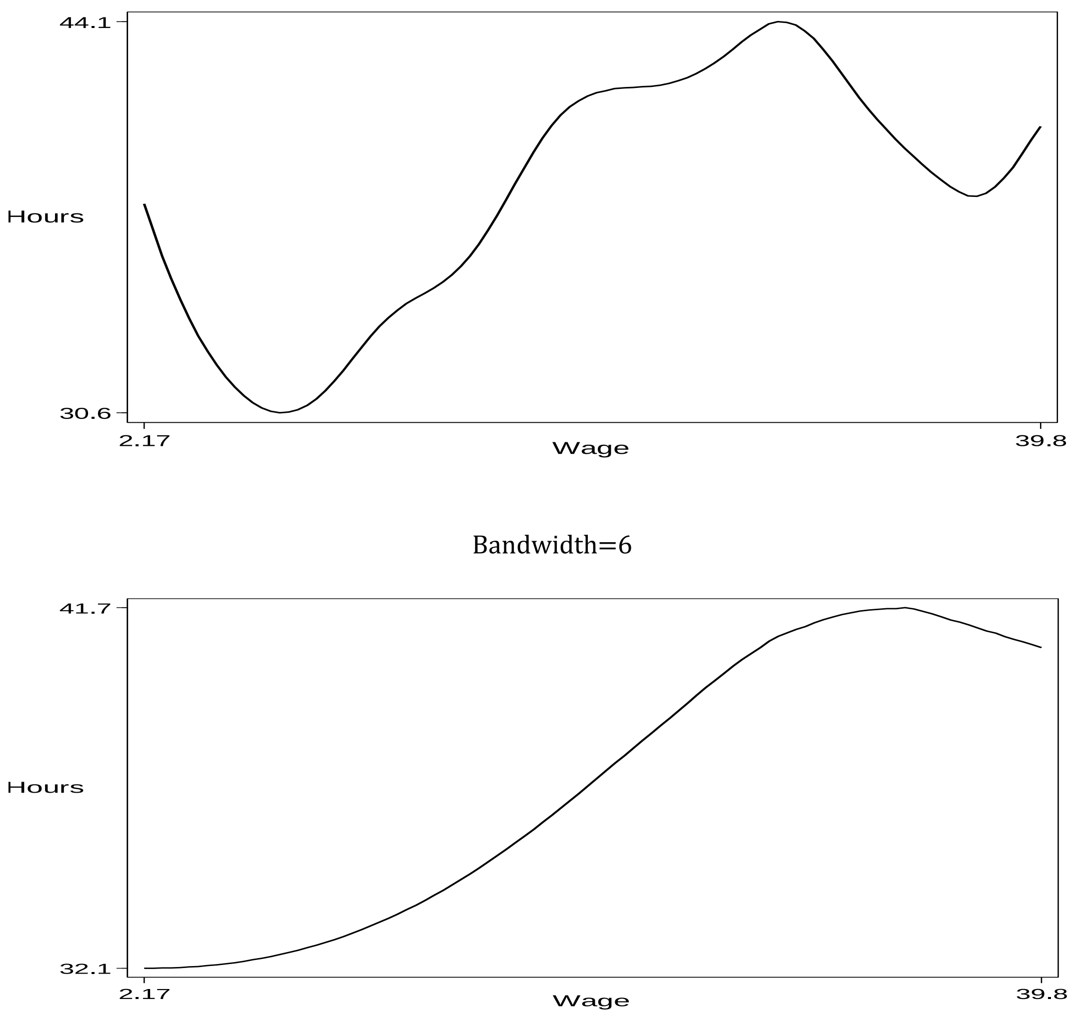
The conclusion emerging from the semi-parametric analysis, therefore, is that the female labour supply in the UK exhibits backward-bending behaviour at upper and forward-falling behaviour at lower wage rates. However, unlike the positively sloped segment, which holds true regardless of the bandwidth used, the backward-bending and forward-falling segments tend to flatten out as the bandwidth of the kernel estimates increases. This raises questions about the statistical significance of these segments, an issue investigated in the parametric analysis which follows.

[5] These non-parametric regression results should be interpreted with caution since no allowance is made for endogeneity of the wage rate or for sample selection.

Figure 2: Kernel estimates for working hours

Bandwidth=2

Bandwidth=6



3.3 Parametric estimation

The parameters of the BB-FF labour supply model (2.5) are estimated using a Tobit model (Tobin, 1958) and information about the personal and family characteristics of the individual, including dichotomous (dummy) variables for age, health condition, marital status, number of dependent children, age of youngest child and whether wife works; and continuous variables for mortgage payments (log), hourly wage rate (log), the inverse of hourly wage rate, and the ratio of non-labour income to hourly wage rate. The constant $\gamma(z_i)$, the effects of the wage rate $\alpha(z_i)$ and $\delta(z_i)$, and the effect of non-labour income $\beta(z_i)$ on working hours are allowed to vary with all the aforementioned characteristics.

Significant for $\alpha(z_i)$, $\delta(z_i)$, and $\beta(z_i)$ turn up to be the dummy variables for age, number of dependent children and age of youngest child; and for $\gamma(z_i)$ all the characteristics above plus the health condition, marital status, mortgage payments and whether the spouse works.

For dealing with the endogeneity of wages we use the two-step instrumental variables (IVs) estimator proposed by Smith and Blundell (1986).[6] The variables used as instruments are the years of work experience and their square, dummies for the education level and dummy for being above or below the statutory minimum wage. The exogenous variables used are those in the hour's equation: dummies for the age group, health condition, number of children, age of youngest child, marital status, and spouse working status; and the continued variable for the amount of (log) mortgage payments.

Table 1 reports results obtained from the estimation of the BB-FF model (2.5); and, for comparison, also those obtained from the BDM model, which is shown in the previous section to result from imposing the restriction $\delta(z_i) = 0$ on the BB-FF model. The difference between estimates obtained from the two models is reported and statistically tested in the last two columns of the table. The estimated coefficients of $\alpha(z_i), \beta(z_i), \gamma(z_i)$, and $\delta(z_i)$ in the hours' equation are calculated at the average of characteristics; while the effect of each characteristic on working hours is calculated at the average of hourly wage and non-labour income. The parameters estimates corresponding to the interactions of characteristics with the logarithmic hourly wage, one over hourly wage and the ratio of non-labour income to the wage rate are reported in Table B2 of the Appendix.

As shown in Table 1 the coefficients for the log wage rate and the ratio of non-labour to wage rate are statistically significant in both the BB-FF and the BDM model; and the coefficient of the ratio of non-labour income to the wage rate is negative, as required by theory. Notably, $\delta(z_i)$, the coefficient capturing the forward-falling part of the labour supply curve, is also statistically significant. This implies that subsistence female labour supply behaviour does exist and the use of the BB-FF specification is justified. As we shall see later, this feature of the BB-FF model can have crucial implications when investigating

[6] First, suspicious for endogeneity variables are regressed on all exogenous variables in the hour's equation and the instruments. Then, a Tobit model of the working hours on the exogenous and endogenous variables and the residuals obtained from the first step is estimated and the coefficients of the residuals are tested. The results of this test [F(30,2345)=3.402, p-value=0] suggest rejection of the wage exogeneity.

policy issues that rely on knowing the labour supply behaviour of female persons at the low end of the wage distribution.

Table 1: Parameter estimates of the BB-FF and BDM labour supply functions

Variables	BB-FF model		BDM model		Difference	
	Coefficient	St. Error	Coefficient	St. Error	Coefficient	St. Error
Constant	-56.264***	13.935	19.750***	3.471	-76.014***	12.170
Hourly wage (log)	26.214***	4.136	4.004***	1.243	22.210***	3.555
1/Hourly wage	243.041***	42.054				
Non-labour income / Hourly wage	-0.133***	0.021	-0.109***	0.021	-0.024***	0.007
Age 35-44	-0.941	1.123	0.390	1.082	-1.331	1.158
Age 45-54	-3.941***	1.168	-3.204***	1.139	-0.737**	0.300
Age 55-64	-5.263***	1.733	-3.831**	1.681	-1.432***	0.518
Health Condition: Fair/Bad	-2.620**	1.042	-3.136***	1.034	0.516**	0.218
Married	-3.795***	1.082	-2.449**	1.059	-1.346***	0.400
Separated/Divorced/Widowed	2.989**	1.245	3.370**	1.235	-0.381	0.322
Dependent children: One	-0.595	1.475	-1.190	1.462	0.595	1.718
Dependent children: Two	-3.028	1.874	-4.357***	1.840	1.329*	0.796
Dependent children: Three or more	-8.197***	2.787	-10.735***	2.712	2.538**	1.211
Youngest child aged 0-2	-14.054***	1.949	-14.306***	1.930	0.253	3.572
Youngest child aged 3-5	-9.152***	2.065	-8.386***	2.043	-0.766	24.218
Youngest child aged 6-12	-3.900**	1.755	-3.966**	1.742	0.066	0.250
Spouse works	2.766***	1.073	2.698***	1.057	0.068	0.110
Mortgage payments (log)	0.333***	0.101	0.445***	0.097	-0.112	0.164
Number of observations			2327			

Note: *, **, *** denote significance at 10%, 5%, 1% level, respectively.

Commenting on other parameter estimates reported in Table 1, the constant term, $\gamma(z_i)$, in the BB-FF model does not have a very informative role; it represents the number of hours an individual with 'average' characteristics is willing to work when the effects of log hourly wage, inverse hourly wage and the ratio of non-labour to hourly wage offset each other. Based on the estimation results there is no real valued wage rate for this to happen, although, the negative constant term suggests that females spend their time at non-labour market activities. In the BDM model the constant term is positive and represents the number of hours the average female is working when the wage rate is at the minimum (6.10 euro), since at this wage rate the effect of log hourly wage and the ratio of non-labour to hourly wage offset each other.

A change in the hourly wage has a multiple effect on the working hours of individuals through the log hourly-wage term, the 1/hourly-wage term and the non-labour income to hourly-wage term; and can be either positive or negative. On average an increase of the hourly wage by 1 euro increases the working hours of females by 0.66 hours. The corresponding figure of BDM model is 0.50 hours. The non-labour to hourly-wage term has a statistically significant and negative effect on working hours. In particular, when the ratio of non-labour to labour income increases by 1 unit the labour supply, on average, decreases by 0.13 hours in the BB-FF model and by 0.11 hours in the BDM model. This difference between the two models is statistically significant, so that one can conclude that a slightly more responsive female labour supply to changes in non-labour income is implied by the BB-FF than the BDM model.

Commenting further on the results obtained from the BB-FF model we see that age has a negative effect on the working hours of females. Compared to women in the youngest age group those in: (i) the 45-54 and 55-64 age groups work 3.9 and 5.3 less hours, respectively; and (ii) the 35-44 age groups work the same number of hours. The health condition also has a negative effect on working hours: females with fair or bad health work about 2.6 hours less compared to those with good or very good health. Married females work by 3.8 hours less than unmarried ones; whereas those divorced, separated or widowed work by 3 hours more. Also females work 2.8 hours more when their spouse is working compared to those whose spouse is jobless.

Mothers with one or two children work the same hours as females with no children. On the other hand mothers with three or more children work 8.2 hours less than females with no children. The age of children also has a significant effect on working hours: females whose youngest child is in the 0-2, 3-5 and 6-12 age groups work 14.1, 9.2 and 3.9 hours less, respectively, compared to females with no children or with children older than 12.

Comparing the effects of characteristics estimated by the BB-FF and BDM models we observe many similarities, but also some interesting differences. In particular, the two models give similar results about the effect of children's age, spouse working status and mortgage payments; whereas, the size of the effect (but not the sign) of age, number of children, health condition and marital status is significantly different between the two models. More precisely, the BDM model depicts a higher negative effect of the number of children and bad health on the working hours of females than the BB-FF model; while age and being married appear to depict a higher negative effect on working hours in the BB-FF than the BDM model.

4. Discussion

4.1 The shape of female labour supply

Table 2 shows the distribution of female labour supply based on the BB-FF and BDM models. According to the BDM model about 70 percent of females have strictly increasing and 30 percent backward bending labour supply; whereas according to the BB-FF model 20.6 percent have strictly positive, 7 percent inverse U-shape, and 72.4 percent U-shape labour curve that incorporates a forward-falling and positively sloped segment. These results suggest that the assumption that labour supply cannot have negative slope at low wage rates is too restrictive for studying the labour supply behaviour of females, insofar as it excludes the shape which evidently describes more accurately the labour supply behaviour of the majority of females.

Table 2: labour supply shapes

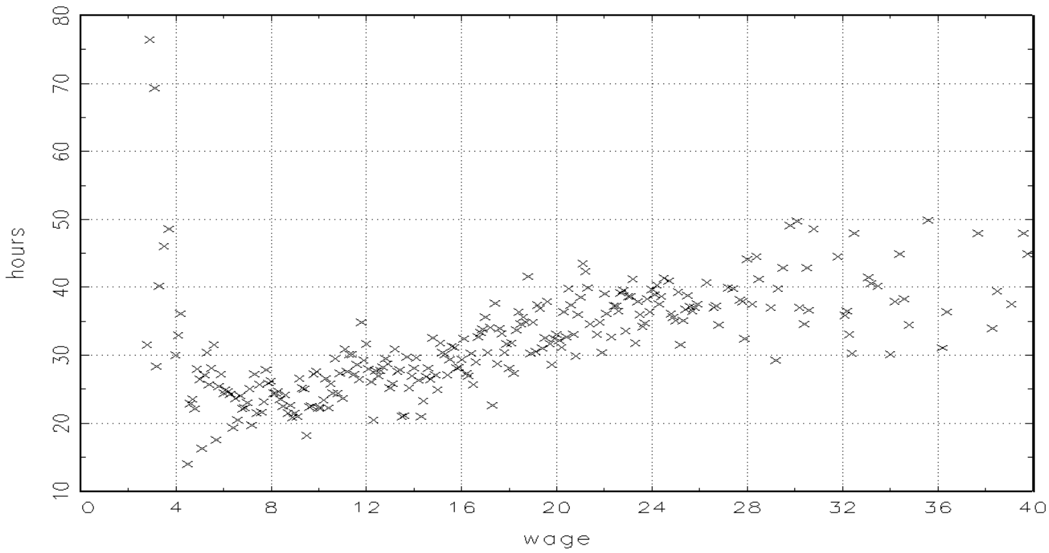
	BB-FF model	BDM model
% of females with U-shape labour supply	72.4%	
% of females with strictly positive labour supply	20.6%	70.1%
% of females with strictly negative labour supply	0.0%	
% of females with inverse U-shape labour supply	7.0%	29.9%
Number of observations	2327	2327
U-shape wage threshold	7.90	
Inverse U-shape wage threshold	28.26	36.8

The last two rows of Table 2 report the wage rate for which the slope of the labour supply changes sign in the case of a U-shape or an inverse U-shape curve. According to BB-FF model females with a U-shape labour supply are on the positively sloped part of this shape for wages above 7.9 euro per hour, and on the negatively sloped (forward falling) part for wages below this wage threshold. Females with an inverse U-shape labour supply are on the positively sloped part of this shape for wages below 28.3 euro, and on the negatively sloped part (backward bending) for wages above this wage threshold. The BDM model can only be backward-bending and, according to this model, the wage rate at which the slope of the labour supply changes from positive to negative is 37 euro.

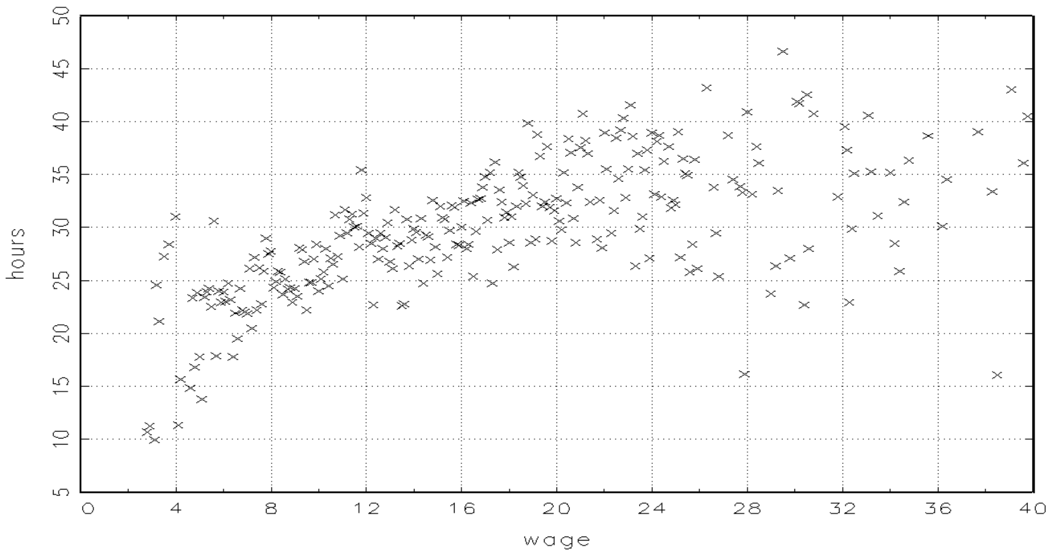
Figure 3 shows a scatter plot for the wage rate and the corresponding working hours predicted by the BB-FF and BDM models. Again, the difference between the two models relates to labour supply at low wage rates: the BB-FF model predicts that females reduce their labour supply up to a minimum of around 25 hours per week, after which further reduction in wages increases their labour supply; whereas, the BDM model predicts that reduction in wages always results in reduction of labour supply. Furthermore, the scatter plots in Figure 3 show that the variance of predictions obtained from the BB-FF model is lower than that obtained from the BDM model, a result conforming with our finding that the BB-FF model is a better overall fit to the data.

Figure 3: Labour supply predictions

BB-FF model



BDM model



4.2 Behavioural and policy implications

Table 3 reports the wage elasticities of female labour supply calculated from the parameter estimates of the BB-FF and BDM models. The elasticities are presented at different percentiles of the wage distribution to highlight variation of labour supply behaviour at the extremes of this distribution. More specifically, we calculate the mean wage elasticity for individuals with wages at the lowest 5, between 5 and 50, between 50 and 95, and at the top 5 percent of the wage distribution.

Table 3: Uncompensated elasticity by wage groups

	BB-FF		BDM		Difference	
	Coefficient	St. Error	Coefficient	St. Error	Coefficient	St. Error
Lowest 5 percent	-0.472***	0.173	0.481***	0.079	-0.953***	0.158
5 to 50 percent	0.123	0.101	0.438***	0.060	-0.314***	0.052
50 to 95 percent	0.585***	0.069	0.229***	0.044	0.356***	0.071
Highest 5 percent	0.507***	0.064	0.158***	0.040	0.350***	0.053
All	0.321***	0.064	0.331***	0.050	-0.010***	0.003

Notes: *, **, *** denote significance at 10%, 5%, 1% level, respectively.

For the lowest 5 percent of the wage distribution the mean elasticity calculated from the parameter estimates of the BB-FF model is negative and statistically significant. On the other hand the mean elasticity for females between 5 and 50 percent of the wage distribution is not statistically significant. Also the mean elasticity for the top half of the wage distribution is positive and statistically significant. The wage elasticities calculated from the parameter estimates of the BDM model are positive over the whole range of the wage distribution and decrease as we move from low to high wage percentiles.

The elasticity differences between the two models for different percentiles of the wage distribution are reported (with their standard error) in the last two columns of Table 3. It emerges from these differences is that the BDM model tends to overstate at low and understate at high wages the response of females to wage changes compared to the BB-FF model. However, on average these differences tend to cancel out, with the mean wage elasticity difference between the two models being very small.

To illustrate the usefulness of considering forward-falling labour supply behaviour in the analysis of policy reforms affecting the wage rate of low-paid females we use the above empirical findings to simulate the impact of a hypothetical change in minimum wage on female labour supply. Notably, in our sample there are about 2.5% of female employees with hourly wage rate below the national minimum wage in the UK.[7] As said in the introduction about 1% of UK employees were below the national minimum wage in April 2009 (Low Pay Commission, 2010), and this figure appear to be rising over time (le Roux et al., 2013). Employer practices to avoid paying the minimum wage include the excuse that employees are paid tips, payment by work-piece rather than hours worked and the labelling of employees as apprentices, volunteers or interns. In addition, employers can under-report the employee hours by paying cash-in-hand.

The assumptions we make for the simulation of the effects on female labour supply are: (i) an increase of the statutory minimum wage from 6.83 to 7.27 euro, the latter being the 2013 national minimum wage; and (ii) perfect compliance with the statutory minimum wage (i.e. individuals in the sample paid below the minimum wage are assumed to earn the minimum wage). In Table 4 we report the effect of this change on the working hours of females with wage rates in the lowest five percent of the wage distribution obtained from the BB-FF and BDM models.[8]

Table 4: The effect of an increase in the minimum wage on working hours

	BB-FF model			BDM model		
	Hours	Hrs under min. wage	Difference	Hours	Hrs under min. wage	Difference
With wage in the lowest 5%	27.4	23.6	-3.81	22.9	24.7	1.85
	(1.307)	(0.636)	(1.054)	(0.933)	(0.625)	(0.408)
All	27.9	27.7	-0.21	27.8	28.0	0.13
	(0.346)	(0.352)	(0.064)	(0.343)	(0.346)	(0.026)

Note: Standard errors in brackets.

It is clear that, as regards labour supply behaviour at low wage rates, the two models come-up with contradictory results: based on the BB-FF model the hypothesised increase in minimum wage would result in a decrease of the working hours of females in the lowest five per cent of the wage distribution by almost 4 hours; the corresponding change predicted for the same minimum wage change by the BDM model is an increase of the

[7] The national minimum wage in the UK was introduced in 1999 and determines the minimum amount employees should be paid per hour, depending on their age. In 2010, minimum wage for individuals aged 21 or above was 6.83 euro, for individuals aged between 18 and 20 was 5.67 euro, for individuals aged under 18 was 4.19 euro and for apprentice was 2.88 euro.

[8] The minimum wage policy affects mainly individuals who have wage rate in the lowest 5 percent of the wage distribution.

working hours by about 2 hours. On aggregate, however, the predictions of the two models differ only by about 0.3 hours.

The upshot of the discussion in this section is that the data provide evidence that the added flexibility of the BB-FF model, through allowing subsistence labour supply behaviour, can be an advantage over models not having this flexibility for the impact assessment of policies affecting female employees at the lower end of wage distribution. However, the same is not found to hold true when the objective of the impact assessment is to assess the effect of policy reforms on aggregate female labour supply.

5. Conclusion

The labour supply behaviour of individuals at the lower end of the wage distribution has attracted very little attention in the literature, despite the fact that these individuals are often at the centre of social and economic policy interest. This paper attempts to fill this literature gap by proposing a simple parametric extension of the popular backward-bending Blundell-Duncan-Meghir (BDM) semi-logarithmic model to also allow for forward-falling (subsistence) labour supply behaviour.

The proposed backward-bending and forward-falling (BB-FF) labour supply model satisfies all the fundamental principles of consumer behaviour, including integrability (i.e. it can yield analytical solutions for the indirect utility and expenditure functions), so it can be used not only for behavioural but also for welfare analysis of reforms affecting individuals at all percentiles of the wage distribution; it can also be easily estimated using simple econometric methods. Notably, the BB-FF nests the DBM model. This is particularly convenient because it enables the proposed model to be empirically validated using as benchmark the long-established as classic labour supply model in the literature.

The empirical analysis in the paper draws on UK data to estimate the labour supply behaviour of female workers. The results obtain show that the BB-FF model fits the data better than the BDM model, as forward-falling labour supply behaviour appears to be statistically significant among female workers paid very low wages. However, the aggregate response to changes in average wage obtained from the two models does not appear to be markedly different.

Finally, in order to illustrate the policy implications of our findings we use the wage elasticities obtained from the two models to simulate the effects of an increase of minimum wage on female labour supply. The results suggest that, on aggregate, the predicted change in female labour supply does not differ decidedly between the two models. For females in the lowest five per cent of the wage distribution, however, the two models yield conflicting effects: an increase in minimum wage would decrease labour supply according to the BB-FF model and increase it according to the BDM model. These results reinforce the point made throughout the analysis is the paper, i.e. the use of proposed BB-FF model can be mandatory for accurate empirical analysis of female labour supply at the low end of wage distribution; but may not add too much empirical accuracy when the investigation is focused simply on aggregate female labour supply.

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Appendix A: Properties of the BB-FF semi-log labour supply function

A1.1 Expenditure function

$$e(w,p,U) = Up\left(\frac{w}{p}\right)^{-\beta} - \frac{w}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right) + \gamma - \frac{\alpha}{\beta+1}\right) + wT - \frac{\delta p}{\beta}$$

1) $e(w,p,U)$ is continuous in (w,p) for $w > 0$ and $p > 0$

The function $e(w,p,U)$ is continuous if it is continuous at every point in its domain $(\mathbb{R}^{++} \times \mathbb{R}^{++})$ and it is continuous at a point $(w_n, p_n) \in \mathbb{R}^{++} \times \mathbb{R}^{++}$ if

$$\lim_{(w,p)\to(w_n,p_n)} e(w,p,U) = e(w_n,p_n,U)$$

$$\lim_{(w,p)\to(w_n,p_n)} e(w,p,U) = \lim_{(w,p)\to(w_n,p_n)} \left[Up\left(\frac{w}{p}\right)^{-\beta} - \frac{w}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right) + \gamma - \frac{\alpha}{\beta+1}\right) + wT - \frac{\delta p}{\beta}\right]$$
$$= U \lim_{(w,p)\to(w_n,p_n)} p^{\beta+1} \lim_{(w,p)\to(w_n,p_n)} w^{-\beta}$$
$$- \frac{1}{\beta+1} \lim_{(w,p)\to(w_n,p_n)} w \lim_{(w,p)\to(w_n,p_n)} \left(\alpha \lim_{(w,p)\to(w_n,p_n)} (logw - logp) + \gamma - \frac{\alpha}{\beta+1}\right)$$
$$+ T \lim_{(w,p)\to(w_n,p_n)} w - \frac{\delta}{\beta} \lim_{(w,p)\to(w_n,p_n)} p$$
$$= Up_n\left(\frac{w_n}{p_n}\right)^{-\beta} - \frac{w_n}{\beta+1}\left(\alpha log\left(\frac{w_n}{p_n}\right) + \gamma - \frac{\alpha}{\beta+1}\right) + w_nT - \frac{\delta p_n}{\beta} = e(w_n,p_n,U)$$

We have used the following properties of the limits:

- the limit of a sum of functions is the sum of the limits of the functions, and
- the limit of a product of functions is the product of the limits of the functions;

and of the continuous functions of two variables:

- the sum of a finite number of continuous functions is a continuous function,
- the product of a finite number of continuous functions is a continuous function, and
- the quotient of two continuous functions is a continuous function wherever the denominator is non-zero

2) $e(w,p,U)$ is non-decreasing in p and w

$$\frac{\partial e(w,p,U)}{\partial p} = (\beta+1)U\left(\frac{w}{p}\right)^{-\beta} + \frac{\alpha}{\beta+1}\left(\frac{w}{p}\right) - \frac{\delta}{\beta} = c^h(w,p,U) \geq 0$$

$$\frac{\partial e(w,p,U)}{\partial w} = -\beta U\left(\frac{w}{p}\right)^{-\beta-1} - \frac{1}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right) + \gamma - \frac{\alpha}{\beta+1}\right) - \frac{w}{\beta+1}\left(\frac{\alpha}{w}\right) + T$$
$$= T - \beta U\left(\frac{w}{p}\right)^{-\beta-1} - \frac{1}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right) + \gamma - \frac{\alpha}{\beta+1} + \alpha\right)$$
$$= T - \left(\beta U\left(\frac{w}{p}\right)^{-\beta-1} + \frac{1}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right) + \gamma + \frac{\alpha\beta}{\beta+1}\right)\right) = L^h(w,p,U) \geq 0$$

For $e(w,p,U)$ to be non-decreasing in p and w we need $c^h(w,p,U)$ and $L^h(w,p,U)$ to be non-negative i.e. the demanded consumption good and leisure time to be non-negative.

3) $\underline{e(w,p,U)}$ is homogeneous of degree 1 in $\underline{(w,p)}$

Assume λ is positive scalar then

$$e(\lambda w,\lambda p,U) = U\lambda p\left(\frac{\lambda w}{\lambda p}\right)^{-\beta} - \frac{\lambda w}{\beta+1}\left(\alpha log\left(\frac{\lambda w}{\lambda p}\right)+\gamma-\frac{\alpha}{\beta+1}\right)+\lambda wT-\lambda\frac{\delta p}{\beta}$$
$$= \lambda\left(Up\left(\frac{w}{p}\right)^{-\beta} - \frac{w}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right)+\gamma-\frac{\alpha}{\beta+1}\right)+wT-\frac{\delta p}{\beta}\right)$$
$$= \lambda e(w,p,U)$$

4) $\underline{e(w,p,U)}$ is concave in $\underline{(w,p)}$

A continues function is concave iff its hessian matrix is negative semidefinite for (w, p) , i.e. the Hessian matrix (H) has a negative first principal minor(H_1) determinant and a nonnegative second principal minor (H_2) determinant.

$$H = \begin{pmatrix} \frac{\partial^2 e(w,p,U)}{\partial w^2} & \frac{\partial^2 e(w,p,U)}{\partial w\partial p} \\ \frac{\partial^2 e(w,p,U)}{\partial p\partial w} & \frac{\partial^2 e(w,p,U)}{\partial p^2} \end{pmatrix}$$

$$H_1 = \frac{\partial^2 e(w,p,U)}{\partial w^2} = \frac{\partial\left(L^h(w,p,U)\right)}{\partial w} = -\left(-\beta(\beta+1)\frac{1}{p}U\left(\frac{w}{p}\right)^{-\beta-2}+\frac{\alpha}{\beta+1}\left(\frac{1}{w}\right)\right)$$
$$= \frac{\beta(\beta+1)}{p}U\left(\frac{w}{p}\right)^{-\beta-2} - \frac{\alpha}{\beta+1}\left(\frac{1}{w}\right)$$

$$\frac{\partial^2 e(w,p,U)}{\partial p^2} = \frac{\partial\left(c^h(w,p,U)\right)}{\partial p} = \beta(\beta+1)U\left(\frac{w}{p}\right)^{-\beta}\left(\frac{1}{p}\right) - \frac{\alpha}{\beta+1}\left(\frac{w}{p}\right)\left(\frac{1}{p}\right)$$
$$= \left(\frac{w}{p}\right)^2\left(\frac{\beta(\beta+1)}{p}U\left(\frac{w}{p}\right)^{-\beta-2} - \frac{\alpha}{\beta+1}\left(\frac{1}{w}\right)\right)$$

$$\frac{\partial^2 e(w,p,U)}{\partial w\partial p} = \frac{\partial\left(L^h(w,p,U)\right)}{\partial p} = \frac{\partial\left(c^h(w,p,U)\right)}{\partial w} = \frac{\partial^2 e(w,p,U)}{\partial p\partial w}$$
$$= \left(-\beta(\beta+1)U\left(\frac{w}{p}\right)^{-\beta-1}\left(\frac{1}{p}\right)+\frac{\alpha}{\beta+1}\left(\frac{1}{p}\right)\right) =$$
$$= \left(-\frac{w}{p}\right)\left(\frac{\beta(\beta+1)}{p}U\left(\frac{w}{p}\right)^{-\beta-2} - \frac{\alpha}{\beta+1}\left(\frac{1}{w}\right)\right)$$

$$H_2 = \frac{\partial^2 e(w,p,U)}{\partial w^2}\frac{\partial^2 e(w,p,U)}{\partial p^2} - \left(\frac{\partial^2 e(w,p,U)}{\partial w \partial p}\right)^2$$

$$= \left(\frac{w}{p}\right)^2 \left(\frac{\beta(\beta+1)}{p} U \left(\frac{w}{p}\right)^{-\beta-2} - \frac{\alpha}{\beta+1}\left(\frac{1}{w}\right)\right)^2$$

$$- \left(\left(-\frac{w}{p}\right)\left(\frac{\beta(\beta+1)}{p} U \left(\frac{w}{p}\right)^{-\beta-2} - \frac{\alpha}{\beta+1}\left(\frac{1}{w}\right)\right)\right)^2 = 0$$

For the proposed expenditure function to be concave we need $\frac{\partial^2 \mathrm{e(w,p,U)}}{\partial \mathrm{w}^2} = \frac{\partial\left(\mathrm{L^h(w,p,U)}\right)}{\partial \mathrm{w}} = \frac{\partial\left(T - \mathrm{h^h(w,p,U)}\right)}{\partial \mathrm{w}} = -\frac{\partial\left(\mathrm{h^h(w,p,U)}\right)}{\partial \mathrm{w}} < 0$ i.e. the Hicksian labour supply be increasing in the wage rate.

A1.2 Indirect utility, Hicksian and Marshallian functions

According to the Shephard's lemma, the derivative of the expenditure function with respect to the wage rate give the Hicksian demand for leisure $\left(L^h(w,p,U)\right)$ which is equal to the total endowment of hours minus the Hicksian labour supply $\left(T - h^h(w,p,U)\right)$. Analytically we have that

$$h^h(w,p,U) = T - L^h(w,p,U) = T - \frac{\partial e(w,p,U)}{\partial w} \Rightarrow$$

$$h^h(w,p,U) = \beta U \left(\frac{w}{p}\right)^{-\beta-1} + \frac{1}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right) + \gamma + \frac{\alpha\beta}{\beta+1}\right)$$

By inverting the expenditure function and setting $\mathrm{e}\left(\mathrm{w,p,u(w,p,M)}\right) = \mathrm{M}$, i.e. the definition of the minimum expenditure necessary for an individual i to reach the utility level $\mathrm{U} = \mathrm{u(w,p,y)}$, we can obtain the indirect utility function.

$$M = e\left(w,p,u(w,p,M)\right) = u(w,p,M)p\left(\frac{w}{p}\right)^{-\beta} - \frac{w}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right) + \gamma - \frac{\alpha}{\beta+1}\right) + wT - \frac{\delta p}{\beta} \Rightarrow$$

$$u(w,p,M) = \frac{1}{p}\left(\frac{w}{p}\right)^{\beta}\left(M + \frac{w}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right) + \gamma - \frac{\alpha}{\beta+1}\right) - wT + \frac{\delta p}{\beta}\right) \Rightarrow$$

$$u(w,p,y) = \frac{1}{\beta+1}\left(\frac{w}{p}\right)^{\beta+1}\left((\beta+1)\frac{y}{w} + \alpha log\left(\frac{w}{p}\right) + \gamma - \frac{\alpha}{\beta+1} + \frac{\delta(\beta+1)}{\beta\frac{w}{p}}\right)$$

where $M = y + wT$ is the full income and y the non-labour income.

Substituting in the Hicksian equation above, the indirect utility function (A3) we obtain the Marshallian labour supply function

$$h^m(w,p,M) = h^h\big(w,p,u(w,p,M)\big)$$

$$= \beta\left(\frac{w}{p}\right)^{-\beta-1}\frac{1}{\beta+1}\left(\frac{w}{p}\right)^{\beta+1}\left((\beta+1)\frac{y}{w}+\alpha log\left(\frac{w}{p}\right)+\gamma-\frac{\alpha}{\beta+1}+\frac{\delta(\beta+1)}{\beta\frac{w}{p}}\right)$$

$$+\frac{1}{\beta+1}\left(\alpha log\left(\frac{w}{p}\right)+\gamma+\frac{\alpha\beta}{\beta+1}\right) \Rightarrow$$

$$h^m(w,p,M) = h^m(w,p,y) = \beta\frac{y}{w}+\frac{1}{\beta+1}(\beta+1)\,\alpha log\left(\frac{w}{p}\right)+\frac{1}{\beta+1}(\beta+1)\gamma+\frac{\delta}{\frac{w}{p}} \Rightarrow$$

$$h^m(w,p,y) = \beta\frac{y}{w}+\alpha log\left(\frac{w}{p}\right)+\gamma+\frac{\delta}{\left(\frac{w}{p}\right)}$$

A1.3 Elasticities

The income elasticities is defined as $s^{inc} = (\partial h^m/\partial y)(y/h)$ and given by

$$s^{inc} = \left(\frac{\beta}{w}\right)\left(\frac{y}{h}\right) = \beta\left(\frac{y}{wh}\right)$$

The compensated elasticity is defined as $s^c = \big(\partial h^h/\partial w\big)(w/h)$ and given by

$$s^c = -\left(\frac{\beta(\beta+1)}{p}\left(\frac{w}{p}\right)^{-\beta-2}U-\frac{\alpha}{\beta+1}\left(\frac{1}{w}\right)\right)\left(\frac{w}{h}\right)$$

$$= -\left(\frac{\beta}{p}\left(\frac{w}{p}\right)^{-1}\left((\beta+1)\frac{y}{w}+\alpha log\left(\frac{w}{p}\right)+\gamma-\frac{\alpha}{\beta+1}+\frac{\delta(\beta+1)}{\beta\frac{w}{p}}\right)-\frac{\alpha}{\beta+1}\left(\frac{1}{w}\right)\right)\left(\frac{w}{h}\right)$$

$$= -\left(\frac{\beta}{w}\left(h+\frac{y}{w}-\frac{\alpha}{\beta+1}+\frac{\delta}{\beta\frac{w}{p}}\right)-\frac{\alpha}{\beta+1}\left(\frac{1}{w}\right)\right)\left(\frac{w}{h}\right)$$

$$= -\left(\beta h+\beta\frac{y}{w}-\frac{\alpha\beta}{\beta+1}+\frac{\delta}{\frac{w}{p}}-\frac{\alpha}{\beta+1}\right)\left(\frac{1}{h}\right) \Rightarrow$$

$$s^c = \frac{\alpha}{h}-\left(\frac{\beta y+\delta p}{wh}\right)-\beta$$

The uncompensated elasticity is defined as $s^u = (\partial h^m/\partial w)(w/h)$ and given by

$$s^u = \left(-\beta\frac{y}{w^2}+\alpha\frac{1}{w}-\frac{\delta p}{w^2}\right)\left(\frac{w}{h}\right) = \frac{\alpha}{h}-\left(\frac{\beta y+\delta p}{wh}\right)$$

Appendix B: Data and empirical results

Table B1: Descriptive Statistics, Females

Variable	Obs	Mean	Std. Dev.	Min	Max
Hours	2327	28.077	16.338	0.000	84.000
Age25-34	2327	0.228	0.420	0.000	1.000
Age 35-44	2327	0.295	0.456	0.000	1.000
Age 45-54	2327	0.345	0.475	0.000	1.000
Age 55-64	2327	0.132	0.338	0.000	1.000
Primary/Lower secondary	2327	0.090	0.286	0.000	1.000
Upper secondary	2327	0.458	0.498	0.000	1.000
Tertiary	2327	0.452	0.498	0.000	1.000
Years of experience	2327	19.661	10.611	0.000	46.000
Health condition: Good	2327	0.875	0.330	0.000	1.000
Health condition: Fair/Bad	2327	0.125	0.330	0.000	1.000
Single	2327	0.206	0.405	0.000	1.000
Married	2327	0.650	0.477	0.000	1.000
Separated/Divorced/Widowed	2327	0.144	0.351	0.000	1.000
Number of dep. children: None	2327	0.511	0.500	0.000	1.000
Number of dep. children: One	2327	0.228	0.420	0.000	1.000
Number of dep. children: Two	2327	0.205	0.403	0.000	1.000
Number of dep. children: Three or more	2327	0.056	0.230	0.000	1.000
Youngest child aged between 0 and 2	2327	0.132	0.338	0.000	1.000
Youngest child aged between 3 and 5	2327	0.088	0.283	0.000	1.000
Youngest child aged between 6 and 12	2327	0.164	0.371	0.000	1.000
Youngest child aged between 13 and 16	2327	0.104	0.306	0.000	1.000
Spouse works	2327	0.719	0.450	0.000	1.000
Mortgage payments (log)	2327	4.804	4.009	0.000	10.633
Hourly wage (log)	2327	2.470	0.461	1.037	4.412
Inverse of hourly wage (1/Hourly wage)	2327	0.093	0.041	0.012	0.355
Ratio of non-labour to hourly wage	2327	36.933	42.650	0.000	585.673

Figure B1: Scatter plot of hours and wages

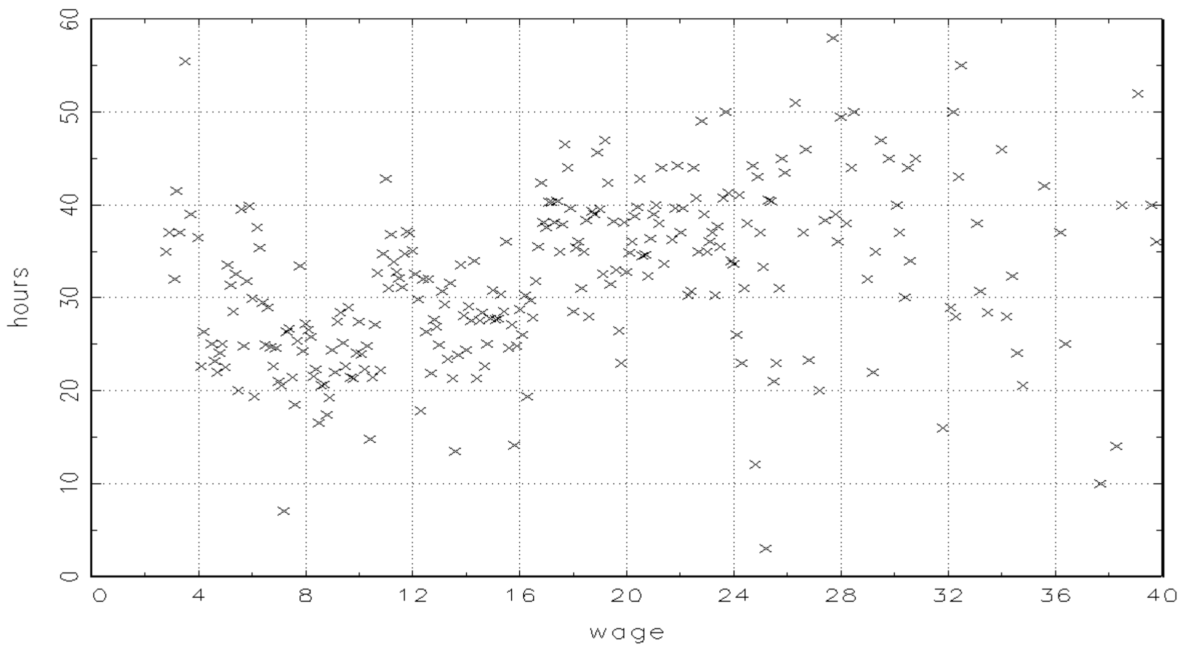

Table B2: Labour supply equation

Variables	Coefficient	St. Error
Constant	16.281	(43.875)
Age 35-44	64.963	(43.263)
Age 45-54	3.118	(47.890)
Age 55-64	-65.899	(56.768)
Health condition: Fair/Bad	-2.693***	(1.042)
Married	-3.900***	(1.082)
Separated/Divorced/Widowed	3.071**	(1.245)
Number of dep. children: One	-89.749**	(38.359)
Number of dep. children: Two	-139.562***	(50.584)
Number of dep. children: Three or more	-168.565***	(64.798)
Youngest child aged between 0 and 2	-113.960**	(51.498)
Youngest child aged between 3 and 5	-109.531**	(55.100)
Youngest child aged between 6 and 12	-22.928	(44.263)
Spouse works	2.843***	(1.073)
Mortgage cayments (log)	0.342***	(0.101)
Hourly wage (log)	8.752	(13.311)
Hourly wage (log)*Age 35-44	-20.555	(13.018)
Hourly wage (log)*Age 45-54	-3.451	(14.339)
Hourly wage (log)*Age 55-64	17.216	(17.011)
Hourly wage (log)*Number of dep. children: One	23.715**	(11.07)
Hourly wage (log)*Number of dep. children: Two	36.736**	(14.698)
Hourly wage (log)*Number of dep. children: Three or more	45.349**	(18.939)
Hourly wage (log)*Youngest child aged between 0 and 2	28.640*	(15.148)
Hourly wage (log)*Youngest child aged between 3 and 5	30.275*	(15.974)
Hourly wage (log)*Youngest child aged between 6 and 12	7.684	(12.977)
1/Hourly wage	19.245	(123.467)
1/Hourly wage*Age 35-44	-150.396	(123.042)
1/Hourly wage*Age 45-54	-8.764	(137.877)
1/Hourly wage*Age 55-64	177.202	(165.000)
1/Hourly wage*Number of dep. children: One	336.441***	(124.109)
1/Hourly wage*Number of dep. children: Two	519.234***	(162.566)
1/Hourly wage*Number of dep. children: Three or more	526.108**	(207.616)
1/Hourly wage*Youngest child aged between 0 and 2	234.788	(158.232)
1/Hourly wage*Youngest child aged between 3 and 5	223.156	(170.478)
1/Hourly wage*Youngest child aged between 6 and 12	-51.838	(138.278)
Ratio of non-labour to labour income	-0.188***	(0.052)
Ratio of non-labour to labour income*Age 35-44	-0.031	(0.039)
Ratio of non-labour to labour income*Age 45-54	0.059	(0.045)
Ratio of non-labour to labour income*Age 55-64	0.039	(0.072)
Ratio of non-labour to labour income*Number of dep. children: One	-0.023	(0.078)
Ratio of non-labour to labour income*Number of dep. children: Two	-0.074	(0.083)
Ratio of non-labour to labour income*Number of dep. children: Three or more	-0.026	(0.088)
Ratio of non-labour to labour income*Youngest child aged between 0 and 2	0.186**	(0.077)
Ratio of non-labour to labour income*Youngest child aged between 3 and 5	0.123	(0.079)
Ratio of non-labour to labour income*Youngest child aged between 6 and 12	0.129*	(0.070)
Number of observations	2327	

Notes: 1. standard errors in brackets

2. *, **, *** significant at 10%, 5%, 1% significance level, respectively.

B.3 Empirical analysis

Table 1: Labour market participation equation, Females

Variables	Coefficient	St. Error
Age 35-44	-0.475***	(0.108)
Age 45-54	-1.161***	(0.148)
Age 55-64	-1.387***	(0.209)
Upper Secondary	0.348***	(0.127)
Post secondary-Tertiary	0.634***	(0.131)
Years of experience	0.088***	(0.014)
Years of experience, squared	-0.000	(0.000)
Health condition: Fair/Bad	-0.181	(0.115)
Married	-0.347***	(0.115)
Separated/Divorced/Widowed	0.074	(0.157)
Number of dep. children: One	-0.143	(0.151)
Number of dep. children: Two	-0.263	(0.167)
Number of dep. children: Three or more	-0.723***	(0.196)
Youngest child aged between 0 and 2	-1.077***	(0.168)
Youngest child aged between 3 and 5	-0.725***	(0.171)
Youngest child aged between 6 and 12	-0.206	(0.153)
Spouse works	0.430***	(0.113)
Non-labour income (log)	-0.124***	(0.021)
Mortgage payments (log)	0.028***	(0.010)
Constant	1.117***	(0.216)
Number of observations	2327	

Notes: 1. standard errors in brackets
2. *, **, *** significant at 10%, 5%, 1% significance level, respectively.

Table 2: Wage prediction equation, Females

Variables	Coefficient	St. Error
Age 35-44	0.054	(0.036)
Age 45-54	0.051	(0.042)
Age 55-64	0.016	(0.055)
Upper secondary	0.168***	(0.044)
Post-secondary,Tertiary	0.596***	(0.045)
Years of experience	0.015***	(0.005)
Years of experience, squared	-0.000**	(0.000)
Married	0.014	(0.029)
Separated/Divorced/Widowed	-0.012	(0.038)
Constant	1.900***	(0.063)
Number of observations	2327	

Notes: 1. standard errors in brackets
2. *, **, *** significant at 10%, 5%, 1% significance level, respectively.