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Information Aggregation with Runoff Voting

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Abstract

A majority of independent voters wants to choose the alternative that better matches the state of the world, but may disagree on its identity due to private information. When we have an arbitrary number of alternatives and also sophisticated partisan voters exist in the electorate, the election of the correct alternative is a real challenge. Building upon McLennan (1998) and Barelli et al. (2017) we show that runoff voting –one of the most intuitive electoral systems– achieves asymptotically full information equivalence. That is, when the society is large, it can lead to the election of the correct alternative under fairly general assumptions regarding the information structure and partisans' preferences.

Keywords: runoff voting; information aggregation; partisan voters; Condorcet jury theorem.

JEL classification: D71, D72

1 Introduction

Consider a group of like-minded voters who wish to make a correct decision, but who might disagree on which the correct decision is due to private information. These common value voters wish their pieces of (possibly conflicting) information to be efficiently aggregated. But can they achieve their goal by the means of simple voting procedures?

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Since Condorcet and his celebrated jury theorem we know that when the available alternatives are only two, the society is composed exclusively of such independent voters, and voters vote sincerely –i.e. according to their private information–, then, in a great variety of cases, the plurality rule leads to the right outcome. Unfortunately, as Austen-Smith and Banks (1996) pointed out, this kind of behavior is rarely an equilibrium. Fortunately, as McLennan (1998) demonstrated, even when there is no sincere voting equilibrium, there is always an equilibrium that properly aggregates information. This intuition has been recently shown by Barelli et al. (2017) to also hold in environments with multiple alternatives and general information structures.

Alas, in real societies truth-seeking voters are not alone: Groups of partisan voters –i.e. individuals that support certain candidates for their private reasons– also participate in the elections. This makes the task of the common value voters even more complicated. They need to solve not only the information aggregation problem that they face among them, but also to overcome the effect of partisan voters on the election's outcome. As it should be expected, this is not possible by the means of simple plurality voting even when the group of independents constitutes a majority: as long as there are multiple alternatives and a non-degenerate fraction of partisan voters, there are information structures that lead to a divided majority and to the election of an alternative that is not the best match to the state of the world (see, for instance, Bouton et al., 2016).

This apparent deadlock has attracted the interest of economic theory. Recently, a series of papers identified approval voting as the most efficient among all scoring rules in terms of information aggregation (see, e.g., Goertz and Maniquet, 2011; Bouton and Castanheira, 2012; Ahn and Oliveros, 2016). Among others, approval voting allows majority voters to surpass the obstacles presented by the existence of partisan minorities and leads to full information equivalence –i.e. to the implementation of the correct alternative– with a probability that converges to one as the society grows large.[1] While these results present an elegant solution to the described information aggregation problem, they involve a voting mechanism that is not –until now– widely adopted by collective entities; and rely on partisans behaving in an rather unambiguous manner.

In this paper we turn attention to one of the most commonly used class of electoral systems –the runoff rules– and we investigate their information aggregation properties in general information

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[1]Actually, Bouton and Castanheira (2012) proved that approval voting also achieves coordination equivalence (i.e. all equilibria lead to the same outcome) in the standard divided majority framework with three alternatives (see, e.g., Myerson and Weber, 1993).

environments. According to the system that we focus on, voting takes place in multiple rounds, and in each round the least voted alternative of the last round gets eliminated, until we have a unique winner. Given that this rule involves several stages, it should provide more opportunities to the independent majority to effectively coordinate and elect the correct alternative, than simple plurality rule (Piketty, 2000). Indeed, under very specific assumptions regarding the information structure this has already been proven to be the case. In a divided majority framework with three alternatives, Martinelli (2005) demonstrated that when the information structure is precisely symmetric –and, subsequently, sincere voting is enough to lead large societies to elect the correct alternative–, then sincere voting becomes asymptotically incentive compatible, and hence information is aggregated efficiently.[2]

Notice, though, that sincere voting does not help a group of common value voters aggregate their information efficiently in a great variety of contexts. Consider for instance, that we have three candidates, A, B, and C, three corresponding states of the world (i.e. each candidate is the best one in a different state of the world), and that the partisans supporting C are fewer than the partisans of the other two alternatives. If in the first state independent voters are assigned type A with probability one, in the second state they are assigned type B with probability one and in the third state they are assigned type A with probability 60%, type B with probability 30% and type C with probability 10%, then sincere voting will lead to candidate C being eliminated from the first round whenever she is the correct candidate![3] Can runoff voting properly aggregate information in cases like this one?

In this paper we undertake the task of providing a general answer and we show that, indeed, runoff voting can lead to full information equivalence under general assumptions regarding the information environment. Our analysis is the first to establish that this intuitive rule is superior in terms of information aggregation compared to other applied rules –like the plurality rule– in the presence of multiple partisan groups, and achieves this by employing an alternative methodological approach that makes the study of relevant questions more efficient.

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[2]To be fair Martinelli (2005) does not only show that sincere voting is an equilibrium, but moreover, that in this particular setup, all equilibria of a Poisson voting model converge to sincere voting.

[3]In this literature (e.g. Martinelli, 2005; Bouton and Castanheira, 2012; etc.) independent voters are assigned types randomly, and the distribution of these types depends on the state of the world. That is, the type of an independent voter represents her information regarding the state of the world.

First, we propose to study these questions by modifying the results of McLennan (1998) in a way that accommodates the existence of expressive partisan voters (i.e. voters who do not engage in strategic reasoning). The McLennan (1998) argument applies to games in which all players have the same utility over final outcomes and can be summarized in the following way: any strategy profile that maximizes the players' common utility is, necessarily, an equilibrium profile as well (see, Ahn and Oliveros, 2016, for a recent application of this approach). Evidently, all models that consider information aggregation problems in the presence of partisan voters, cannot directly employ this approach. Indeed, Bouton and Castanheira (2012) and Martinelli (2005) have followed alternative paths in order to produce relevant results. We demonstrate that the McLennan (1998) approach can be applied even when partisan voters exist by studying an "ex-ante" version of the model that we are interested in, and by proving that these two games are strategically equivalent. The "ex-ante" game is such that all voters share common preferences and, after they select their strategies, nature moves and might alter their votes to each of the available alternatives with exogenously given probabilities. This equivalence allows us to argue that if a common strategy followed by all independent voters induces the election of the correct alternative, then a (Bayes-Nash) equilibrium with similar properties should also exist. Utilizing recent findings of Barelli et al. (2017) we tailor such a strategy and establish our first main result.[4]

We then proceed by adding in the model rational partisan voters with arbitrary policy preferences. Extending the analysis in such a direction is important since, to our knowledge, all existing approaches consider partisan voters whose actions are unaffected by the preferences of the rest of the voters –e.g. given their preferences and the voting rule at hand, they have a unique dominant strategy (see, e.g., Goertz and Maniquet, 2011, Bouton and Castanheira, 2012). While this makes sense in some frameworks, it is, arguably, a limitation that we would like to break free from: partisan voters –even if they need not be as sophisticated as common value ones– may adjust their voting behavior in response to their expectations regarding the behavior of the rest of the voters. To accommodate the extra assumptions of all voters being rational (i.e. they best-respond to the beliefs that they hold) and of this rationality being common knowledge (i.e. each player believes

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[4]One should stress here that this is not the first study with a random realization of voters' types –i.e. every player can be either an independent or a partisan one– which establishes its results referring to McLennan (1998). Bouton et al. (2018) also do that, but in their model common-value voters can be pivotal only conditional on all voters being assigned to the common-values' type. Hence, their application of McLennan (1998) is direct and does not require the "ex-ante" version of the game proposed here.

that all other players are rational) we require that partisan voters use rationalizable strategies (in the spirit of Bernheim, 1984 and Pearce, 1984) and do not behave in a naïve sincere manner. Notice that rationalizable strategies are not best responses to any (potentially unreasonable beliefs) but involve a high level of sophisticated reasoning. Indeed, they require that the beliefs that players have are consistent with the assumption that the other players are also rational, and hence they are also best responding to reasonable beliefs. For instance, if a group of voters is expressive or has a dominant strategy (e.g. if common value voters never use a particular action when they get a certain signal), then all other partisan voters take this information into account and properly adjust their strategies. In this richer and more convoluted setup, we still find that the common value majority can reach efficient decisions, establishing our second result.

Our contribution is, hence, threefold. First, we demonstrate that one of the most intuitive and widely used voting systems –runoff voting– can lead to efficient outcomes in the presence of partisan voters under general assumptions regarding the space of alternatives and the information structure. Second, we propose a modelling approach, which can help us explore more questions and voting rules in this interesting environment, in a tractable and intuitive way. Finally, we introduce for the first time instrumental partisan voters, which arguably complicates the analysis but also makes it more relevant to settings of applied interest.

Overall, these findings combine and strengthen the case for democratic decision making: truth-seeking majorities may achieve full information equivalence by the means of simple voting rules, even when: a) there are many alternatives, b) the information structure is general, and c) they have to face sophisticated minorities with complex objectives. To our knowledge this is the first paper that makes this claim, and, probably, this is an observation of independent and wider interest.

In what follows, we first present our benchmark model –i.e. in which partisan voters are essentially parametric– and results (Section 2), we then derive our main results assuming instrumental partisan voters with general preferences (Section 3) and, finally, we briefly discuss the results (Section 4) and we conclude (Section 5).

2 The benchmark model

The original game: We consider a society of $n > 1$ individuals, given by $N = \{1, ..., n\}$, that has

to choose an alternative from the set $X = \{x_1, ..., x_k\}$ through an election that is performed using runoff voting.

Before providing formal definitions, let us briefly describe the voting process. In runoff voting there are $k-1$ different voting rounds, with typical round $r \in \{1, \ldots, k-1\}$. In the first round, each voter gives a vote to an alternative of her choosing –or abstains– and the alternative that receives the fewest votes is eliminated. In the second round, each voter gives a vote to one of the remaining alternatives –or abstains– and the least voted alternative is eliminated, and so on until a unique alternative remains, which is proclaimed the winner of this procedure. All ties are broken with equiprobable draws, though the choice of tie–break rule is completely inconsequential for our results. After each voting round voters are only let know of the alternative that is eliminated and not of the complete vote distribution. This is the only piece of information that the voters need for the result to be obtained. Therefore, our results straightforwardly go through any kind of generalization that provides more details regarding the voting outcome of each round.

The voting process can be described as a dynamic game, whose nodes can be determined by the sequence of alternatives that have been eliminated and the voters' prior actions. However, given that each voter knows only the sequence of eliminated alternatives and her own prior actions, at each round the information set of agent i contains all nodes that have followed a common sequence of eliminated alternatives, from now on called *history*, and the same order of voter i's choices so far. Formally, all histories can be described using a $k-1$ dimension vector h that denotes the sequence of the alternatives that have been eliminated so far in the order that this has happened. For instance, a history of $r < k-1$ voting rounds would be of the form $h = (h_1, \ldots, h_r, 0, \ldots, 0)$, where the 0s denote that no alternative has been eliminated in those rounds yet. Let $\hat{H}$ denote the set of all histories (including those corresponding to terminal nodes where a winner has been chosen). Moreover, the actions of voter i are described by a $k-1$ dimension vector $a^i = (a_1^i, \ldots, a_r^i, 0, \ldots, 0)$, with $\hat{A}^i$ denoting the set of all actions. An action corresponds to the alternative the voter voted in favor of, or to abstention. It is also useful to define the set of histories and the set of actions associated to non–terminal nodes, as these jointly characterize the set of information sets in which a voter makes a choice. Formally, let $H = \{h \in \hat{H} : h_{k-1} = 0\}$ and $A^i = \{a^i \in \hat{A}^i : a_{k-1}^i = 0\}$. Finally, an information set h^i of player i is a pair of vectors h and a^i, i.e. $h^i = (h, a^i)$ and a voter has to take an action in every $h^i \in H^i := H \times A^i$, which are all the information sets of voter i,

except those corresponding to terminal nodes.

For an $h = (h_r)_{r=1}^{k-1} \in H$, let $e(h) = \{l \in \{1, \ldots, k\} : l \neq h_r \text{ for all } r \in \{1, \ldots, k-1\}\}$, i.e. the set of the subscripts of remaining alternatives in history h. This will be called an *election round* and note that multiple histories may correspond to the same election round. Then, $E = \{e(h) : h \in H\}$ denotes the set of all possible election rounds, when looking only at the remaining alternatives. Given these definitions, let A_h and A_e be the set of actions available to (all) the voters in a history $h \in H$ or in an election round $e \in E$ respectively, including abstention. Formally, $A_h = \{x_l \in X : l \in e(h)\} \bigcup \{\varnothing\}$ and $A_e = \{x_l \in X : l \in e\} \bigcup \{\varnothing\}$, where $\varnothing$ denotes abstention. Note that, A_h is common for all voters for a history h, hence the set of available actions at an information set $(h, a^i) \in H^i$ is A_h for all i.

In the beginning of the game, nature chooses privately the state of the world by conducting a random draw from a multinomial distribution with support $\bar{X} = \{X_1, \ldots, X_k\}$ (i.e. the state of the world X_l is drawn with probability $q_l > 0$): when X_c is chosen by nature, it means that the corresponding alternative x_c is the correct one.

Then, each player i is assigned a type t^i. There are $k + s$ possible types and let the set of all types be $T = \{t_1, \ldots, t_k, t_{k+1}, \ldots, t_{k+s}\}$. For $l \in \{1, \ldots, k\}$, voters of type t_l are partisans of alternative x_l and are essentially expressive. That is, they increase their utility by voting for the alternative that corresponds to their type in as many rounds as possible. For $\hat{s} \in \{1, \ldots, s\}$, voters of type $t_{k+\hat{s}}$ are independent and enjoy a positive utility if the correct alternative wins and zero otherwise. These voters and the main focus of our analysis.[5]

Each voter's type is selected privately through an independent random draw and $w(t|X_c)$ denotes the probability that a random voter is of type t given that the state of nature is X_c. We assume the following:

For $l \in \{1, \ldots, k\}$, $w(t_l|X_c) = p_l$ for all $X_c \in \bar{X}$ and

For $\hat{s} \in \{1, \ldots, s\}$, $w(t_{k+\hat{s}}|X_c) = w_c(\hat{s})$ such that $\sum_{\hat{s}=1}^{s} w(t_{k+\hat{s}}|X_c) = 1 - \sum_{l=1}^{k} p_l = p_I$

Throughout this section, we assume that $p_I > \max_{j,l=1,\ldots,k} \{p_j - p_l\}$, which is trivially satisfied whenever $p_I > 1/2$. Moreover, denote by $\widetilde{w_c}(\hat{s}) := \widetilde{w}(t_{k+\hat{s}}|X_c)$ the probability that a random voter is of type

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[5] The analysis can be extended to environments where the preferences of the independent voters are such that in each state of the world the utility they receive from each alternative being elected is different, with the maximum utility being received when the correct alternative is elected.

$k + \hat{s}$ given that the state of nature is X_c and also given that she is independent, i.e. that her type is in $\{t_{k+1}, \ldots, t_{k+s}\}$ and let $\widetilde{w_c} = (\widetilde{w_c}(\hat{s}))_{\hat{s}=1}^s$. Note that, for each $\hat{s} \in \{1, \ldots, s\}$, $\widetilde{w_c}(\hat{s}) = w_c(\hat{s})/p_I$.

We consider a generic type space that is rich enough with respect to the number of alternatives. By rich enough we mean that the different types of independent voters are at least as many as the alternatives, i.e. $s \geqslant k$. The only condition on the distribution of types is that the vectors $\widetilde{w_c}$ are linearly independent, which is a generic property given that $s \geqslant k$. The different types of independent voters essentially capture the different beliefs that independent voters might have about the true state of the world. In most relevant studies (e.g. Martinelli 2005; Bouton and Castanheira 2012) it is assumed that $s = k$. By allowing richer type spaces we introduce the possibility that some independent voters might have better information regarding the state of the world than others. To our knowledge this realistic generalization of the model has been used so far only in environments without partisan voters (e.g. Barelli et al. 2017).

Therefore, let $\Sigma^i := \prod\limits_{(h,a^i) \in H^i} A_h$, then a voter's pure strategy $\sigma_i : T \to \Sigma^i$ is a vector function that determines the choice of voter i in all information sets $h^i = (h, a^i) \in H^i$ (allowing for abstention), given the type of the voter. That is, a pure strategy $\sigma_i = (\sigma_i^{h^i})_{h^i \in H^i}$ means that voter i chooses action σ_i^h in information set h^i. The restriction in the set of alternatives captures the fact that voters cannot vote for an alternative that has already been eliminated and the definition allows a voter to condition her strategy not only on type and sequence of eliminated alternatives, but also on her own prior voting behavior. Then $\sigma = (\sigma_i)_{i \in N}$ is a pure strategy profile and the set of all pure strategy profiles is $\Sigma = \prod\limits_{i \in N} \Sigma^i$.

Note that, the fact that the type space is finite and that each type appears with strictly positive probability allows us to consider without loss of generality that voters choose their strategy before observing their type, i.e. they essentially choose a strategy for each type they might end up having.

Analogously, a voter's mixed strategy $\mu_i : T \to \Delta(\Sigma^i)$ describes the probability with which voter i chooses each of her available pure strategies. That is, for some $\sigma_i = \left(\sigma_i^{h_1^i}, \sigma_i^{h_2^i}, \ldots, \sigma_i^{h_{|H|}^i} \right)$, $\mu_i(\sigma_i|t)$ denotes the probability with which voter i chooses $\sigma_i^{h_1^i}$ in node h_1^i and $\sigma_i^{h_2^i}$ in node h_2^i and $\ldots$ and $\sigma_i^{h_{|H|}^i}$ in node $h_{|H|}^i$, given t, whereas $\mu_i(t) := (\mu_i(\sigma_i|t))_{\sigma_i \in \Sigma_i}$ denotes the vector of probabilities with which each pure strategy is used by voter i given t. Given these, $\mu = (\mu_i)_{i \in N}$ is a (mixed) strategy profile and the set of all (mixed) strategy profiles is $M = \prod\limits_{i \in N} \Delta(\Sigma^i)$.

It is useful here to also define behavior strategies over election rounds $e \in E$, which take into account only the alternatives that participate in a given election and ignore both the order in which the excluded alternatives were eliminated and the behavior of the voter in other election rounds. That is, a behavior (mixed) strategy of player i in election $e \in E$ denoted $b_i^e : T \to \Delta(A_e)$ is a probability distribution over the possible choices in A_e in the given election round, given the voter's type. More generally, defining $B := \prod_{e \in E} \Delta(A_e)$, a behavior strategy of voter i is $b_i = (b_i^e)_{e \in E} \in B$. Finally, $b = (b_i)_{i \in N}$ is a behavior strategy profile and the set of behavior strategy profiles is B^n. A crucial difference with standard mixed strategies is that in behavior strategies the actions in different elections are stochastically independent.

The reason we have defined behavior strategies here is that in some cases it would be simpler to characterize some strategies in the form of behavior strategies, thus essentially considering each election round separately. However, for this to be possible and useful, there should be some sort of "equivalence" between behavior strategies and mixed strategies. Kuhn (1953) defined two strategies as being *equivalent* for some player if they yield the same payoffs to everyone for any strategy of the other players. Given this notion of equivalence, the following result is obtained immediately.

Lemma 1 (Kuhn (1953)) *For an arbitrary behavior strategy $b_i \in B$, consider the mixed strategy μ_i that satisfies $\mu_i(\sigma_i|t) = \prod_{(h,a^i) \in H^i} b_i^{e(h)}(\sigma_i^{h^i}|t)$. Then, b_i is equivalent to μ_i.*[6]

In each $h \in H$, each voter $i \in N$ makes a choice $a_i^h \in A_h$ to cast a vote or abstain according to her strategy and her prior behavior. Let $V_l^h = |\{i \in N : a_i^h = x_l\}|$ be the number of votes that alternative x_l receives in h. Then in the first round, always $h^1 = (0, 0, \ldots, 0)$, the alternative η_1 is eliminated, which is the alternative that receives the fewest votes. If there are multiple such alternatives, then one of them is chosen uniformly at random to be eliminated. That is, for each alternative $x \in \arg\min_{l \in \{1, \ldots, k\}} V_l^{h^1}$, $\eta_1 = x$ with probability $1 / \left| \arg\min_{l \in \{1, \ldots, k\}} V_l^{h^1} \right|$. The second round is then $h^2 = (\eta^1, 0, \ldots, 0)$ and the alternative that is eliminated is η^2. Recursively, the r-th round is $h^r = (\eta^1, \eta^2, \ldots, \eta^{r-1}, 0, \ldots, 0)$ and the alternative that is eliminated is η^r. Finally, when reaching

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[6]In fact, Kuhn (1953) showed that in games of perfect recall the equivalence holds on the other direction as well, i.e. for all mixed strategies there exists some equivalent behavior strategy. However, the information sets over which we have defined behavior strategies here do not allow perfect recall, as they do not take into account both the order in which excluded alternatives were eliminated and the past behavior of the voter.

a terminal node $h^{k-1} = (\eta^1, \eta^2, \ldots, \eta^{k-1})$ the election is completed and $\hat{x}_v = x_{e(h^{k-1})}$ is the society's chosen alternative.

Hence, a mixed strategy profile μ induces a probability distribution over chosen alternatives, $x_v(\mu) : M \to \Delta(X)$, where $x_v = (x_v^l)_{l=1}^k$ and $x_v^l(\mu)$ denotes the probability that alternative x_l is chosen by the society if the voters choose a mixed strategy profile μ. We denote by $W_n(\mu)$ the ex–ante probability that the correct alternative (the one that matches the correct state) is chosen by a population of n agents who play according to a strategy profile μ.[7]

The independent voters' utility $U_I : X \times X \to \mathbb{R}$ depends on the chosen alternative $\hat{x}_v \in X$ and the correct alternative $x_c \in X$, i.e. $U_I(\hat{x}_v, x_c) = \begin{cases} 1 \text{ if } \hat{x}_v = x_c \\ 0 \text{ otherwise} \end{cases}$. For a strategy profile $\mu \in M$, the ex–ante expected utility of a voter i conditional on ending up being independent (i.e $t^i \in \{t_{k+1}, \ldots, t_{k+s}\}$) is equal to $EU_I^i(\mu) = W_n(\mu)$, which is the ex–ante probability of the correct alternative being chosen.

On the contrary, we consider that partisans are essentially agents of their preferred alternative. That is, the utility function of a partisan voter of type j is such that she votes for alternative x_j whenever it is available. On the other hand, in any election round $h \in H$ such that $x_j \notin A_h$ the voter is completely indifferent among all the available alternatives, including abstention. In these cases, we assume that the voter abstains from the election round.[8] Henceforth, m_j denotes the strategy in which *a voter votes for alternative x_j in all $(h, a^i) \in H^i$ such that $x_j \in A_h$ and abstains in all $(h, a^i) \in H^i$ such that $x_j \notin A_h$*. It is apparent that this is a pure strategy with a trivial equivalent behavior strategy. Hence, for a strategy profile $\mu \in M$, the expected utility of a voter i conditional on ending up being a partisan voter of type j is equal to $EU_j^i(\mu) = \begin{cases} 1 \text{ if } \mu_i = m_j \\ 0 \text{ otherwise} \end{cases}$.

Overall, the ex–ante expected utility of a voter i is equal to $EU^i(\mu) = \sum_{j=1}^k p_j EU_j^i(\mu) + p_I EU_I^i(\mu)$. Recall that the voter can condition her strategy on her type, and a strategy profile μ is an equilibrium if it maximizes the expected utility of every given voter when the other voters play according to μ.

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[7]Ex–ante here means that the calculation of this probability is done before the true state of the world is realized.

[8]This assumption will have an effect on the results only quantitatively, as it will affect the fraction of the independent voters in the society that will be needed for a correct choice to be possible.

The modified game: We now define a modified version of the previous game, with the major difference being that all voters are independent, i.e. the set of types in the modified game is $\widetilde{T} = \{t_{k+1}, \ldots, t_{k+s}\}$. This means that, in principle, all of them intend to vote so that the likelihood that the correct alternative is chosen is maximized. We will show that this modified game has, essentially, the same equilibria as the original game. In order to keep the exposition simple, we will redefine only the parts of the game that are substantially different between the two versions.

Consider that all n voters of society N are independent: $t^i \in \widetilde{T} = \{t_{k+1}, \ldots, t_{k+s}\}$ for all $i \in N$. That is, they all care to choose the correct alternative. Each voter's type is once again selected privately through an independent random draw and the probability that a random voter is of type $k + \hat{s}$ given that the state of nature is X_c is now $\widetilde{w_c}(\hat{s}) = w_c(\hat{s})/p_I$, thus equal to the conditional probabilities defined in the original game. Hence, pure strategies $\tilde{\sigma}_i : \widetilde{T} \to \Sigma^i$, mixed strategies $\tilde{\mu}_i : \widetilde{T} \to \Delta(\Sigma^i)$ and behavior strategies $\tilde{b}_i = (\tilde{b}_i^e)_{e \in E}$ for $\tilde{b}_i^e : \widetilde{T} \to \Delta(A_e)$ are all functions of the restricted type space. Strategy profiles are defined as before and Lemma 1 extends directly to the modified game.

After the voters choose their strategies, nature randomly censors some voters (i.e. alters their strategies) as follows: if a voter i has chosen a strategy μ_i then her strategy remains μ_i with probability p_I or her strategy is changed to m_l with probability p_l for $l \in \{1, \ldots, k\}$. The probabilities $p_1, \ldots, p_k$ are the same as the probabilities of different partisan types in the original game, hence assumption $p_I > \max_{j,l=1,\ldots,k} \{p_j - p_l\}$ is retained in the modified game.

Importantly, although some voters might end up mimicking the strategy of partisan voters, the ex–ante expected utility of each voter $i \in N$ in the modified game is equal to $EU_M^i(\mu) = W_n(\mu)$, i.e. all voters seek to maximize the ex–ante probability that the correct alternative is chosen.

A strategy profile (thus, also an equilibrium) of the original game is equivalent to a strategy profile of the modified game, if for all types of independent voters the strategy of each voter in the original game conditional on being independent of a certain type is identical to the strategy of this voter in the modified game conditional on being of the same type. However, note that, a voter's choice has an effect on the outcome of the modified game only if her strategy is not changed by nature, therefore the voters in the modified game maximize their expected utility conditional on not being censored by nature.

Lemma 2 *A strategy profile $\widetilde{\mu}$ is an equilibrium of the modified game if and only if the strategy profile μ^*, such that $\mu_i^*(t_j) = \begin{cases} \widetilde{\mu}_i(t_j), & \text{if } t_j \in \{t_{k+1}, \dots, t_{k+s}\} \\ m_j & , \text{if } t_j \in \{1, \dots, k\} \end{cases}$, is an equilibrium of the original game.*

Lemma 2 ensures that there exists a one–to–one relationship between the equilibria of the original and the modified game. This will be useful for establishing some of our subsequent results, because in the modified game all voters have aligned and truth–seeking preferences, despite the fact that their realized strategies might differ. For this reason, we prove the following two lemmas for the modified game, but it is straightforward to restate them for the original game, with the proofs being essentially identical.

We first prove that in the modified game exists a behavior strategy such that a voter who uses it and whose strategy is not altered by nature is more likely to vote in favor of the correct rather than any other alternative, whenever this alternative has not been eliminated yet (Lemma 3). The result follows mainly from Theorem 2 and Corollary 3 in Barelli et al. (2017) and, in the Appendix, we provide some clarifications on why this is the case.[9]

Lemma 3 *For all voters $i \in N$, there exists a behavior strategy $\hat{b} = (\hat{b}^e)_{e \in E}$ such that for all election rounds $e \in E$ and for all alternatives $x_l \in A_e \backslash \{\varnothing\}$ it holds that $\sum\limits_{\hat{s}=1}^{s} \hat{b}^e(x_l | t_{k+\hat{s}}) \widetilde{w}_l(\hat{s}) > \sum\limits_{\hat{s}=1}^{s} \hat{b}^e(x_j | t_{k+\hat{s}}) \widetilde{w}_l(\hat{s})$ for all $j \neq l$.*

Note that, in the modified game the behavior strategies, thus also the one described in Lemma 3, affect the voters' actual behavior only if they are not censored by nature. It is also clear that an equivalent result holds for the original game, conditioning on strategies of voters who end up being independent.

In fact, this result would be sufficient to ensure that that the argument of McLennan (1998) is also true for runoff elections in a society where all voters are independent and seek the best alternative, or equivalently where nature does not alter any strategy. Essentially, this happens because if all voters were independent and more likely to vote in favor of the correct alternative

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[9]In fact, Barelli et al. (2017) allow both the state space and the type space to be general measurable spaces, keeping the set of alternatives finite. This generalization could also be made here. The signals in Barelli et al. (2017) play the same role as the types in our modified game.

(as they do with the strategy provided in Lemma 3), then as the population increases without bound the vote share of the correct alternative should eventually become the highest in each election round that it is still present. This, in turn, guarantees that it would be selected by a sufficiently large society almost surely. The formal proof of this argument is a simplified version of the proof of the following result (Lemma 4). Yet, this is not enough in the current setup, as the presence of partisans requires that the independent voters not only support the correct alternative more than any other alternative, but that they also overcome the discrepancies arising by the fixed voting behavior of the partisans. However, as proven in the following lemma, a properly tailored strategy can solve this problem.

Lemma 4 *Consider a sequence of elections in the modified game in which the population increases without bound. Then, for each n there exists a strategy profile $\hat{\mu}^n$ in which all voters choose a common strategy b^* such that $W_n(\hat{\mu}^n) \to 1$ as $n \to \infty$.*

Note again that the result could also be proven for the original game, considering a strategy profile such that all voters would vote according to b^* when independent and according to m_l. when partisans of type t_l (for $l = 1, \ldots, k$).

The proposed behavior strategy is essentially a convex combination of two strategies: a) voting in favor of the alternative with the lowest expected fraction of partisans up to the point that its expected vote-share matches the expected vote-share of the second-to-last alternative, and b) voting according to a strategy provided in Lemma 3, which makes voting in favor of the correct alternative more likely than any other. As the population increases, the Strong Law of Large Numbers (SLLN) guarantees that the actual vote shares approach their expected values, which are determined by the probabilities with which nature alters voters' strategies in favor of each alternative and the chosen strategy profile.

This is a strongly positive result for the voters. Namely, as long as they can counterbalance the disadvantage of alternatives that are unlikely to be supported through nature's intervention, they can essentially guarantee the selection of the correct alternative in sufficiently large societies. However, the described strategy profile does not necessarily constitute an equilibrium. Nonetheless, notice that in the modified game voters have common preferences, therefore due to McLennan

(1998) the existence of such a common strategy is sufficient to ensure that there will also be an equilibrium strategy with the same property.[10] Formally,

Lemma 5 *Consider a sequence of elections in the modified game in which the population increases without bound. Then, for each n there exists an equilibrium strategy profile $\tilde{\mu}^n$ such that the sequence $W_n(\tilde{\mu}^n) \to 1$ as $n \to \infty$.*

Yet, if $\tilde{\mu}^n$ is an equilibrium strategy profile of the modified game, then by Lemma 2 we can also characterize a corresponding strategy profile μ^{n*} that would be an equilibrium of the original game. In our main result, we show that in the respective equilibrium of the original game the probability that the correct alternative is elected goes to 1 as the population increases as well. The result follows from the proof of Lemma 2.

Proposition 1 *Consider a sequence of elections in the original game in which the population increases without bound. Then, for each n there exists an equilibrium strategy profile μ^{n*} such that $W_n(\mu^{n*}) \to 1$ as $n \to \infty$.*

3 Sophisticated partisans

Extended Game: In this section, we analyze an extended version of the previous game in which we allow non-independent voters to have more complex objectives and behavior. They are called "sophisticated partisans" since their conduct will now be affected by how they expect other voters to behave (hence, sophisticated), but their preferences are still state-independent (hence, partisans). In what follows, we present only those parts of the model that are different from the previous section. More specifically, consider a society of n individuals, $N = \{1, \ldots, n\}$, who have to choose an alternative from the set $X = \{x_1, \ldots, x_k\}$ using runoff voting. The state space, histories, election rounds and strategies are defined as before and the timing is the same as in the original game. The only difference is on the types.

There are $m + s$ possible types, given by $\hat{T} = \{\hat{t}_1, \ldots, \hat{t}_m, \hat{t}_{m+1}, \ldots, \hat{t}_{m+s}\}$. Voters of types in $\{\hat{t}_{m+1}, \ldots, \hat{t}_{m+s}\}$ are again independent and enjoy positive utility when the correct alternative is

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[10]Quoted by McLennan (1998): "If, along some sequence of voting environments, the probability of a correct decision under any sequence of aggregators goes to one, then the probability of a correct decision under any optimal mixed strategy will also converge to unity."

selected and zero otherwise. On the contrary, voters of type $\hat{t}_l$ have a common utility function U_l that depends on the outcome and describes the order and intensity of their preferences over different election outcomes. Importantly, all voters are outcome-oriented and even voters of the same type are allowed to choose different strategies.

Each voter's type is selected privately through an independent random draw and $\hat{w}(t|X_c)$ denotes the probability that a random voter is of type t given that the state of nature is X_c. We assume the following:

For $l \in \{1, \ldots, m\}$, $\hat{w}(\hat{t}_l|X_c) = \hat{p}_l$ for all $X_c \in \bar{X}$ and

For $\hat{s} \in \{1, \ldots, s\}$, $\hat{w}(\hat{t}_{m+\hat{s}}|X_c) = w_c(\hat{s})$ such that $\sum_{\hat{s}=1}^{s} \hat{w}(\hat{t}_{m+\hat{s}}|X_c) = 1 - \sum_{l=1}^{m} \hat{p}_l = p_I$

We consider $p_I > 1/2$, which is a slightly stronger assumption than the one in the previous section. Under this assumption the results hold for any combination of preferences of the other types and expected shares of each type in the society. Yet, this is a very conservative assumption and given U_l and p_l (for $p = 1, \ldots, l$) the results would hold under less conservative assumptions. The assumption that $s \geqslant k$ and the linear independence of w_c are again assumed to hold.

The behavior of partisan voters is considered to be more sophisticated in this section. More specifically, both partisan and independent voters are restricted to use *rationalizable strategies* (for formal definitions see Bernheim, 1984 and Pearce, 1984). That is, each voter best–responds to some reasonable conjecture regarding what other voters' strategies are and rationality is common knowledge. Rationalizability is considered here at the interim stage, that is each voter chooses a strategy for each potential type, which will be rationalizable conditional on the type realization. On top of that, for independent voters we retain the same level of sophistication as in the original game. That is, we consider independent voters to use strategies that are not only rationalizable, but are also best responses to the strategies used by the remaining players.[11]

Because of symmetry, the set of rationalizable strategies is the same for all voters of the same type, i.e. if some conjecture is reasonable for one voter of some type, then it must be reasonable for all voters of the same type and the best–responses to this conjecture should obviously be the same for all these voters, given that they share the same expected utility function. Let R_l be the set of rationalizable strategies for voters of type $\hat{t}_l$ and $R := R_1 \times \cdots \times R_{m+s}$.

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[11]Observe that, such strategies are always guaranteed to be rationalizable, as they best–respond to some reasonable conjecture about the other players' strategies.

Moreover, due to the finiteness of the current game, if we consider the game's agent normal form representation, in which each type of each voter is considered as a different agent,[12] we can directly adopt from Pearce (1984) the following result, adapted to fit our notation:

Remark 1 (Pearce (1984)) *For each type $\hat{t}_l$, the set of rationalizable strategies R_l is nonempty and contains at least one pure strategy.*

To avoid confusion, recall that in the original model, we have defined strategies to be chosen ex–ante –i.e. before the realization of voters' types– but to depend on the type realization. Hence, each voter has been considered to choose a vector of voting decisions for each possible history and each possible type realization. This is merely a choice that simplifies our formal analysis and, given that types, histories and actions are finite, we could have equivalently defined strategies at the interim stage where voters choose their strategies upon observing their type. The same can be done here. Namely, we consider that each voter chooses ex–ante a type–dependent strategy for all histories such that this strategy is rationalizable conditional on the voter's type realization. Formally, this means that $\mu_i(\hat{t}_l) \in R_l$ for all $\hat{t}_l \in \hat{T}$ and all $i \in N$.

According to the above, a strategy profile $\mu^{E,n}$ is an equilibrium of the Extended Game if $\mu_i^{E,n} \in R$ and $EU_I^i(\mu_i^{E,n}, \mu_{-i}^{E,n}|\hat{t}_l) \geqslant EU_I^i(\mu_i, \mu_{-i}^{E,n}|\hat{t}_l)$ for all $\mu_i \in \Delta(\Sigma^i)$ and $l \in \{m+1, \ldots, m+s\}$ and for all $i \in N$.[13]

Note that, voters can choose any strategy that is rationalizable conditional on their type. There might be partisan voters who vote in favor of some alternative in some history and in favor of another alternative in some other history and it is even possible that two voters choose a different strategy even conditional on being of the same type. In fact, the same type of partisan voters might change voting strategy even in the same history depending on the size of the electorate.

In practice, for a society of size n, let mr^n denote a collection of strategies for all voters that

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[12]Usually, an agent normal form representation would treat each information set of each player as a different agent. This is not needed here, given that a strategy conditional on a voter's type, describes the vector of her voting decisions in all histories.

[13]While this is in general more permissive than the equilibrium notion employed in the previous section, when partisans are merely expressive –as in the original game– the two equilibrium concepts coincide. Hence, the current extension is an unambitious generalization of the benchmark model.

are rationalizable when a voter is of some partisan type. Formally, let

$$mr^n = (mr_1^n, \ldots, mr_n^n) = \left(\left(mr_1^n(\hat{t}_1), \ldots, mr_1^n(\hat{t}_m) \right), \ldots, \left(mr_n^n(\hat{t}_1), \ldots, mr_n^n(\hat{t}_m) \right) \right) =$$

$$= \left(\left(\mu_1(\hat{t}_1), \ldots, \mu_1(\hat{t}_m) \right), \ldots, \left(\mu_n(\hat{t}_1), \ldots, \mu_n(\hat{t}_m) \right) \right)$$

such that $\mu_i(\hat{t}_l) \in R_l$ for all $i \in N$ and $l \in \{1, \ldots, m\}$ and let MR^n be the set that contains all mr^n.

Remark 2 *For each $mr^n \in MR^n$ there is some equilibrium $\mu^{E,n}$ such that the voters' strategies are consistent with mr^n, i.e. $\mu^{E,n}$ is such that $mr_i^n = \left(\mu_i^{E,n}(\hat{t}_1), \ldots, \mu_i^{E,n}(\hat{t}_m) \right)$ for all $i \in N$.*

Remark 2 is apparent by observing that taking mr^n as given makes the behavior of partisan voters essentially parametric and the notion of equilibrium for independent voters becomes the same as in the original game.

In what follows, we show that for any infinite sequence of collections $\{mr^n\}$, there is a sequence of equilibria $\{\mu^{E,n}\}$ such that $W_n(\mu^{E,n}) \to 1$ as $n \to \infty$. For simplicity of notation, we provide the formal analysis for type–symmetric collections of strategies mr^n, i.e. mr^n such that $mr_i^n = mr_j^n$, for all $i, j \in \{1, \ldots, n\}$. Given this, let $\widehat{MR}^n$ be the set of all type–symmetric collections in MR^n. We add no further restriction on the relation between mr^n and $mr^{n'}$ for $n \neq n'$. Note that, the result extends straightforwardly to the remaining cases.

Proposition 2 *Consider any sequence $\{mr^n\}_{n=2}^{\infty}$, with $mr^n \in \widehat{MR}^n$ for all n, and a sequence of elections in the extended game in which n increases without bound. Then, for each n there exists an equilibrium strategy profile $\mu^{E,n}$ such that $mr_i^n = (\mu_i^{E,n}(\hat{t}_1), \ldots, \mu_i^{E,n}(\hat{t}_m))$ for all $i \in N$ and $W_n(\mu^{E,n}) \to 1$ as $n \to \infty$.*

The proof follows similar arguments as in the case of expressive partisans, as we can prove it by constructing appropriate modified games. The main difference with respect to the previous section is that partisan voters may use more complex strategies, than simply voting in favor of a particular alternative or abstaining and, in fact, *they may even change their strategies depending on the preferences of other voters and on the size of the electorate.* Yet, for each mr^n, we can calculate the probability $p_k^{h,n}$ with which a partisan voter votes in favor of alternative k in history h when the size of the electorate is n. The fact that $p_k^{h,n}$ may change depending on the size of the electorate, precludes us from being able to use the SLLN to prove the existence of a sequence

of strategy profiles for which the probability of electing the correct alternative converges to 1. Nevertheless, we can still overcome this problem, by employing a different and more constructive approach that makes use of the Berry–Esseen Theorem. In particular, the Berry–Esseen theorem allows us to bind the exact probabilities of average vote differences in elections within societies of different sizes and then show that these probabilities converge to values that guarantee the survival of the correct alternative in each election round.

This is the main issue we need to address differently. Other than this, we can proceed as before. In a nutshell (formal proof in the Appendix), for each mr^n we can define the respective modified extended game. Each equilibrium of this modified game can be mapped to an equilibrium (in the sense defined above) of the extended game for which partisan voters use the rationalizable strategies described by the modified game. This proof is essentially identical to that of Lemma 2. Subsequently, the behavior strategies of Lemma 3 are defined over histories, rather than election rounds, as the probabilities with which the non–independent voters support each alternative might be different in different histories associated with the same election round. Using these revised strategies and the Berry–Esseen Theorem, we prove for the sequence of modified extended games the existence of a sequence of strategy profiles for which the probability that the correct alternative is elected converges to 1 as the population increases without bounds. This result is the analog of Lemma 4, yet the analysis required is substantially different. After proving this, the result follows immediately. A similar argument as in Lemma 5 guarantees the existence of a sequence of equilibrium strategy profiles for which the probability that the correct alternative is selected converges to 1 in the sequence of modified extended games. Finally, as in Proposition 1, we establish the existence of a sequence of equilibrium strategy profiles of the extended game that guarantees the convergence to 1 of the probability that the society elects the correct alternative.

One should note here that the assumption that partisan voters are allowed to use any rationalizable strategy is not unambiguously superior or inferior compared to assuming that they best-respond to the actual strategies used by other players. On the one hand it proves tractable given the proposed modification of the McLennan (1998) argument, and provides a very strong result regarding the efficiency of runoff voting. Indeed, by studying the modified games we were able to show that independent voters have a way to implement the correct alternative for every rationalizable behavior of partisan voters; and such behaviors are substantially larger in number

compared to the ones that they could adopt according to any standard equilibrium notion. On the other hand, the asymmetries between independent and partisan voters in terms of sophistication calls for extreme caution in the interpretation of our findings: while independent voters seem to be able to neutralize an intuitive class of rational partians, this does not necessarily mean that they can withstand the effect of all kinds of elaborate behaviors. That is, despite the fact that the current approach represents a clear advance with respect to existing models in terms of generality (number of alternatives and information structure) and partisan sophistication, one needs to bear in mind that there are still aspects (e.g. equilibrium uniqueness, alternative partisan behavior) that call for additional investigation.

4 Discussion

The current literature on runoff rules is primarily focused, not on the multi-round version that we consider here, but on the two-round variant (e.g. Martinelli, 2011; Bouton, 2013; Bouton and Gratton, 2015). Given, though, that these papers consider a three-alternative setup –in which the sequential runoff rule coincides with the two-round runoff procedure– their discussions and results are directly relevant to the present study. A first common observation of these studies is that the argument that runoff voting can be efficient simply because sincere behavior of majority voters in the first round will allow these voters to better coordinate around the correct alternative, is not really strong. As Bouton (2013) and Bouton and Gratton (2015) argue, sincere voting in the first round is almost never an equilibrium strategy profile. Moreover, Duvergerian equilibria in which all majority voters discard their information and coordinate behind the same alternative exist quite generally. Finally, as Martinelli (2011) notes, it is far from obvious that runoff voting will achieve "completely successful information aggregation" in general setups. All these observations show that our main result was not something that the current literature took for granted, and hence, the value of our analysis is not only in terms of providing a formal proof, but also in terms of providing novel insights on the asymptotic efficiency of runoff voting in general settings; about which little was known so far.

What is also noteworthy is the fact that unlike previous studies which consider potentially different populations of voters in each round of the voting procedure (e.g. Bouton, 2013), in

our model we assume that the same population of voters participates in all rounds, but they are allowed to abstain instrumentally in every round they wish to. One could argue that there are merits in both modelling approaches –and indeed there are– but what is crucial to add here is that our results go through even if one assumed that in each round each of the players were assigned a new type with a non-degenerate probability. Hence, while we consider the current environment to be suitable for the analysis of the question in hand, we note that the results still hold under alternative and, potentially, equally plausible scenarios.

Finally, we observe that our analysis qualifies, at least partially, to single-round variations of the runoff rule; sometimes referred to as instant runoff, alternative vote or ranked choice voting. In specific, if we are in a standard divided majority framework (i.e. there are three alternatives, A, B, and C; the alternatives that can be a match for the true state of the world are only A or B; and C is supported by a minority of partisan voters who are indifferent between A and B), then one can easily adapt our general arguments and establish that there is always an equilibrium in which full information equivalence is achieved even under such one round procedures.

5   Concluding remarks

Beyond runoff voting –and perhaps more importantly– we have demonstrated for the first time that full information equivalence can be achieved by a majority of independent voters in complex environments with multiple alternatives, sophisticated partisan voters with arbitrary preferences, and general informational assumptions. Indeed, we arrived to this conclusion by focusing on the case in which the society employs a runoff rule for collective decisions. Our modelling approach, though, allows us to study the performance of alternative voting rules in the same context. For instance, it is possible to adapt the current analysis and further establish that approval voting also leads to asymptotic efficiency in the same general environment.

Given that neither the runoff rule nor approval voting admits a unique equilibrium in general settings (e.g. Bouton, 2013 discusses this issue of equilibrium multiplicity of runoff voting and Goertz and Maniquet, 2011 present an example in which approval voting leads to inefficient outcomes), it appears as the natural next step to try to compare the performance of these two rules with respect to alternative criteria. To this end, empirical/experimental approaches could become

valuable, as the main motivating force behind all this literature is to detect mechanisms that enhance efficiency in the presence of disagreements (e.g. private information) and other obstacles (e.g. partisan voters).

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6 Mathematical Appendix

Proof of Lemma 2: Let $\tau_n = (t^1, \ldots, t^n)$ be a realization of types in the original game and let $\mathcal{T}_n$ denote the set of all possible realizations of types in a population of n voters. Given that the draws of types are i.i.d., the probability that this particular realization is drawn is equal to $P(\tau_n) = P(t^1) \times \cdots \times P(t^n)$. Now, let $\phi_n = (f^1, \ldots, f^n)$ be a combined realization of types and nature's draws in the modified game defined as follows: For $j \in \{1, \ldots, k\}$, $f^i = f_j$ denotes a realization for which $t^i \in \widetilde{T}$ and the nature has altered i's strategy to m_j, whereas for $j \in \{k+1, \ldots, k+s\}$, $f^i = f_j$ denotes a realization for which $t^i = t_j$ and the nature has not altered i's strategy. For each voter i, f^i takes values in a set $F = \{f_1, \ldots, f_{k+s}\}$, which apparently has the same cardinality as T. Then, let Φ_n denote the set of all possible realizations ϕ_n. As before, $P(\phi_n) = P(f^1) \times \cdots \times P(f^n)$.

Now, let us define $\Phi_n^{i,j} = \{\phi_n \in \Phi_n : f^i = f_j\}$. Then, we can rewrite the expected utility of voter i in the modified game for a strategy profile $\widetilde{\mu}$ as follows: $EU_M^i(\widetilde{\mu}) = \sum_{j=1}^{k+s} \sum_{\phi_n \in \Phi_n^{i,j}} P(\phi_n) EU_M^i(\widetilde{\mu}|\phi_n)$. But notice that for all $j \in \{1, \ldots, k\}$, a realization in $\phi \in \Phi_n^{i,j}$ is such that the strategy of voter i is changed to m_j, therefore the respective part of the objective function cannot be affected by the voter's choice. Hence, for $\widetilde{\mu}$ to be an equilibrium, it should hold that $\widetilde{\mu}_i(t_j) \in \arg\max_{\mu_i(t_j)} \sum_{\phi_n \in \Phi_n^{i,j}} P(\phi_n) EU_M^i(\mu_i, \widetilde{\mu}_{-1}|\phi_n)$ for each $j \in \{k+1, \ldots, k+s\}$, where $\widetilde{\mu}_{-i}$ means that all players except i play according to $\widetilde{\mu}$.

Similarly, for each $j \in \{1, \ldots, k+s\}$, let us define $\mathcal{T}_n^{i,j} = \{\tau_n \in \mathcal{T}_n : t^i = t_j\}$ to be the sets of all realizations of types in the original game for which voter i is of type j. Recall that, in the original game, $EU^i(\mu) = \sum_{j=1}^{k} p_j EU_j^i(\mu) + p_I EU_I^i(\mu)$, which in the current notation can be rewritten as $EU^i(\mu) = \sum_{j=1}^{k} \sum_{\tau_n' \in \mathcal{T}_n^{i,j}} P(\tau_n') EU_j^i(\widetilde{\mu}|\tau_n') + \sum_{j=k+1}^{k+s} \sum_{\tau_n \in \mathcal{T}_n^{i,j}} P(\tau_n) EU_I^i(\widetilde{\mu}|\tau_n)$. Hence, first, it is apparent that for a strategy μ^* to be an equilibrium of the original game it must be that $\mu_i^*(t_l) = m_l$ for $l \in \{1, \ldots, k\}$. That is, a voter when partisan of type l finds it optimal to play according to m_l. Second, μ^* should also satisfy for each $j \in \{k+1, \ldots, k+s\}$ that $\mu_i^*(t_j) \in \arg\max_{\mu_i(t_j)} \sum_{\tau_n \in \mathcal{T}_n^{i,j}} P(\tau_n) EU_I^i(\mu_i, \mu_{-1}^*|\tau_n)$.

Consider a strategy profile $\widetilde{\mu}$ in the modified game and a respective strategy profile μ^* in the original game such that for all $i \in N$ it holds that $\mu_i^*(t_j) = \begin{cases} \widetilde{\mu}_i(t_j) \text{ , if } t_j \in \{t_{k+1}, \ldots, t_{k+s}\} \\ m_j \quad \text{ , if } t_j \in \{1, \ldots, k\} \end{cases}$.

We will show that a player i does not have a profitable deviation from $\widetilde{\mu}_i$ when all other voters behave according to $\widetilde{\mu}$ in the modified game if and only if she does not have a profitable deviation from μ_i^* when all players behave according to μ^* in the original game. The second branch of the strategy for the original game corresponds to realizations in which the voter is a partisan of some type t_j, for which it is clear that the optimal strategy is m_j. Therefore, it is sufficient to show the equivalence for realizations of types in $\{t_{k+1}, \ldots, t_{k+s}\}$ in both games.

To do that, note that there is a one–to–one function $g : \mathcal{T}_n \to \Phi_n$ that maps to each element (of the finitely many) of $\mathcal{T}_n$ exactly one element of Φ_n and vice versa. Namely, for each $\tau_n = (t^1, \ldots, t^n) \in \mathcal{T}_n$ there exists exactly one $\phi_n = (f^1, \ldots, f^n) \in \Phi_n$ such that for all l it holds that if $t^l = t_j$ then $f^l = f_j$, and vice versa. Moreover, if $g(\tau_n) = \phi_n$ then $P(\tau_n) = P(\phi_n)$. This is because, on one hand the probability of a voter i being a partisan of type $j \in \{1, \ldots, k\}$ in the original game $(t^i = t_j)$ is the same as the probability of the voter's strategy being altered by nature to m_j in the modified game $(f^i = f_j)$, as it is equal to p_j in both cases. On the other hand, the probability of a voter i being an independent voter of type $j \in \{k + 1, \ldots, k + s\}$ in the original game is the same as the probability of the voter being of the same type in the modified game and her strategy not being altered by nature.

Take an arbitrary pair (τ_n, ϕ_n) such that $g(\tau_n) = \phi_n$ and observe that the actual strategies of all players except i conditional on these realizations are necessarily the same in the two games. This is because, each voter with a type in $t^l \in \{t_{k+1}, \ldots, t_{k+s}\}$ in the original game has the same type in the modified game and her strategy is not altered by nature, as $f^l = f_j$ for $j \in \{k + 1, \ldots, k + s\}$ implies that $t^l = t_j$ and nature has not altered her strategy. Whereas, each voter with a type $t^l = t_j \in \{t_1, \ldots, t_k\}$ in the original game can have any type $t^{l'}$ in the modified game, but for sure her strategy is altered to m_j, which means that eventually both players use the same strategy. Moreover, for all $j \in \{k + 1, \ldots, k + s\}$ the optimal strategy of voter i in the original game is such that $\mu_i^*(t_j) = \underset{\mu_i(t_j)}{\arg\max} \sum_{\tau_n \in \mathcal{T}_n^{i,j}} P(\tau_n) EU_I^i(\mu_i, \mu_{-1}^* | \tau_n)$, whereas in the modified game is such that $\widetilde{\mu}_i(t_j) = \underset{\mu_i(t_j)}{\arg\max} \sum_{\phi_n \in \Phi_n^{i,j}} P(\phi_n) EU_M^i(\mu_i, \widetilde{\mu}_{-1} | \phi_n)$. But notice that the equivalence in the eventual strategies of all other voters and the fact that $EU_I^i(\cdot | \tau_n) = EU_M^i(\cdot | \phi_n)$ –as in both cases the voter's objective is to maximize W_n– for all pairs (τ_n, ϕ_n) such that $g(\tau_n) = \phi_n$ implies that $\underset{\mu_i(t_j)}{\arg\max} \sum_{\tau_n \in \mathcal{T}_n^{i,j}} P(\tau_n) EU_I^i(\mu_i, \mu_{-1}^* | \tau_n) = \underset{\mu_i(t_j)}{\arg\max} \sum_{\phi_n \in \Phi_n^{i,j}} P(\phi_n) EU_M^i(\mu_i, \widetilde{\mu}_{-1} | \phi_n)$, for all

24

$t_j \in \{t_{k+1}, \ldots, t_{k+s}\}$, and for μ_{-1}^* and $\widetilde{\mu}_{-1}$ defined as above. Therefore, for all $t_j \in \{t_{k+1}, \ldots, t_{k+s}\}$, if $\mu_i^*(t_j)$ is optimal in the original game then $\widetilde{\mu}_i(t_j) = \mu_i^*(t_j)$ is optimal in the modified game and, vice versa, if $\widetilde{\mu}_i(t_j) = \mu_i^*(t_j)$ is optimal in the modified game then $\mu_i^*(t_j)$ is optimal in the original game, which completes the argument,

Proof of Lemma 3: We use the analysis in Barelli et al. (2017) to show that such a behavior strategy exists for the first election round, call it e_0, in which all alternatives participate. That is, there exists a behavior strategy $\hat{b} = (\hat{b}^e)_{e \in E}$ such that, in election round e_0, $\sum_{\hat{s}=1}^{s} \hat{b}^{e_0}(x_l|t_{k+\hat{s}})\widetilde{w}_l(\hat{s}) >$ $\sum_{\hat{s}=1}^{s} \hat{b}^{e_0}(x_j|t_{k+\hat{s}})\widetilde{w}_l(\hat{s})$ for all alternatives $x_l \in X$ and for all $j \neq l$. If this is the case, consider some other round $e \in E$ and for each $\hat{s} \in \{1, \ldots, s\}$ let $\beta_{\hat{s}}^e = \sum_{x \notin A_e} \hat{b}^{e_0}(x|t_{k+\hat{s}})$, i.e. $\beta_{\hat{s}}^e$ is the sum of probabilities with which a voter would vote during e_0 in favor of alternatives that do not participate in e when following $\hat{b}$. Then for each $x_l \in A_e \backslash \{\varnothing\}$ and for each $\hat{s} \in \{1, \ldots, s\}$ let $\hat{b}^e(x_l|t_{k+\hat{s}}) = \hat{b}^{e_0}(x_l|t_{k+\hat{s}}) + \frac{\beta_{\hat{s}}^e}{|X \backslash A_e|}$, where $|\cdot|$ denotes the cardinality of the set. This is still a probability distribution, as we have just rearranged the probabilities calculated for e_0. Moreover, it keeps the same order in all inequalities obtained for e_0 that are still relevant in e, as we still add the same factors in both sides. Therefore, it is sufficient to show why the result holds for $e_0 \in E$.

Barelli et al. (2017) et al. consider a single round of voting and allow the state space and the signal space (which is the analogous of the type space of our modified game) to be general measurable spaces, with each alternative being strictly preferred than the others in some subset of the state space. In our discrete environment, each alternative is strictly preferred by the voters –by the independent voters in the original game, or by all voters in the modified game– in exactly one state.

In their Theorem 2 the authors provide a condition that guarantees that the probability that the correct alternative does not get the highest vote share goes to 0 as $n \to \infty$. They prove this by constructing a strategy for which the ex–ante probability that a voter votes in favor of the correct alternative is strictly higher than any other alternative. Subsequently, in their Corollary 3 (which follows from their Lemma 2 when the state space and the signal space are discrete) the authors prove that the condition that was required for Theorem 2 holds if two conditions are satisfied: (1) The cardinality of the signal space is weakly larger than the cardinality of the state space and this is weakly larger than the number of alternatives. In our case, the state space has the same

cardinality as the number of alternatives, hence the requirement boils down to $s \geqslant k$, i.e. the type space in the modified game having larger cardinality than the set of alternatives, (2) If the conditional vectors of probabilities of observing some signal given a state are linearly independent. Here, this translates to the vectors $\widetilde{w_c}$ (of conditional probabilities of different types) being linearly independent. In fact, these are the only two conditions we had imposed on the (otherwise generic) type space.

Proof of Lemma 4: We start by proving that as the population increases without bound the correct alternative does not get eliminated in any election round that participates almost surely. Formally, let the correct alternative be x_c and consider an election round $e \in E$ such that $x_c \in A_e$. Then, let $\{z_n^{c,e}\}$ be a sequence of random variables, where $z_n^{c,e}$ takes value 0 if x_c gets eliminated in the election round e when there are n voters and takes value 1 if it does not. Then, we show that $z_n^{c,e} \to 1$ almost surely.

The proof is constructive, as we present explicitly the mixed strategy that guarantees the result. In fact, we construct a behavior strategy that does so and we know by Lemma 1 that there exists a mixed strategy that is equivalent to this behavior strategy.

Let the correct alternative be x_c and consider an election round $e \in E$ such that $x_c \in A_e$ and for which $P_e := \{p_l\}_{l \in e}$ is the set of probabilities that a voter's strategy is changed to m_l for all $l \in e$, p_{last} is the smallest element of P_e, x_{last} is the corresponding alternative and p_{sl} is the second smallest element of P_e. Then, for $\rho = \frac{p_{sl} - p_{last}}{p_I}$ and for all $t_{k+\hat{s}} \in \widetilde{T}$ consider the following behavior strategy:

$$
b^{e*}(x_l | t_{k+\hat{s}}) = \begin{cases} (1 - \rho)\hat{b}^e(x_l | t_{k+\hat{s}}) \text{ if } x_l \neq x_{last} \\ \rho + (1 - \rho)\hat{b}^e(x_l | t_{k+\hat{s}}) \text{ if } x_l = x_{last} \end{cases}
$$

where $\hat{b}^e$ is the behavior strategy obtained in Lemma 3. The assumption $p_I > \max_{j,l=1,\ldots,k} \{p_j - p_l\}$ guarantees that $\rho \in [0, 1)$. Recall that we consider the modified game, hence this will be the actual strategy that the voter will follow only if she is not chosen by nature, which occurs with probability p_I. Otherwise, with probability p_l (for each $l = 1, \ldots, k$) a voter's strategy is changed to m_l, which means that the voter votes in favor of alternative x_l if $x_l \in A_e$ and abstains otherwise.

Consider a sequence of random vectors $\{v_n\}_{n=2}^{\infty}$ where v_n is a vector of length $k + 1$ such that $v_n(l) = 1$ if voter n votes for alternative x_l and $v_n(l) = 0$ otherwise, for $l = 1, \ldots, k$, whereas

$v_n(k+1) = 1$ if voter n abstains and $v_n(k+1) = 0$ otherwise. These random vectors take values whose distribution depends on the election round and the strategies followed by the voters, which we omit from subsequent notation. For a given election round and if all voters use the same strategy (which we consider to be the case here) the value of each of these random vectors depends on the choice of nature, the types and the randomization device that determines which action is actually chosen given a mixed strategy. Note that, ex–ante all these three factors of uncertainty are the same for each voter, therefore v_i are i.i.d. random vectors.

Given this, we can also define a sequence of $(k+1) \times (k+1)$ random matrices $\{dv_n\}_{n=1}^{\infty}$ where $dv_n(j, l) = v_n(j) - v_n(l)$. These random matrices are also i.i.d. and describe the expected difference in votes at the individual level. Finally, recall that $\{z_n^{c,e}\}_{n=1}^{\infty}$ is a sequence of random variables, such that $z_n^{c,e}$ takes value 0 if x_c gets eliminated in the election round e when there are n voters and takes value 1 if it does not. This could be restated as follows: $z_n^{c,e}$ takes value 0 if $\frac{1}{n} \sum_{i=1}^{n} dv_i(c, l) \leq 0$ for all $l \in e$ and takes the value 1 otherwise, as an alternative gets eliminated if it receives the lowest number of votes among all available alternatives.

Note that, $z_n^{c,e}$ are not i.i.d, yet the relation between $z_n^{c,e}$ and dv_n guarantees that if $dv_n(c, l) \xrightarrow{\text{a.s.}} dv_e(c, l)$ for some $l \in e$ and some $dv_e(c, l) > 0$ then $z_n^{c,e} \xrightarrow{\text{a.s.}} 1$. To prove this, consider a sequence $\{dv_n(c, l)\}_{n=1}^{\infty}$ that converges almost surely to some positive number $dv_e(c, l)$, or equivalently $P\left[\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} dv_i(c, l) = dv_e(c, l)\right] = 1$. If a sequence satifies the equality $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} dv_i(c, l) = dv_e(c, l)$, then for all $\epsilon > 0$, exists $n > 0$ such that $dv_e(c, l) - \epsilon < \frac{1}{n} \sum_{i=1}^{n} dv_i(c, l) < dv_e(c, l) + \epsilon$. Hence, given that $dv_e(c, l) > 0$, if we consider $\hat{\epsilon} = dv_e(c, l)/2 > 0$, there is some $\hat{n} = n(\hat{\epsilon})$ such that for all $n > \hat{n}$ holds that $\frac{1}{n} \sum_{i=1}^{n} dv_i(c, l) > 0$. This in turn means that for all $n > \hat{n}$ also holds that $z_n^{c,e} = 1$, hence $\lim_{n\to\infty} z_n^e = 1$. But, $P\left[\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} dv_i(c, l) = dv_e(c, l)\right] = 1$, hence also $P\left[\lim_{n\to\infty} z_n^{c,e} = 1\right] = 1$, i.e. $z_n^{c,e} \xrightarrow{\text{a.s.}} 1$.

Therefore, it suffices to show that (for the given strategy profile) there is some $l \in e$ such that $dv_n(c, l) \xrightarrow{\text{a.s.}} dv_e(c, l)$ and $dv_e(c, l) > 0$. The first part follows immediately from the Strong Law of Large Numbers (SLLN), given that $\{dv_n(c, l)\}_{n=1}^{\infty}$ is a sequence of i.i.d. random variables and $dv_e(c, l)$ is the expected value of each $dv_n(c, l)$. The expression of the expected value depends on which is the correct alternative and for this reason we prove it for all different cases.

First, let $x_c = x_{last}$ and $x_l = x_{sl}$, then

$$dv_e(l, sl) = p_I \left[\rho + (1 - \rho) \sum_{\hat{s}=1}^{s} \hat{b}^e(x_{last}|t_{k+\hat{s}}) \tilde{w}(t_{k+\hat{s}}|X_{last}) \right] + p_{last} - p_I \left[(1 - \rho) \sum_{\hat{s}=1}^{s} \hat{b}^e(x_{sl}|t_{k+\hat{s}}) \tilde{w}(t_{k+\hat{s}}|X_{last}) \right] - p_{sl} =$$

$$= p_I (1 - \rho) \sum_{\hat{s}=1}^{s} \left[\hat{b}^e(x_{last}|t_{k+\hat{s}}) - \hat{b}^e(x_{sl}|t_{k+\hat{s}}) \right] \tilde{w}(t_{k+\hat{s}}|X_{last}) > 0$$

Second, let $x_c = x_{sl}$ and $x_l = x_{last}$, then

$$dv_e(sl, l) = p_I \left[(1 - \rho) \sum_{\hat{s}=1}^{s} \hat{b}^e(x_{sl}|t_{k+\hat{s}}) \tilde{w}(t_{k+\hat{s}}|X_{sl}) \right] + p_{sl} - p_I \left[\rho + (1 - \rho) \sum_{\hat{s}=1}^{s} \hat{b}^e(x_{last}|t_{k+\hat{s}}) \tilde{w}(t_{k+\hat{s}}|X_{sl}) \right] - p_{last} =$$

$$= p_I (1 - \rho) \sum_{\hat{s}=1}^{s} \left[\hat{b}^e(x_{sl}|t_{k+\hat{s}}) - \hat{b}^e(x_{last}|t_{k+\hat{s}}) \right] \tilde{w}(t_{k+\hat{s}}|X_{sl}) > 0$$

Third, let $x_c = x_j \notin \{x_{last}, x_{sl}\}$ and $x_l = x_{sl}$, then

$$dv_e(j, sl) = p_I \left[(1 - \rho) \sum_{\hat{s}=1}^{s} \hat{b}^e(x_j|t_{k+\hat{s}}) \tilde{w}(t_{k+\hat{s}}|X_j) \right] + p_j - p_I \left[(1 - \rho) \sum_{\hat{s}=1}^{s} \hat{b}^e(x_{sl}|t_{k+\hat{s}}) \tilde{w}(t_{k+\hat{s}}|X_j) \right] - p_{sl}$$

$$= (p_j - p_{sl}) + p_I (1 - \rho) \sum_{\hat{s}=1}^{s} \left[\hat{b}^e(x_j|t_{k+\hat{s}}) - \hat{b}^e(x_{sl}|t_{k+\hat{s}}) \right] \tilde{w}(t_{k+\hat{s}}|X_j) > 0$$

where all sums are strictly positive by Lemma 3 and in the last case $p_j \geqslant p_{sl}$ by definition.

Therefore, we have shown that as the population grows the correct alternative is not eliminated in some arbitrary election round almost surely.

We can now proceed to prove that $W_n(\hat{\mu}^n) \to 1$ as $n \to \infty$, where recall that $\hat{\mu}^n$ is a strategy profile in a society of n voters in which all voters choose behavior strategy b^* and $W_n(\hat{\mu}^n)$ is the ex–ante probability that the correct alternative is selected by a population of n voters who vote according to the strategy profile $\hat{\mu}^n$. We denote by $W_n(\hat{\mu}^n|X_l)$ the (interim) probability that the correct alternative is selected conditional on the true state being X_l. Given this, for each n, we have that $W_n(\hat{\mu}^n) = \sum_{l=1}^{k} W_n(\hat{\mu}^n|X_l) q_l$. We prove the result by contradiction.

More specifically, for the result not to hold there must exist an infinite subsequence of elections along which $W_n(\hat{\mu}^n|X_c) \leqslant 1 - \delta < 1$ for some alternative x_c and some $\delta > 0$.[14] For each element of this subsequence there must exist an election round e_n in which x_c is eliminated with probability

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[14] The number of alternatives is finite, hence every infinite subsequence along which the inequality holds must have an infinite sub–subsequence along which the inequality holds for the same alternative.

at least $\delta > 0$ and a history leading to e_n with probability at least $\delta > 0$. These election rounds may differ, but given that both the election rounds and the possible histories are finite, there must exist an infinite sub–subsequence along which x_c is eliminated with probability at least $\delta > 0$ in the same election e.

Overall, for the result not to hold there must exist some alternative x_c, some election round e and some subsequence of elections along which alternative x_c is eliminated in election round e with probability at least $\delta > 0$ when it is the correct alternative. But this means that there is a subsequence along which $z_n^{c,e}$ is equal to 0 with probability at least $\delta > 0$, which completes the contradiction, as we have already shown that for all alternatives $x_c \in X$ and for all election rounds $e \in E$, $z_n^{c,e} \xrightarrow{\text{a.s.}} 1$.

Proof of Lemma 5: The compactness of the (mixed extension of the) strategy space, the continuity of the ex–ante common expected utility of the voters in the modified game on voters' strategies, and the finite number of types and voters guarantee that there exists a strategy profile that maximizes ex–ante the common expected utility function of the players in the modified game. From McLennan(1998) we have that this optimal strategy profile is also an equilibrium strategy profile. Since the optimal strategy profile gives to the voters of the modified game at least as large expected utility as $\hat{\mu}^n$, according to which they all employ b^*, (by the mere fact that it is the optimal one), and since this optimal strategy profile is also an equilibrium one, it follows that, for every fixed n, there exists an equilibrium $\tilde{\mu}^n$ in the modified game such that $W_n(\tilde{\mu}^n) \geqslant W_n(\hat{\mu}^n)$. Given that $W_n(\hat{\mu}^n) \to 1$ as $n \to \infty$, it must be the case that $W_n(\tilde{\mu}^n) \to 1$ as $n \to \infty$ as well.

Proof of Proposition 1: Recall that the expected utility of a voter i in the modified game with n voters who play according to a strategy profile $\tilde{\mu}^n$ is equal to $EU_M^i(\tilde{\mu}^n) = W_n(\tilde{\mu}^n)$, hence $EU_M^i(\tilde{\mu}) \to 1$ as $n \to \infty$ as well and recalling that $EU_M^i(\mu) \in [0,1]$ for all μ and following the notation of Lemma 2 it must also hold that $EU_M^i(\tilde{\mu}^n | \phi_n \in \Phi_n^{i,j}) \to 1$ as $n \to \infty$ for all $j \in \{k+1, \ldots, k+s\}$. Hence, $\sum_{j=k+1}^{k+s} P(\phi_n \in \Phi_n^{i,j}) EU_M^i(\tilde{\mu}^n | \phi_n \in \Phi_n^{i,j}) \to \sum_{j=k+1}^{k+s} P(\phi_n \in \Phi_n^{i,j}) = p_I$ as $n \to \infty$. Moreover, for equilibrium strategy profiles $\tilde{\mu}^n$ and μ^{n*} in the modified and the original game respectively, we showed that $\sum_{j=k+1}^{k+s} P(\phi_n \in \Phi_n^{i,j}) EU_M^i(\tilde{\mu}^n | \phi_n \in \Phi_n^{i,j}) = \sum_{j=k+1}^{k+s} P(\tau_n \in T_n^{i,j}) EU_I^i(\mu^{n*} | \tau_n \in T_n^{i,j})$, where $\sum_{j=k+1}^{k+s} P(\tau_n \in T_n^{i,j}) EU_I^i(\mu^{n*} | \tau_n \in T_n^{i,j}) = p_I EU_I^i(\mu^{n*})$. Therefore, $p_I EU_I^i(\mu^{n*}) \to p_I$, as $n \to \infty$, or equivalently $EU_I^i(\mu^{n*}) \to 1$ as $n \to \infty$, which can be rewritten as

29

$W_n(\mu^{n*}) \to 1$ as $n \to \infty$.

Proof of Proposition 2: We prove the result using a series of lemmas that are analogous to those that led to the proof of Proposition 1. Namely, Lemmas 6, 7 and 8 are the analogs of Lemmas 2, 4 and 5 respectively. Several of those lemmas have proofs which are identical, or essentially identical, to their analogs in the previous section. In order to avoid unnecessary repetitions, we provide a complete proof only for Lemma 7, which is substantially different and for the rest we only highlight the details that are different compared to their analogs. We do the same for the part of the proof that corresponds to Proposition 1, which is also a part of this proof. Finally, Lemma 3 still holds here and is used with a minor modification on the definition of behavior strategies.

We are now ready to proceed to the proof. We prove the result for any sequence $\{mr^n\}_{n=2}^{\infty}$, which means that we need to define one modified game for each mr^n. Namely, let us define the mr^n–*modified extended game* as follows: All voters are independent, i.e. the set of types is $\{\hat{t}_{m+1}, \ldots, \hat{t}_{m+s}\}$ and the probability that a random voter is of type $t_{m+\hat{s}}$ is $\widetilde{w_c}(\hat{s}) = w_c(\hat{s})/p_I$. After voters choose their strategies, nature randomly censors some voters as follows: if a voter i has chosen a strategy μ_i then her strategy remains μ_i with probability p_I or her strategy is changed to $mr_i^n(\hat{t}_l)$ with probability $\hat{p}_l$. The rest of the game is defined just as the modified original game of Section 2.

Lemma 6 *A strategy profile $\widetilde{\mu}^{E,n}$ is an equilibrium of the mr^n–modified extended game if and only if the strategy profile $\mu^{E,n}$, such that $\mu_i^{E,n}(\hat{t}_j) = \begin{cases} \widetilde{\mu}_i^{E,n}(\hat{t}_j), & \text{if } \hat{t}_j \in \{\hat{t}_{m+1}, \ldots, \hat{t}_{m+s}\} \\ mr_i^n(\hat{t}_j) & , \text{if } \hat{t}_j \in \{\hat{t}_1, \ldots, \hat{t}_m\} \end{cases}$, is an equilibrium of the extended game.*

The proof is essentially identical to the proof of Lemma 2, thus it is omitted. The only point that requires some attention has to do with the choices of partisan voters.[15] More specifically, in the original game (hence, in the proof of Lemma 2) the behavior of partisan voters is a solution to the maximization of a state–independent and non–outcome related expected utility function, whereas in the extended game their behavior is state independent and outcome related, but need

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[15]There are also some changes in subscripts, due to the different number of partisan types between the original and the extended game.

only satisfy rationalizability. The latter allows us to assign to each partisan type a specific (possibly mixed) strategy in the extended game, without having to consider possible deviations from this strategy. Hence, we need to consider deviations from the proposed equilibrium strategy only for non–partisan voters of the extended game (exactly as we did in Lemma 2). This difference is taken into account by the notion of equilibrium we use in the extended game, thus also in Lemma 6.

Moving forward, Lemma 3 is also useful here. The only modification we need to do is to consider behavior strategies defined over histories rather than election rounds, as the strategies of non–independent voters may be different in different histories that correspond to the same election round $e \in E$. Nevertheless, we can easily modify the previously used behavior strategy by defining $\widetilde{b} = (\widetilde{b}^h)_{h \in H}$ such that $\widetilde{b}^h = \hat{b}^{e(h)}$ for all $h \in H$, for $\hat{b}$ as defined in Lemma 3, which needs no further modification.

Lemma 7 *Consider any sequence $\{mr^n\}_{n=1}^{\infty}$, with $mr^n \in \widehat{MR}^n$ for all n, and a sequence of elections in the respective mr^n–modified extended games in which n increases without bound. Then, for each n there exists a strategy profile $\hat{\mu}^{E,n}$ in which all voters choose a common strategy b^{n*} such that $W_n(\hat{\mu}^{E,n}) \to 1$ as $n \to \infty$.*

This result is the only one whose proof is essentially different from that of the respective result of the previous section. If we assume for a moment that Lemma 7 holds, then the result follows immediately. Namely, the lemma below follows:

Lemma 8 *Consider any sequence $\{mr^n\}_{n=2}^{\infty}$, with $mr^n \in \widehat{MR}^n$ for all n, and a sequence of elections in the respective mr^n–modified extended games in which n increases without bound. Then, for each n there exists an equilibrium strategy profile $\tilde{\mu}^{E,n}$ such that the sequence $W_n(\tilde{\mu}^{E,n}) \to 1$ as $n \to \infty$.*

The proof is identical to the one of Lemma 5, using the strategy profiles $\hat{\mu}^{E,n}$ instead of $\hat{\mu}^n$ and the behavior strategies b^{n*} instead of b^*. Both are based on the same argument of McLennan (1998). Now, given Lemma 8, Proposition 2 follows, with the proof being essentially identical to that of Proposition 1.[16] Hence, it remains to prove Lemma 7.

Proof of Lemma 7: Similarly to Lemma 4, we show that for an arbitrary correct alternative x_c and an arbitrary history $h \in H$ such that $x_c \in A_h$ the imposed sequence of strategy profiles $\hat{\mu}^{E,n}$

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[16]Up to modifications in subscripts due to the different number of partisan types.

satisfies that the probability that x_c is not eliminated at this history goes to 1 as the population grows. Formally, we define again the sequence of random variables $\{z_n^{c,h}\}$ where $z_n^{c,h}$ takes value 0 if x_c gets eliminated in history h when there are n voters and takes value 1 if it does not. We show that $z_n^{c,h} \xrightarrow{\text{P}} 1$ as $n \to \infty$.

Proving this is sufficient to ensure that $W_n(\hat{\mu}^{E,n}) \to 1$ as $n \to \infty$, for the same reason that it was in Lemma 4. Namely, if we assume for the moment that $z_n^{c,h} \xrightarrow{\text{P}} 1$ as $n \to \infty$, then we show that $W_n(\hat{\mu}^{E,n}) \to 1$ as $n \to \infty$ by contradiction.

First, recall that $\hat{\mu}^{E,n}$ is a strategy profile in a society of n voters in which all voters choose behavior strategy $b^{n*} = (b^{h,n*})_{h \in H}$ and let $W_n(\hat{\mu}^{E,n}|X_l)$ be the (interim) probability that the correct alternative is selected conditional on the true state being X_l. Given this, for each n, we have that $W_n(\hat{\mu}^{E,n}) = \sum\limits_{l=1}^{k} W_n(\hat{\mu}^{E,n}|X_l)q_l$.

Then, if $W_n(\hat{\mu}^{E,n}) \nrightarrow 1$ as $n \to \infty$ there must exist an infinite subsequence of elections along which $W_n(\hat{\mu}^{E,n}|X_c) \leqslant 1 - \delta < 1$ for some alternative x_c and some $\delta > 0$. For each element of this subsequence there must exist a history h_n that is reached with probability at least $\delta > 0$ and in which x_c is eliminated with probability at least $\delta > 0$. By finiteness of H, this subsequence must have an infinite sub–subsequence along which x_c is eliminated with probability at least $\delta > 0$ in the same history h.

Hence, overall there must exist some alternative x_c, some history h and some subsequence of elections along which alternative x_c is eliminated in history h with probability at least $\delta > 0$ when it is the correct alternative. But, this means that there is a subsequence along which $z_n^{c,h}$ is equal to 0 with probability at least $\delta > 0$, which completes the contradiction, as for all alternatives $x_c \in X$ and all $h \in H$, $z_n^{c,h} \xrightarrow{\text{P}} 1$ as $n \to \infty$. Therefore, $z_n^{c,h} \xrightarrow{\text{P}} 1$ as $n \to \infty$ implies $W_n(\hat{\mu}^{E,n}) \to 1$ as $n \to \infty$.

Let us then prove that $z_n^{c,h} \xrightarrow{\text{P}} 1$ as $n \to \infty$. As explained before, the strategy that will be used by the voters needs to be stated at the history level, as the strategies of non–independent voters might be different in different histories that correspond to the same election round $e \in E$. Namely, the strategy $\hat{\mu}_i^{E,n}$ will be based on the behavior strategy $\hat{b} = (\hat{b}^e)_{e \in E}$ whose existence we proved in Lemma 3. As already mentioned, we can easily modify the previously used behavior strategy

to the history level by defining $\widetilde{b} = (\widetilde{b}^h)_{h \in H}$ such that $\widetilde{b}^h = \hat{b}^{e(h)}$ for all $h \in H$, for $\hat{b}$ as defined in Lemma 3.

Moreover, note that each collection of strategies mr^n –to which nature can change the voters actual strategies– induces some probabilities each alternative is voted with, in each history h and for each n. Namely, for each $h \in H$, consider all pairs $\tilde{h} = (h, a)$ and all pure strategies such that $\sigma^{\tilde{h}} = x_k$. The ex–ante probability that a voter $i \in N$, in a society of n voters, will vote for alternative x_k in history h conditional on nature changing her strategy towards $mr_i^n(\hat{t}_l)$ is equal to $P(i \text{ votes } x_k \text{ in } h | \hat{t}_l, n) = \sum_{\tilde{h}} mr_i^n(\sigma^{\tilde{h}} | \hat{t}_l)$. For symmetric collections $mr^n \in \widehat{MR}^n$, we can denote this by $r_k^{h,n}(\hat{t}_l)$, without conditioning also on i. Therefore, $p_k^{h,n} = \sum_{l=1}^{m} r_k^h(\hat{t}_l)\hat{p}_l$ is the ex–ante probability that a voter will vote in favor of alternative k in history h in a society of n voters, conditional on her strategy being affected by nature. Obviously, $p_k^{h,n}$ depends on mr^n.

Now, for a history h, a correct alternative $x_c \in A_h$ and a society consisting of n voters, consider the collection of random matrices $\{dv_{i,n}^{c,h}\}_{i=1}^n$, which describes the difference in votes at the individual level. Given that all voters are ex–ante identical, these are i.i.d. with expected value $\overline{dv_{i,n}^{c,h}}$. Crucially though, this expected value may depend on the size of the society. Therefore we cannot use arguments based on Laws of Large Numbers. However, we can still restate $z_n^{c,h}$ in relation to the sequences $\{dv_{i,n}^{c,h}\}_{i=1}^n$ as follows: $z_n^{c,h}$ takes value 0 if $\frac{1}{n}\sum_{i=1}^{n} dv_{i,n}^{c,h}(c,l) \leq 0$ for all $l \in e(h)$ and takes the value 1 otherwise, as an alternative gets eliminated if it receives the lowest number of votes among all available alternatives. But, given this restatement, the proof of the result boils down to proving that there is a sequence $\{l_n^h\}_{n=1}^\infty$ such that $\lim_{n \to \infty} P\left(\sum_{i=1}^{n} \frac{dv_{i,n}^{c,h}(c,l_n^h)}{n} \leq 0\right) = 0$. We will prove this using the Berry–Esseen Theorem.

Namely, let the correct alternative be x_c and consider a history $h \in H$ such that $x_c \in A_h$. Let $P^{h,n} = \{p_1^{h,n}, \ldots, p_k^{h,n}\}$, where $p_{last}^{h,n}$ is the smallest element in $P^{h,n}$ and $x_{last}^{h,n}$ is the corresponding alternative and $p_{sl}^{h,n}$ is the second smallest element of $P^{h,n}$. Then, for $\rho^{h,n} = \frac{p_{sl}^{h,n} - p_{last}^{h,n}}{p_I}$ consider the following behavior strategy:

$$b^{h,n*}(x_l | \hat{t}_{m+\hat{s}}) = \begin{cases} (1 - \rho^{h,n})\hat{b}^h(x_l | \hat{t}_{m+\hat{s}}) \text{ if } x_l \neq x_{last}^{h,n} \\ \rho^{h,n} + (1 - \rho^{h,n})\hat{b}^h(x_l | \hat{t}_{m+\hat{s}}) \text{ if } x_l = x_{last}^{h,n} \end{cases}$$

Importantly, note that for $\sum_{l=1,\ldots,k} p_l^{h,n} = 1 - p_I$ and given that $p_I > 1/2$ we get that $p_{sl}^{h,n} - p_{last}^{h,n} \leqslant 1 - p_I < 1/2$ for all n. In fact there is some $\delta > 0$ (that does not depend on n) such that $p_I = \frac{1}{2} + \delta$, therefore $(1 - \rho^{h,n}) \geqslant \frac{4\delta}{1+2\delta} > 0$ for all h and all n. Now, given the property of $\hat{b}$ described in Lemma 3 and through identical calculations as the ones we performed in Lemma 4 (to show that the expected values were positive), let us define $\zeta := \left(\frac{1}{2} + \delta\right) \frac{4\delta}{1+2\delta} \gamma > 0$, where $\gamma = \min_{l \in A_h} \sum_{\hat{s}=1}^{s} \left[\hat{b}^{e(h)}(x_c|\hat{t}_{m+\hat{s}}) - \hat{b}^{e(h)}(x_l|\hat{t}_{m+\hat{s}})\right] \hat{w}(\hat{t}_{m+\hat{s}}|X_c) > 0$. Then, we get that for each n there is some alternative $x_l \in A_h$ such that $\overline{dv_{i,n}^{c,h}(c,l)} \geqslant \zeta > 0$, where crucially ζ is bounded strictly away from zero and its value does not depend on n. Let us denote this alternative by l_n^h. This observation is crucial because it guarantees that for each n there is some alternative compared to which the correct alternative receives strictly more votes in expectation and the difference is larger than some lower bound $\epsilon > 0$ that does not depend on the size of the society.

Now, consider the random variable $\widehat{dv_{i,n}^{c,h}}(c,l_n^h) = dv_{i,n}^{c,h}(c,l_n^h) - \overline{dv_{i,n}^{c,h}(c,l_n^h)}$ and the sequence of random variables $\{\widehat{dv_{i,n}^{c,h}}(c,l_n^h)\}_{i=1}^n$. The elements of the sequence are i.i.d. as well, as we have just subtracted from each random variable their common mean. The latter guarantees that the new mean is equal to zero. Moreover, it is apparent that they have strictly positive variance σ^2 that is also bounded away from zero for all n and finite third moment κ bounded above by 1.[17] Therefore, by Berry–Esseen Theorem we know that there is some $C > 0$ such that for all x it holds that $\left| \mathbb{P}\left(\frac{\sum_{i=1}^n \widehat{dv_{i,n}^{c,h}}(c,l_n^h)}{n} \frac{\sqrt{n}}{\sigma} \leqslant x \right) - \Phi(x) \right| \leqslant \frac{C\kappa}{\sigma^3 \sqrt{n}}$, where Φ is the cdf of the standard normal distribution. Then,

$$0 \leqslant \mathbb{P}\left(\frac{\sum_{i=1}^n dv_{i,n}^{c,h}(c,l_n^h) - \overline{dv_{i,n}^{c,h}(c,l_n^h)}}{n} \frac{\sqrt{n}}{\sigma} \leqslant x \right) \leqslant \Phi(x) + \frac{C\kappa}{\sigma^3 \sqrt{n}}, \text{ for all } x \Rightarrow$$

$$0 \leqslant \mathbb{P}\left(\frac{\sum_{i=1}^n dv_{i,n}^{c,h}(c,l_n^h)}{n} \leqslant x \frac{\sigma}{\sqrt{n}} + \overline{dv_{i,n}^{c,h}(c,l_n^h)} \right) \leqslant \Phi(x) + \frac{C\kappa}{\sigma^3 \sqrt{n}}, \text{ for all } x \Rightarrow$$

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[17]The random variable $\widehat{dv_{i,n}^{c,h}}(c,l_n^h)$ has the same variance as $dv_{i,n}^{c,h}(c,l_n^h)$, as they differ by a constant. Yet, the latter takes values in $\{-1,1\}$ by construction and we know that it has a strictly positive mean, which guarantees that the variance is strictly positive, unless of course if $dv_{i,n}^{c,h}(c,l_n^h) = 1$ for all i, in which case the result holds trivially. The fact that $dv_{i,n}^{c,h}(c,l_n^h) \in \{-1,1\}$ is also sufficient to guarantee that $\kappa \leqslant 1$.

$$0 \leqslant \mathbb{P}\left(\frac{\sum\limits_{i=1}^{n} dv_{i,n}^{c,h}(c, l_n^h)}{n} \leqslant 0\right) \leqslant \Phi\left(-\frac{\sqrt{n}}{\sigma}\overline{dv_{i,n}^{c,h}(c, l_n^h)}\right) + \frac{C\kappa}{\sigma^3\sqrt{n}} \Rightarrow$$

$$0 \leqslant \mathbb{P}\left(\frac{\sum\limits_{i=1}^{n} dv_{i,n}^{c,h}(c, l_n^h)}{n} \leqslant 0\right) \leqslant \Phi\left(-\frac{\sqrt{n}}{\sigma}\zeta\right) + \frac{C\kappa}{\sigma^3\sqrt{n}}$$

where the third inequality is obtained by substituting $x = -\frac{\sqrt{n}}{\sigma}\overline{dv_{i,n}^{c,h}(c, l_n^h)}$ and the last inequality holds because $\overline{dv_{i,n}^{c,h}(c, l_n^h)} \geqslant \zeta > 0$.

But, note that since the last inequality holds for all n and $\lim\limits_{n \to \infty}\left[\Phi\left(-\frac{\sqrt{n}}{\sigma}\zeta\right) + \frac{C\kappa}{\sigma^3\sqrt{n}}\right] = 0$, by Squeeze Theorem we get that it must also hold that

$$\lim\limits_{n \to \infty} \mathbb{P}\left(\frac{\sum\limits_{i=1}^{n} dv_{i,n}^{c,h}(c, l_n^h)}{n} \leqslant 0\right) = 0$$

which completes the proof.