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Restoring monotone power in the CUSUM test.

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Abstract

This paper shows that a near-stationarity boundary condition for heteroskedastic and autocorrelation consistent estimators can solve the problem of non-monotone power of the CUSUM test for a single break in the mean of a weakly dependent process.

JEL classifications: C12;C22.

Keywords: Heteroskedastic and autocorrelation consistent estimator; structural break test.

1 Introduction.

The CUSUM test for structural breaks is consistent and has good local asymptotic properties for given fixed values in the relevant set of alternative hypotheses (e.g. Kramer and Ploberger, 1990). However in finite samples, its power function can be non-monotone and even reach a zero value as the alternative considered is further away from the null value. This was shown by Perron (1991) and Vogelsang (1999) for a family of tests. We focus on the CUSUM test but the results presented here apply to other tests (e.g. Vogelsang, 1999).

The non-monotone power is due to the variance estimate which scales the CUSUM statistic. The estimated variance is based on the demeaned observed process or the errors. In the general case of dependence this is the spectral density function at zero frequency or a Heteroskedastic and Autocorrelation Consistent (HAC) estimator. The HAC is evaluated under the null hypothesis which implies that under the alternative the variance yields an inflated estimate. This leads to a

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small scaled CUSUM and thereby loss of power. There are two related sources of overestimation: (i) The variance is usually based on least-squares or recursive residuals that are contaminated by the shift under the alternative. As the shift gets larger the variance increases. (ii) When there is dependence in the process the power problem is exacerbated because the shift induces a bias of the autoregressive coefficient towards one (Perron, 1989) thereby inflating the variance estimator.

This paper shows that a simple near-stationarity boundary condition for the HAC estimator restores the monotone power of the CUSUM test for a mean shift in a weakly dependent process. This is inspired by Andrews (1991) and Sul, Phillips and Choi (2005). We show that this boundary condition solves the overestimation problem of the variance under the change-point alternative and preserves the $\sqrt{T}$ consistency of the HAC estimator. Simulation and empirical evidence support this method.

2 HAC estimators for the CUSUM test.

Consider the following stochastic process for a univariate time series, y_t :

$$y_t = \mu + u_t,\ \ t = 1, ..., T, \tag{2.1}$$

where u_t is a second-order stationary mean zero error process. The partial sums process $S_t = \sum_{j=1}^{t} u_j$ satisfies the Functional Central Limit Theorem (FCLT), for regularity conditions found, for instance, in Herrndorf (1984), such that $T^{-1/2}S_{[mT]} \to \sigma W(m)$, where $W(m)$ denotes the standard Wiener process defined on $[0, 1]$ and $\sigma^2 = \lim_{T\to\infty} E\left[T^{-1}\left(\sum_{t=1}^{T} u_t\right)^2\right]$.

The CUSUM statistic for detecting structural changes in the mean of y_t in (2.1):

$$CUSUM = \left(\sigma\sqrt{T}\right)^{-1} \sup_{1\le j\le T}\left|\sum\nolimits_{t=1}^{j} y_t - \sum\nolimits_{t=1}^{T} y_t\right| \to \sup|B(m)|. \tag{2.2}$$

converges to the supremum of a Brownian Bridge, $B(m) = W(m) - mW(1)$. Equivalently (2.2) can be expressed in terms of the OLS residuals $\widehat{u}_t^{OLS} = y_t - 1/T\sum_{t=1}^{T} y_t$:

$$CUSUM = \left(\widehat{\sigma}\sqrt{T}\right)^{-1} \sup_{1\le j\le T}\left|\sum\nolimits_{t=1}^{j} \widehat{u}_t^{OLS}\right| \tag{2.3}$$

where $\widehat{\sigma}$ is a consistent estimator under the null of stability. Traditional estimators of σ^2 include the class of non-parametric spectral density estimators given by $\widehat{\sigma}^2 = \sum_{j=-(T-1)}^{T-1} K\left(j/s(T)\right)\widehat{\gamma}_j$, where $K(.)$ is the kernel function, $\widehat{\gamma}_j = T^{-1}\sum_{t=j+1}^{T}\widehat{u}_t\widehat{u}_{t-j}$, $s(T)$ is the bandwidth and $\widehat{\sigma}^2$ is consistent if $s(T)/T \to 0$ and $s(T) \to \infty$ as $T \to \infty$. For instance, Andrews and Monahan (1992) propose the prewhitened estimator $\widehat{\sigma}^2_{PW}$ given by:

$$\widehat{\sigma}^2_{PW} = \widehat{\sigma}^2_{\varepsilon}/(1-\widehat{\rho})^2 \text{ and } \widehat{\rho} = \sum\nolimits_{t=2}^{T} \widehat{u}_t\widehat{u}_{t-1}/\sum\nolimits_{t=2}^{T} \widehat{u}^2_{t-1}, \text{ where} \tag{2.4}$$

$$\widehat{\sigma}^2_{\varepsilon} = \sum\nolimits_{j=-(T-1)}^{T-1} K\left(j/\widehat{s}_{PW}(T)\right)\widehat{\gamma}^{\varepsilon}_j, \quad \widehat{\gamma}^{\varepsilon}_j = T^{-1}\sum\nolimits_{t=j+1}^{T} \widehat{\varepsilon}_t\widehat{\varepsilon}_{t-j} \quad \text{and } \widehat{\varepsilon}_t = \widehat{u}_t - \rho\widehat{u}_{t-1}. \tag{2.5}$$

The bandwidth $\widehat{s}_{PW}(T)$ is based on the AR(1) plug-in method and depends on the parameter $\widehat{\alpha}_{PW}(1)$ given by:

$$\widehat{s}_{PW}(T) = 1.1447(\widehat{\alpha}_{PW}(1)T)^{1/3}, \;\; \widehat{\alpha}_{PW}(1) = 4\widehat{\rho}^2_{\varepsilon}/(1-\widehat{\rho}^2_{\varepsilon})^2, \;\; \widehat{\rho}_{\varepsilon} = \sum\nolimits_{t=2}^{T} \widehat{\varepsilon}_t\widehat{\varepsilon}_{t-1}/\sum\nolimits_{t=2}^{T} \widehat{\varepsilon}^2_{t-1}, \tag{2.6}$$

which uses the autoregressive estimate $\widehat{\rho}_{\varepsilon}$ obtained from $\widehat{\varepsilon}_t$ instead of $\widehat{u}_t$.

The recoloring procedure in prewhitened HAC estimators (2.4) involves $\widehat{\rho}$. Andrews and Monahan (1992) suggest to replace any $\widehat{\rho}$ that exceeds 0.97 by 0.97 and is less than -0.97 by -0.97. Using this rule the literature shows that the CUSUM test exhibits non-monotone power. Andrews (1991) suggests a boundary condition based on the idea of a confidence interval for $\widehat{\rho}$ which can lead to accurate size of a test and reduce its variance. Sul, Phillips and Choi (2005) propose another recoloring rule based on T given by the boundary condition $\widehat{\rho}' = \min[1-1/\sqrt{T}, \widehat{\rho}]$. This represents the maximum allowable value for ρ to be unity minus its asymptotic standard error, $1/\sqrt{T}$. We generalize this boundary to represent deviations from unity by some fixed local coefficient c, in the spirit of the 'stationary order of magnitude' distance from unity in Sul, Phillips and Choi (2005), so that any root preserves near-stationarity and recoloring is based on:

$$\widehat{\rho}' = \min[1-c/\sqrt{T}, \widehat{\rho}]. \tag{2.7}$$

The effects of this condition on the finite properties of the CUSUM for given c are evaluated via simulations in the next section. One could consider $c = 1, 1.28, 1.65$ in (2.7) where $c = 1$ is proposed by Sul, Phillips and Choi (2005) and $c = 1.28$ and 1.65 are the one-sided confidence intervals values for near-stationarity deviations from $\rho = 1$ (given $c > 0$ such that $\rho < 1$) that correspond to the 10% and 5% standard normal probabilities, respectively. On theoretical grounds any $c > 0$ in (2.7) preserves the consistency of the variance estimator under the null and alternative.

In order to show that condition (2.7) yields a consistent variance estimator consider the process (2.1) where for simplicity we assume that u_t is an AR(1):

$$u_t = \rho u_{t-1} + \varepsilon_t, \varepsilon_t \sim NIID(0, \sigma^2_{\varepsilon}). \tag{2.8}$$

The limiting distribution of the CUSUM depends on $\widehat{\rho}$ and $\widehat{\sigma}^2_{\varepsilon}$ that define the long-run variance of u_t in (2.3). In the parametric model (2.8) the least squares estimate of the long-run variance is

$\widehat{\sigma}_u^2 = \widehat{\sigma}_\varepsilon^2/(1-\widehat{\rho})^2$. In the non-parametric setting the estimates of $1/(1-\widehat{\rho})^2$ are used in the final stage of recoloring to obtain the HAC estimator (2.4). Perron (1989) shows that neglected shifts in (2.8) cause $\widehat{\rho} \to 1$ which imply $\widehat{\rho} = 1 + O_p(T^{-1})$, $(1-\widehat{\rho})^2 = O_p(T^{-2})$ and $\widehat{\sigma}_\varepsilon^2/(1-\widehat{\rho})^2 = O_p(T^2)$. Consequently, $\widehat{\sigma}_u$ does not satisfy the $\sqrt{T}$ consistency. On the contrary, the near-stationarity boundary (2.7) controls the order of magnitude of the long-run variance under the alternative of a spurious unit root and yields $\widehat{\sigma}_\varepsilon^2/(1-\widehat{\rho})^2 = O_p(T)$ which is bounded by $\widehat{\sigma}_u^2 = T\widehat{\sigma}_\varepsilon^2/c^2$.

Turning now to the prewhitened HAC estimators we explain the effects of spurious unit root due to neglected breaks. Under the alternative of large shifts u_t is $I(1)$ and ε_t is $I(0)$. Hence $\widehat{\alpha}_{PW}(1) = O_p(T)$ which implies $\widehat{s}_{PW}(T) = O_p(T^{2/3})$. This preserves the consistency of the non-parametric variance $\widehat{\sigma}_\varepsilon^2$ in (2.5) since $\widehat{s}_{PW}(T)/T = O_p(T^{-1/3}) \to 0$ as $T \to \infty$. The problem arises at the recoloring stage since the PW HAC estimator (2.4) involves the spurious unit root $\widehat{\rho} = 1 + O_p(T^{-1})$ and thereby $\widehat{\sigma}_{PW}^2 = O_p(T^2)$ which hurts power. In contrast if we adopt the near-stationarity recoloring rule (2.7) then $\widehat{\sigma}_{PW}^2 = O_p(T)$ which yields a $\sqrt{T}$ consistent $\widehat{\sigma}_{PW}$.

In the same vain we show that if we adopt a HAC estimator *without* prewhitening and condition (2.7) we maintain consistency under the alternative. When there is no prewhitening $\widehat{\alpha}(1) = 4\widehat{\rho}^2/(1-\widehat{\rho}^2)^2$ where $\widehat{\rho}$ is defined in (2.4), $\widehat{s}(T) = 1.1447(\widehat{\alpha}(1)T)^{1/3}$ and $\widehat{\sigma}^2 = \sum_{j=-(T-1)}^{T-1} K\left(j/\widehat{s}(T)\right)\widehat{\gamma}_j$ (Andrews, 1991). Given the spurious unit root $\widehat{\alpha}(1) = O_p(T^2)$ and $\widehat{s}(T) = O_p(T)$ which implies that $\widehat{s}(T)/T = O_p(1)$ and $\widehat{\sigma}^2$ is inconsistent.[1] We suggest using (2.7) as the plug-in estimate in obtaining $\widehat{\alpha}(1)$. This implies that $\widehat{\alpha}(1) = O_p(T)$, $\widehat{s}(T) = O_p(T^{2/3})$ and $\widehat{s}_{PW}(T)/T = O_p(T^{-1/3}) \to 0$ as $T \to \infty$. Hence the near-stationarity boundary (2.7) yields $\sqrt{T}$ consistent long-run variance estimators when used as the plug-in estimate in $\widehat{\alpha}(1)$ for HAC estimators with no prewhitening and when used as a recoloring method for prewhitened HAC estimators.

3 Simulation and empirical results.

A Monte Carlo analysis is performed to evaluate the effects of the near-stationarity boundary (2.7) on the finite sample power of the CUSUM test and on the properties of the HAC estimators. The Monte Carlo design considers the following Data Generating Process:

$$y_t = \mu + \delta D_t + u_t,\ u_t = \rho u_{t-1} + \varepsilon_t,\ t = 1, ..., T, \tag{3.9}$$

where $\varepsilon_t \sim NIID(0,1)$, $D_t = 0$ for $t < \tau$ and 1 otherwise and $\tau = 0.5T$ is the change-point. The size of the break $\delta = 1, 2, ..., 20$ represents the alternative hypotheses and $\delta = 0$ denotes the null hypothesis of stability. In (3.9) we consider $\rho = 0.7$ and 0.9, $T = 100$ and 200. The $H_0 : \delta = 0$ is

[1] Vogelsang (1999) and Crainiceanu and Vogelsang (2001) also show that under the alternative the bandwidth increases at a rate T as the size of the break increases.

examined using the statistic (2.3) for alternative $\widehat{\sigma}^2_{PW}$ in (2.4)-(2.6). For $K(.)$ in (2.5) we use the Quadratic Spectral (QS) and the Bartlett (BT) kernels.[2]

The power functions of the CUSUM test (adjusted for the empirical size) can be found in Figures 1-8. Each figure compares the four recoloring rules, 0.97 and $c = 1, 1.28, 1.65$, in (2.7) applied to the QS and BT kernel HAC estimators. Figures 1-4 refer to $\rho = 0.7$ in (3.9) while the rest to $\rho = 0.9$. Two broad results can be drawn from Figures 1-8: (i) As the size of the break δ increases the power functions for *all* $c = 1, 1.28, 1.65$ in (2.7) approach one, irrespective of the correlation, sample size or kernel. In contrast, as documented previously in the literature, the power functions that correspond to 0.97 recoloring yield power as poor as zero for large δ. (ii) For the alternative values of c in (2.7) we find that there is still some weak evidence of non-monotone power in the CUSUM for $c = 1$ when $\rho = 0.7$ (Figures 1-4), which disappears when $\rho = 0.9$. In general, $c = 1.65$ for the near-stationarity boundary condition (2.7) in $\widehat{\sigma}_{PW}$ yields not only monotone power functions but also relatively better power compared to $c = 1$ and 1.28.

Next we turn to the finite sample efficiency of the HAC estimators. Figures 9-14 show the Mean Absolute Error (MAE) of the HAC estimators (for $\rho = 0.7$ and $T = 200$ for conciseness) as a function of the break, δ. The inflated MAEs for $\widehat{\sigma}_{PW}$ with the 0.97 recoloring under the alternatives are shown in Figures 9 and 10 for the QS and BT, respectively. In contrast, for $\widehat{\sigma}_{PW}$ with $1 - c/\sqrt{T}$ the MAEs in Figures 11-14 are only a fraction of the aforementioned ones. Moreover, the QS for $c = 1.65$ yields the lowest relative MAE compared to the BT kernel and alternative c values.

Robustness checks for the finite sample properties of the CUSUM are also performed. The above results are robust to $T \leq 300$, $\rho \leq 0.9$ and $\tau = 0.7T$ in (3.9) as well as other kernels and data-dependent bandwidths. The above OLS CUSUM results use a HAC estimator for the sample demeaned residuals. The OLS CUSUM results are also robust to recursive demeaning of residuals i.e. $\widehat{u}^{RD}_t = \widehat{u}_t - \overline{\widehat{u}}_t$ where $\overline{\widehat{u}}_t$ is the recursive mean. Similar results are found for the Recursive Least Squares CUSUM test. Finally, the above results are valid for a single break alternative hypothesis.[3]

Summarizing, the simulation results show that the near-stationarity bound (2.7) can restore the monotone power functions of the CUSUM test and yield HAC estimators which are relatively more efficient both under the null and the change-point alternative.

This section concludes with an empirical illustration of the above method using the money market rate of two Asian economies that went through a financial liberalization period in the 1990s. The data source is the International Financial Statistics and the samples for Korea and Thailand are 8/1979-2/2005 and 1/1977-5/2005, $T = 343$ and $T = 341$ monthly data, respectively. The money

[2] All computations were carried in GAUSS using the random number generator RNDNS from the GAUSS library. The reported simulation results are based on 2000 Monte Carlo replications.

[3] These robustness results are available upon request from the author.

market rates (in logs) in Figures 15 and 16 show a large permanent shift in the late 1990s.[4]

The empirical results of the CUSUM test for HAC estimators with prewhitening using alternative boundaries are summarized in Table 1. Whilst $\widehat{\sigma}_{PW}$ with the 0.97 recoloring rule fails to detect a break, the boundary $1 - c/\sqrt{T}$, for all $c = 1, 1.28, 1.65$, estimates July 1998 and September 1998 as the change-points in the money rates of Korea and Thailand, respectively, at the 5% critical level. These results are supported by the two kernel estimates, the QS and BT, and both sample and recursive demeaning methods of residuals. Moreover the sequential sample segmentation method of Bai (1997) does not provide evidence for multiple change-points. The vertical time lines in Figures 15 and 16 show the location of the estimated breaks which are related to reforms in the money market instruments in Korea and to the banking system in Thailand.

4 Final remarks.

Other HAC estimators to overcome the non-monotone power problem of the CUSUM test are proposed in Altissimo and Corradi (2003) where the long-run variance is based on local mean estimates. This method directly deals with the inconsistency of the mean and variance estimates under the alternative. The results here are complementary since an alternative approach is pursued that does not involve local mean estimates but annihilates the divergence of the HAC estimator (caused by $\widehat{\rho} \to 1$ due to the mean shift) via the near-stationarity boundary condition (2.7). This condition also retains $\sqrt{T}$ consistency of the HAC under the alternative hypothesis of a single break. At the same time this is a simple procedure that restores the monotone power of the CUSUM test as shown by the simulation and empirical results.

References.

Altissimo F. and V. Corradi, 2003. Strong rules for detecting the number of breaks in a time series. Journal of Econometrics, 117, 207-244.

Andrews D.W.K., 1991. Heteroskedasticity and autocorrelation consistent covariance matrix estimation. Econometrica, 59, 817-858.

Andrews D.W.K. and J. Monahan, 1992. An improved heteroskedasticity and autocorrelation consistent covariance matrix estimator. Econometrica, 60, 953-966.

Bai J. 1997, Estimating multiple breaks one at a time. Econometric Theory, 13, 315-352.

Crainiceanu C.M. and T.J. Vogelsang, 2001. Spectral density bandwidth choice: Source of nonmonotonic power for tests of a mean shift in a time series. Working Paper.

[4] It is worth noting that other financial variables of emerging economies exhibit a similar behavior.

Herrndorf N., 1984. A functional central limit theorem for weakly dependent sequences of random variables. Annals of Probability, 12, 141-153.

Kramer W. and W. Ploberger, 1990. The local power of the CUSUM and CUSUM of squares tests. Econometric Theory, 6, 335-347.

Perron P., 1989. The great crash, the oil price shock and the unit root hypothesis. Econometrica, 57, 1361-1401.

Perron P., 1991. A test for changes in a polynomial trend function for a dynamic time series. Working Paper.

Sul D., P.C.B. Phillips and C-Y Choi, 2005. Prewhitening bias in HAC estimation. Oxford Bulletin of Economics and Statistics, 67, 517-546.

Vogelsang T.J., 1999. Sources of nonmonotonic power when testing for a shift in mean of a dynamic time series, Journal of Econometrics, 88, 283-299.

Table 1: CUSUM tests for the money market rate using alternative HAC estimators

		KOREA		THAILAND	
HAC estimator:		**Bartlett Kernel (BT)**	**Quadratic Spectral (QS)**	**Bartlett Kernel (BT)**	**Quadratic Spectral (QS)**
Residual Demeaning:	**Recoloring:**				
Sample	**0.97**	**1.076**	**1.241**	**1.051**	**0.967**
	1-1/$\sqrt{T}$	**1.483 (7/1998)**	**1.785 (7/1998)**	**1.529 (9/1998)**	**1.737 (9/1998)**
	1-1.28/$\sqrt{T}$	**1.594 (7/1998)**	**1.868 (7/1998)**	**1.634 (9/1998)**	**2.193 (9/1998)**
	1-1.65/$\sqrt{T}$	**1.662 (7/1998)**	**1.797 (7/1998)**	**1.728 (9/1998)**	**2.717 (9/1998)**
Recursive	**0.97**	**1.078**	**1.244**	**1.059**	**0.972**
	1-1/$\sqrt{T}$	**1.446 (7/1998)**	**1.771 (7/1998)**	**1.541 (9/1998)**	**1.744 (9/1998)**
	1-1.28/$\sqrt{T}$	**1.580 (7/1998)**	**1.844 (7/1998)**	**1.644 (9/1998)**	**2.203 (9/1998)**
	1-1.65/$\sqrt{T}$	**1.645 (7/1998)**	**1.765 (7/1998)**	**1.734 (9/1998)**	**2.728 (9/1998)**

Figure 1: Power functions of the CUSUM test for QSHAC estimators of OLS residuals of an AR(1) with ρ=0.7, T=100

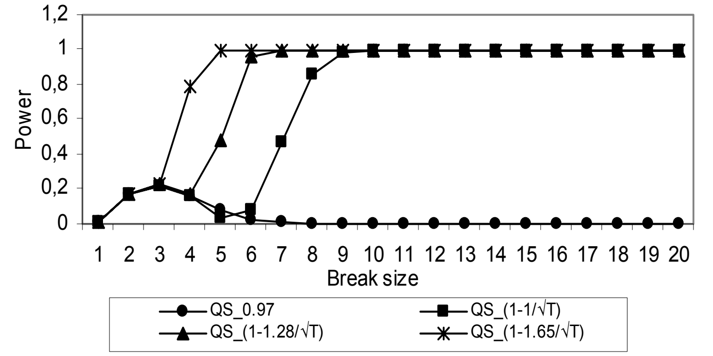

Figure 2: Power functions of the CUSUM test for BTHAC estimators of OLS residuals of an AR(1) with ρ=0.7, T=100

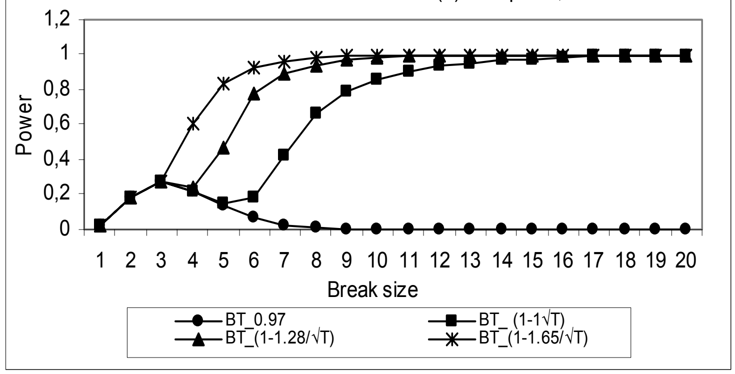

Figure 3: Power functions of the CUSUM test for QSHAC estimators of OLS residuals of an AR(1) with ρ=0.7, T=200

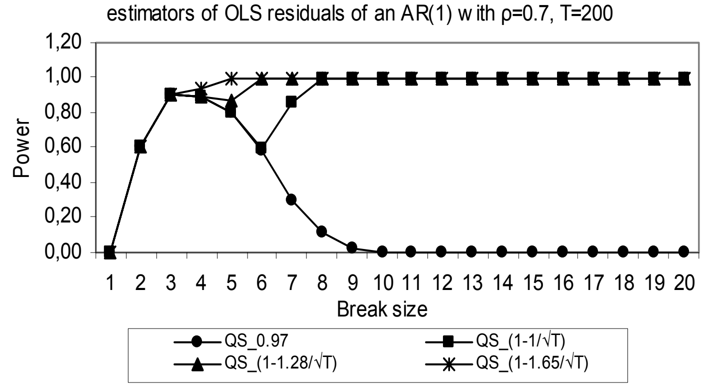

Figure 4: Power functions of the CUSUM test for BTHAC estimators of OLS residuals of an AR(1) with ρ=0.7, T=200

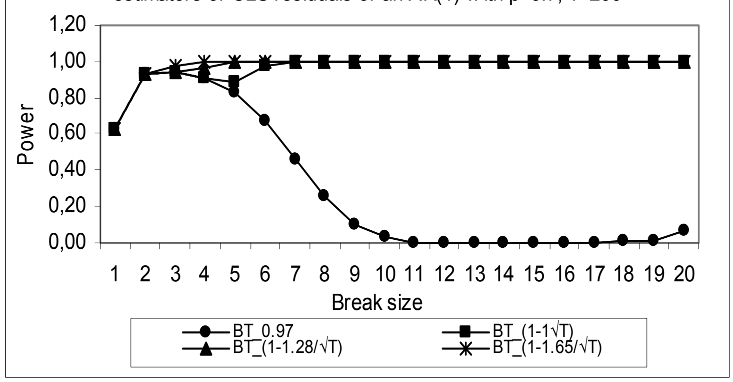

Figure 5: Power functions of the CUSUM test for QSHAC estimators of OLS residuals of an AR(1) with ρ=0.9, T=100

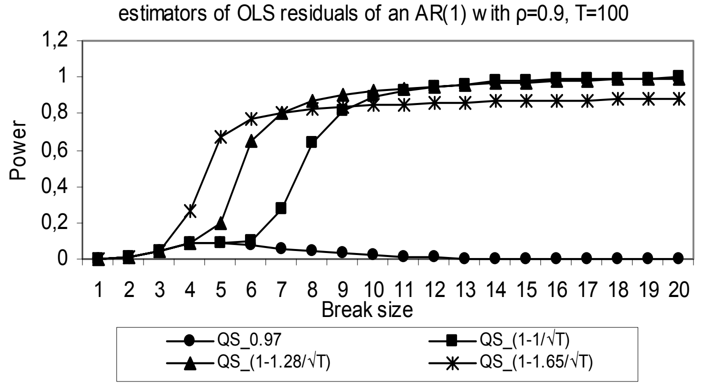

Figure 6: Power functions of the CUSUM test for BTHAC estimators of OLS residuals of an AR(1) with ρ=0.9, T=100

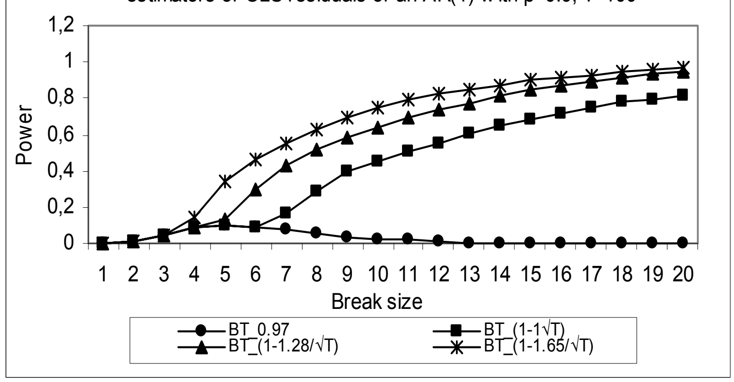

Figure 7: Power functions of the CUSUM test for QSHAC estimators of OLS residuals of an AR(1) with ρ=0.9, T=200

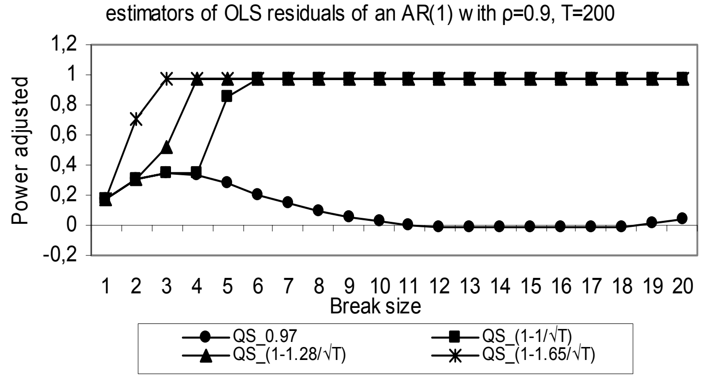

Figure 8: Power functions of the CUSUM test for BTHAC estimators of OLS residuals of an AR(1) with ρ=0.9, T=200

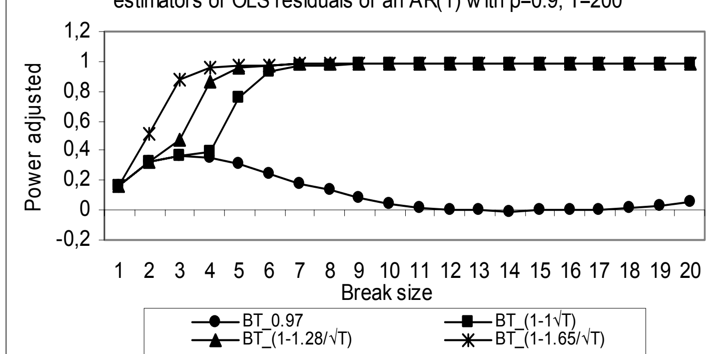

Figure 9: MAE of the QSHAC estimators of OLS residuals of an AR(1), ρ=0.7, T=200

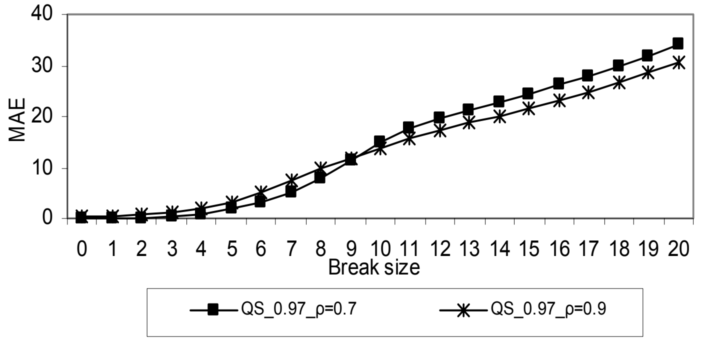

Figure 10: MAE of the BKHAC estimators of OLS residuals of an AR(1), ρ=0.7, T=200

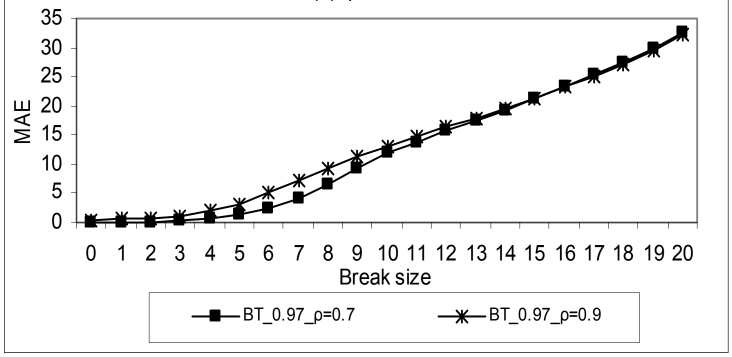

Figure 11: MAE of the QSHAC estimators of OLS residuals of an AR(1), ρ=0.7, T=200

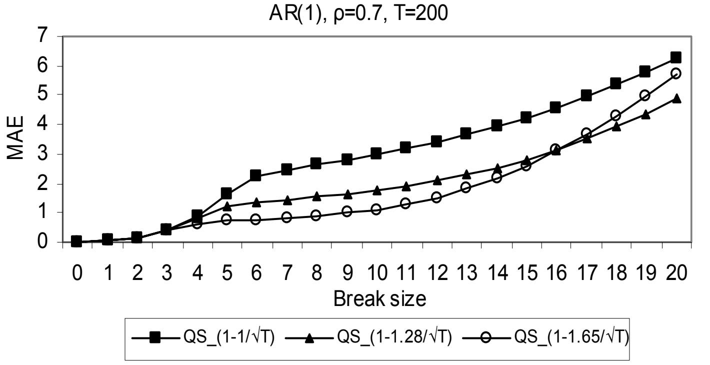

Figure 12: MAE of the BKHAC estimators for OLS residuals of an AR(1) , ρ=0.7, T=200

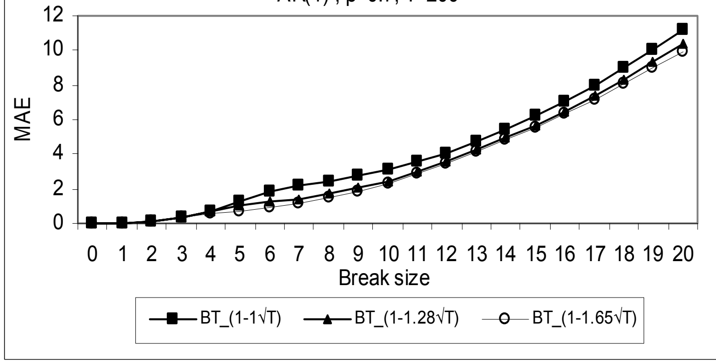

Figure 13: MAE of the QSHAC estimators of OLS residuals of an AR(1), ρ=0.9, T=200

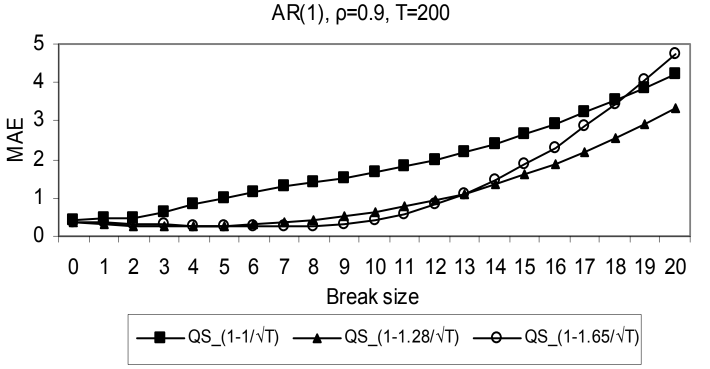

Figure 14: MAE of the BKHAC estimators for OLS residuals of an AR(1) , ρ=0.9, T=200

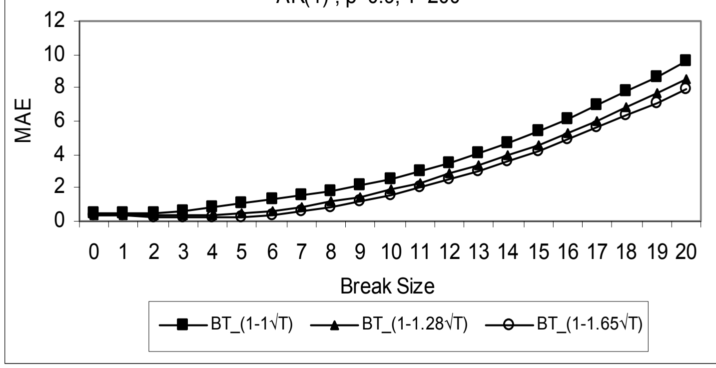

Figure 15: Korea monthly money market rate and estimated break point

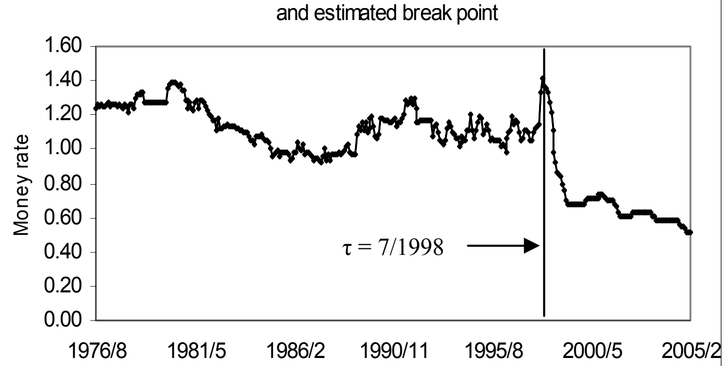

Figure 16: Thailand montly money market rate and estimated break point