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Stability in electoral competition: A case for multiple votes

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Abstract

It is well known that the Hotelling-Downs model generically fails to admit an equilibrium when voting takes place under the plurality rule (Osborne 1993). This paper studies the Hotelling-Downs model considering that each voter is allowed to vote for up to k candidates and demonstrates that an equilibrium exists for a non-degenerate class of distributions of voters' ideal policies - which includes all log-concave distributions - *if and only if* $k \geq 2$. That is, the plurality rule ($k = 1$) is shown to be the unique k-vote rule which generically precludes stability in electoral competition. Regarding the features of k-vote rules' equilibria, first, we show that there is no convergent equilibrium and, then, we fully characterize all divergent equilibria. We study comprehensively the simplest kind of divergent equilibria (two-location ones) and we argue that, apart from existing for quite a general class of distributions when $k \geq 2$, they have further attractive properties - among others, they are robust to free-entry and to candidates' being uncertain about voters' preferences.

Keywords: Hotelling-Downs model; equilibrium; multiple votes.

JEL codes: D70

1 Introduction

Electoral competition consistently attracts the interest both of political scientists and of economists for a very simple reason: it is relevant for shaping political and economic outcomes in every democracy. The standard economic model of electoral competition, though, known as the Hotelling-Downs model of elections (Osborne 1993), provides a negative result: when elections for an office are held under

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the plurality rule among win-motivated candidates, and both *candidacy* and *platform*[1] *selection* are endogenous, an equilibrium fails to exist for almost all distributions of voters' ideal policies. Lack of existence of an equilibrium gives its place to equilibrium multiplicity once we consider the same model with runoff voting (Haan and Volkerink 2001; Brusco et al. 2012; Xefteris 2014). These equilibria have many positive features (for example, they exist for every distribution of voters' ideal policies) but they are also characterized by certain negative ones (for example, they are not robust to candidates being uncertain about voters' preferences).[2] That is, electoral competition in the Hotelling-Downs model under these two popular rules produces *extremely unstable outcomes*: either no equilibrium exists (plurality rule) or equilibria exist but only when candidates have amazingly accurate information regarding voters' preferences (runoff rule).

To our knowledge there are no results regarding electoral competition in the framework of the Hotelling-Downs model under electoral rules other than the two above. It is, hence, not clear whether it is the model's demanding assumptions (namely, the endogeneity of both candidacy and platform choice) that generate instability or it is just an issue related to the characteristics of these two particular rules. In this paper we consider a large family of voting rules which we call k-vote rules. If elections take place under a k-vote rule, each voter is allowed to vote for up to k candidates[3] and the candidate which collects most votes is declared the winner of the elections. These rules take various names in the literature; Cox (1987) names them multiple votes procedures while others (for example, Dellis 2009) call them k-approval voting rules. Our main finding is that *for every k-vote rule, with $k \geq 2$, there exists a non-degenerate class of distributions of voters' ideal policies - which contains every log–concave*[4] *distribution - for which an equilibrium exists.* Hence, the plurality rule is proved to be the unique k-vote rule which precludes stability in electoral competition.

But why is the plurality rule so different compared to other k-vote rules? When voters are allowed to vote for only one candidate, we cannot have in equilibrium three (or more) candidates proposing the same policy. In such a case they would be splitting their constituency (that is, the voters that rank them first) into three equal shares and hence each of them would have incentives to deviate marginally to the left or to the right and get the votes of strictly more than a third of these voters. Moreover, it is impossible that a candidate locates strictly to the left (right) of all other candidates, because by moving to her right (left) she may strictly improve her electoral performance. That is, in equilibrium, exactly two candidates should be making the same most leftist (rightist) policy proposal. If this is the case, though, then the number of their supporters on their left (left semi-constituency)

[1] In the Hotelling-Downs model: a) the policy space is considered to be unidimensional and, in particular, to coincide with the unit interval; and b) a candidate prefers to enter the electoral race if and only if she has a strictly positive probability of being elected (Osborne 1993).

[2] See Matsushima (2007).

[3] A voter may use all the k votes that she has at her disposal or only some of them. The only condition that we impose on voting behavior is that each voter votes for her top-ranked candidate(s); other than that a voter may or may not vote for candidates ranked in lower positions.

[4] One is referred to Bagnoli and Bergstrom (2005) for an account of popular functions that satisfy log-concavity.

should be *precisely identical* to the number of their supporters on their right (right semi-constituency) - otherwise one of the candidates would have incentives to deviate towards the larger semi-constituency. Notice that the unique convergent strategy profile that is compatible with these requirements, is when exactly two candidates enter the race and both locate at the median voter's ideal policy. Of course, this may never be an equilibrium, since any third candidate can get elected with certainty by entering, for example, marginally to the left of the ideal policy of the median voter. Most importantly, in the seminal contribution of Osborne (1993) it is shown that, for a generic distribution of voters' ideal policies, there is no divergent strategy profile that meets these tight conditions. Hence, there is generically no equilibrium in elections under the plurality rule.

When voters have $k \geq 2$ votes, though, $k+1$ candidates may be making the same most leftist (rightist) policy proposal in a divergent strategy profile and still it could be the case that none of them has any incentive to move marginally towards the right or towards the left - even if their left semi-constituency *is not* precisely identical in size to their right semi-constituency. This is so, because, when one does not deviate from the common policy proposal, one is voted by $\frac{k}{k+1}$ of their constituency (which, for $k \geq 2$, is much larger than half of the constituency); while, if their left semi-constituency is *sufficiently similar* (but not necessarily precisely identical) to their right semi-constituency, when one deviates marginally either to the left or to the right, one is voted by nearly half of their initial constituency. Indeed, by deviating marginally to the left (right), a candidate ranks strictly below the other k candidates for all voters with ideal policies to the right (left) of their initial policy proposal and, hence, suffers a severe drop in support. The transition from requiring that the two semi-constituencies are *"precisely identical"* (plurality) to just *"sufficiently similar"* (k-vote rules with $k \geq 2$), gives an indication why equilibrium existence is no longer cut-edge when voters are allowed to vote for more than one candidate.

As far as qualitative features of equilibria are concerned, first, we argue that in equilibrium it is never the case that all active candidates[5] propose the same platform. That is, no k-vote rule admits a convergent equilibrium for any distribution of voters' preferences. Then we fully characterize all divergent equilibria of the game[6] and we exhaustively analyze the simple class of symmetric two-location equilibria (equilibria such that half of the active candidates propose policy y^1 and the rest propose policy y^2). We demonstrate that these equilibria have certain very attractive properties when $k \geq 2$: a) they are robust to free-entry (some potential candidates strategically decide not to enter the electoral race), b) the maximum number of active candidates is independent of the cardinality of the set of potential candidates, c) they are robust to candidates being after multiple offices[7] and, perhaps

[5] An active candidate is someone who declared candidacy and joined the electoral race.

[6] In the paper we focus on a particular case of diverging equilibria. Given that all technical steps that are required for the characterization of every other equilibrium class are very similar to the ones employed for this particular case, we characterize all remaining equilibria in the Appendix.

[7] In M-winner systems (we borrow this terminology from Myerson 1999) each of the M most voted candidates is assigned an office (or gets elected in a M-member committee). Since k-vote rules are often used in such elections (for

more importantly, d) they are robust to candidates' being uncertain about voters' preferences. The first two properties are important because they guarantee that the number of active candidates is *completely endogenous*, the third property extends the scope of the analysis to wider frameworks and the fourth property ensures that our results are relevant for real world elections - equilibria, which exist only when information about voters' preferences is perfect, raise obvious plausibility concerns. We moreover prove that the set of distributions of voters' ideal policies for which such equilibria exist is expanding in k and, in particular, that this set becomes large enough to contain *every log-concave distribution*[8] when k becomes equal to two and *every symmetric distribution* (and every distribution in a neighborhood of each symmetric distribution) when k becomes equal to three. That is, we prove that: a) equilibria exist for a very general class of voters' preference profiles when voters are allowed to vote for more than one candidate and b) by increasing the number of votes that a voter is allowed to cast, we increase the likelihood that electoral competition will reach stability.

Of course, this is not the first paper which studies such k-vote rules: Cox (1987), Dellis (2009), Cahan et al. (2011) and Dellis and Oak (2015) are just some examples of papers which look at electoral competition when voters are allowed to cast more than one vote. In these papers however, either the set of competing candidates is exogenous and candidates are not win-motivated (it is assumed that candidates payoffs are smoothly increasing in their vote-shares) or candidacy is endogenous;[9] but the policy platforms of candidates must coincide with their ideal policies (citizen-candidate models). To the author's knowledge there is no paper which studies equilibrium existence under k-vote rules considering that both candidacy and positioning of win-motivated candidates are endogenous. This is precisely the gap in the literature that this paper aims to fill.

Our analysis can be seen as *a case for multiple votes* since it demonstrates that, by giving the voters the opportunity to vote for more than one candidate, one creates the appropriate circumstances which may lead to stable outcomes. That is, multiple votes procedures are found to be better than the plurality rule in that respect. One should stress here, that other kinds of cases for multiple votes exist in the literature. In the literature of information aggregation, for example, Bouton and Castanheira (2012) recently showed that multiple vote procedures perform better than the plurality rule in that they admit only efficient equilibria, while the plurality rule admits both efficient and inefficient ones. Therefore, our results add to the voices that call for a serious reconsideration of the extent of use of the plurality rule in collective decision making - especially, in the context of representative democracy.

example, in certain Swiss cantons - including Zurich - voters are allowed to vote for as many candidates as the number of the seats of the canton's council), one would care to know whether our stability results apply to cases in which more than one individual is elected or not. As we will argue in the end of the paper, indeed, the identified equilibria qualify for any M-winner system with $M \leq k$.

[8] Log-concave distributions of voters' ideal policies are widely used in electoral competition literature (see, for example, Caplin and Nalebuff 1991).

[9] One is referred to Dutta et al. (2001) for a detailed presentation of the reasons why, for every voting rule, the precise assumption regarding candidacy - endogenous versus exogenous - is a crucial determinant of equilibrium policy outcomes.

In the remainder we present our formal setup (Section 2), we prove that a k-vote rule admits an equilibrium for non-degenerate classes of distributions of voters' ideal policies if and only if $k \geq 2$ and we analyze in depth symmetric two-location equilibria[10] (Section 3), and, finally, we demonstrate that our results are robust to candidates being uncertain about voters' preferences and to candidates being after multiple offices (Section 4).

2 The model

We have a set of win-motivated potential candidates who compete under a k-vote rule for an office. Formally, the set of potential candidates is $N = \mathbb{N}^{+}$. We also have a unit mass of voters, indexed by $\lambda \in [0, 1]$, with standard symmetric Euclidean preferences over $[0, 1]$ that are single-peaked. We denote by $w(\lambda)$ the ideal policy of voter, $\lambda \in [0, 1]$, where $w : [0, 1] \to [0, 1]$ is a strictly increasing bijection of $[0, 1]$ to itself. Hence, the ideal policies of a unit mass of voters are distributed over $[0, 1]$ according to a continuous distribution function $F = w^{-1}$ with a unique median, $m \in (0, 1)$, which is assumed to be common information. Throughout the analysis we use the term admissible to refer to a distribution function that has these properties.

The potential candidates simultaneously choose their strategies in the first stage of the game from the set $[0, 1] \cup \{Out\}$. We are interested only in pure strategies and we denote by $y_i \in [0, 1] \cup \{Out\}$ the choice of candidate $i \in N$ and subsequently by $Y = (y_1, y_2, ...)$, with $Y \in \mathcal{Y} = \prod_{i=1}^{+\infty}([0, 1] \cup \{Out\})$, the potential candidates' strategy profile. We moreover define $A = \{i \in N | y_i \neq Out\}$ with $a = \#A$ (that is, a is the cardinality of the set A). If $i \in A$ then i is called an active candidate or entrant (not just a potential candidate). When considering a particular strategy profile $Y = (y_1, y_2, ...)$ such that a finite number of potential candidates enters the race, we denote by $0 \leq y^1 < y^2 < ... < y^r \leq 1$ the r distinct policy platforms that belong to the strategy profile $Y = (y_1, y_2, ...)$ and by $n(y^j)$ the number of candidates who choose the policy platform y^j. In line with Osborne (1993) we consider that the left semi-constituency of y^j is given by $F(y^j) - F(\frac{y^j+y^{j-1}}{2})$ for $2 \leq j \leq r$ and by $F(y^j)$ for $j = 1$ and that the right semi-constituency of y^j is defined symmetrically. The constituency of y^j is the sum of its left and its right semi-constituencies.

After Y is determined, every voter is assigned a strict ordering of A, $\mathbf{r}(A)$, which will be relevant *only in cases of indifferences.* The expression $i\mathbf{r}(A)j$ means that, according to the strict ordering $\mathbf{r}(A)$, element $i \in A$ ranks higher than $j \in A$ and $\mathbf{r}(A)|_X$ denotes the restriction of $\mathbf{r}(A)$ to $X \subseteq A$. Considering that $\mathbf{R}(A)$ is the set of all strict orderings of the set $A \subseteq N$ and $\mathcal{R} = \cup_{\Gamma \subseteq N}\mathbf{R}(\Gamma)$, we assume that: a) if A is finite, then the fraction of the constituency of y^j that is assigned $\mathbf{r}(A)$ is equal to the fraction of the constituency of y^j that is assigned $\mathbf{r}'(A)$, for every $\mathbf{r}(A), \mathbf{r}'(A) \in \mathbf{R}(A)$ and every

[10] In the end of Section 3 we also discuss possibility of other equilibria and in the Appendix we fully characterize all remaining cases.

$j \in \{1, 2, ..., r\}$,[11] and b) if A is infinite, then the strict ordering assigned to each voter is the one that is compatible with the natural order of the elements of A.[12] All potential candidates know that, after Y is determined, strict orderings of A will be assigned to voters according to the above essentially unbiased procedure and, hence, they take this information into account when they make their strategy choices.

A voter votes for up to $k \in \mathbb{N}^+$ active candidates according to some voting behavior. A voting behavior is understood to be a function which maps ideal policies, potential candidates' strategy profiles and strict orderings of A into subsets of A with cardinality at most equal to k. That is, if a voter with ideal policy w follows voting behavior Ξ when potential candidates' strategy profile is Y and the strict ordering assigned to her is $\mathbf{r}(A)$, then the voter will give exactly one vote to each $i \in \Xi(w, Y, \mathbf{r}(A), k)$, where $\Xi(w, Y, \mathbf{r}(A), k) \subseteq A$ and $\#\Xi(w, Y, \mathbf{r}(A), k) \leq k$, and exactly zero votes to each $i \notin \Xi(w, Y, \mathbf{r}(A), k)$.

Definition 1 ***(Voting behavior)*** *A function $\Xi : [0, 1] \times \mathcal{Y} \times \mathcal{R} \times \mathbb{N}^+ \to \mathcal{N}$, where $\mathcal{N}$ is the set of all subsets of N, is a voting behavior if it satisfies $\Xi(w, Y, \mathbf{r}(A), k) \subseteq A$ and $\#\Xi(w, Y, \mathbf{r}(A), k) \leq k$.*

We allow each voter to follow any voting behavior, Ξ, as long as it is *minimally sincere*. A voter is understood to behave in a minimally sincere way if she votes for her top-ranked candidate(s). This assumption is quite general as: a) it does not require that all k votes are actually used by a voter and b) it allows different voters to behave differently. Moreover, this behavior is in line with voters' behavior in relevant papers, which also study elections in which voters are allowed to vote for more than one candidate. In particular, sincerity notions developed in Brams and Fishburn (1978), Dellis and Oak (2006) and Dellis and Oak (2015) are conceptually compatible with (and they are actually stricter than) minimal sincerity, taking into account of course differences in the contexts of analysis.

We now proceed to introduce formally the notion of minimal sincerity. To do this we first need to define the set of top-ranked active candidates

$$\mathcal{Q}^{top}(w, Y) = \{i \in A \,|\, |y_i - w| \leq |y_j - w|, \forall j \in A\}$$

of a voter with ideal policy $w \in [0, 1]$, when potential candidates use the strategy profile $Y = (y_1, y_2, ...)$.

Definition 2 ***(Minimal sincerity)*** *A voting behavior, Ξ, satisfies minimal sincerity if: a) $\mathcal{Q}^{top}(w, Y) \subseteq \Xi(w, Y, \mathbf{r}(A), k)$ when $\#\mathcal{Q}^{top}(w, Y) \leq k$ and b) $\Xi(w, Y, \mathbf{r}(A), k)$ is composed of the k top elements of $\mathbf{r}(A)|_{\mathcal{Q}^{top}(w,Y)}$ when $\#\mathcal{Q}^{top}(w, Y) > k$, for every admissible w, Y, $\mathbf{r}(A)$ and k.*

[11] This assumption only suggests that voters, when indifferent among a number of candidates, split in a fair manner, and has no other implication on the results.

[12] The specific assumption about the case of an infinite A is only made for the sake of completeness. It is absolutely immaterial as far as our formal analysis is concerned. As we will prove, there are no equilibria with an infinite number of entrants and this result is *completely independent* of how strict orderings are produced when entrants are infinite.

It is straightforward that there are many different voting behaviors that satisfy minimal sincerity ranging from top-voting (a voter votes only for her top-ranked candidates) to full-voting (the voter votes for all the k candidates she likes most). Therefore, by assuming that each voter is minimally sincere, we do not impose that all voters will exhibit the same voting behavior.[13] This suggests that the electorate in this model cannot be fully characterized only by the means of a distribution of ideal policies; one needs a profile of voting behaviors too. Without entering into unnecessary formalities, a profile of voting behaviors is simply a collection of voting behaviors (one for each voter) and we consider that it is common information.[14]

We denote by $v_i(y_1, y_2, ...)$ the vote-mass of potential candidate $i \in N$ given a strategy profile $Y = (y_1, y_2, ...)$, a distribution of ideal policies and a profile of minimally sincere voting behaviors. If $y_i = Out$ then $v_i(y_1, y_2, ...) = 0$ independently of what the other players do. The candidate who receives the largest vote-mass wins. All ties are broken with equiprobable draws. Candidates are purely win-motivated; they maximize the probability of being elected. They moreover strictly prefer the pure strategy Out to any pure strategy which gives them zero election probability and they strictly prefer any strategy which gives them positive election probability to the pure strategy Out.[15]

Given that the set of instrumental players coincides with the set of potential candidates:[16] a) a Nash equilibrium of the described game may be defined only in terms of potential candidates' strategies and b) in what follows we use the terms *player* and *potential candidate* interchangeably. Our game sums up to the following:

Stage 1. Players decide simultaneously strategies from $[0, 1] \cup \{Out\}$ holding perfect information regarding the distribution of ideal policies, F, and the profile of minimally sincere voting behaviors.

Stage 2. A strict ordering of A is assigned to each voter according to the essentially unbiased procedure that we described above.

Stage 3. Each voter votes for up to k active candidates according to a minimally sincere voting behavior.

Stage 4. Vote-masses are computed and players get their payoffs.

[13] This is true for every k except for $k = 1$ (plurality). In that case minimal sincerity coincides with standard sincere voting - one votes for one's top-ranked candidate.

[14] This assumption is without loss of generality because - as it will be clear in the next few paragraphs where we introduce our equilibrium notion - we are interested only in equilibria that are robust to every possible profile of minimally sincere voting behaviors. That is, all our equilibrium analysis continues to hold, even if one assumes that candidates have incomplete information about the profile of minimally sincere voting behaviors.

[15] Alternatively, one could assume that there is a positive cost in declaring candidacy. As it will be evident during the formal analysis, each equilibrium of our game survives such a switch in assumptions if the entry cost is sufficiently small. Hence, we prefer to stick with the original formulation of Osborne (1993). Moreover, notice that the set of strategies of each player is not a convex set and hence existence of an equilibrium in pure strategies cannot be established/ruled-out by the means of standard theorems (for example, Debreu 1952).

[16] A voter in this model is essentially parametric since, given a potential candidates' strategy profile Y, her behavior is fully characterized by her ideal policy, a strict ordering of A and a function Ξ.

We will focus only on Nash equilibria that, given a k-vote rule and a distribution of ideal policies, F, exist for *every possible profile of minimally sincere voting behaviors*. That is, we will describe equilibria whose existence does not hinge on the exact beliefs that candidates hold regarding voters' behavior. We do that because we consider that equilibria which exist only for some minimally sincere voting behaviors do not represent a really robust prediction of the model.

Definition 3 ***(Behaviorally robust equilibrium - BRE)*** *Given a k-vote rule and a distribution of ideal policies, F, a players' strategy profile is a BRE if for every profile of minimally sincere voting behaviors no player has incentives to unilaterally change her strategy.*

Next, we give a formal description of what we call a non-degenerate set of distribution functions. We do this because our aim is to prove that for every $k \geq 2$ an equilibrium exists for a non-degenerate class of voters' preference profiles. Since a class of preference profiles coincides in this model with a set of probability distributions over $[0, 1]$, one needs a proper definition of such a set before stating the results.

Definition 4 ***(Non-degenerateness)*** *A set, S, of admissible distribution functions is non-degenerate if: a) it contains at least two admissible distribution functions, F_1 and F_2, such that $F_1(x) < F_2(x)$ for every $x \in (0, 1)$ and b) it contains every admissible distribution function, F, such that $F(x) \in [F_1(x), F_2(x)]$ for every $x \in [0, 1]$.*

Finally, we comment on why we assume an infinite set of potential candidates. We do that to be able to state our results only as a function of k. One can instead assume that N is finite without adding anything to the intuition that we get from our results. Apart from complicating the conditions, when N is finite and $k \geq \frac{\#N}{2}$, the convergent free-entry equilibria identified by Cox (1987), which are such that all $\#N$ potential candidates enter the race, exist in our game. Obviously, these equilibria raise serious plausibility issues since the number of active candidates has to coincide with the cardinality of the set of potential candidates. That is, the number of active candidates in these convergent equilibria is essentially *exogenous*. They describe a situation in which an extra candidate would get elected with positive probability but there simply is no extra potential candidate to enter. Thus, the assumption $N = \mathbb{N}^+$ is actually an equilibrium refinement tool which helps us focus on equilibria which are free of such concerns.

3 Equilibrium analysis

3.1 Preliminary results

We first present a set of results which describe outcomes that may not be expected in equilibrium. We show that in equilibrium the set of active candidates may not be infinite, that there are generically no equilibria when voting takes place according to the plurality rule and that elections under k-vote rules never admit convergent equilibria.

Lemma 1 *For every $k \in \mathbb{N}^+$, the set of active candidates, A, cannot be infinite in a BRE.*

Proof. This trivially follows from the assumption that each player prefers the pure strategy *Out* to any pure strategy which gives her zero election probability. If infinite players enter and each gets a positive election probability, then it must be the case that $v_i(y_1, y_2, ...) = v$ for every $i \in A$ because candidates who have positive election probability are the ones who tie in the first place. Hence, each player should enjoy the same election probability $p > 0$. But $\sum\limits_{i \in A} p > 1$ when $p > 0$ and A infinite and, hence, A cannot be infinite in any BRE. ■

We note that the substantial implication of this result (namely, that in equilibrium we cannot have an arbitrarily large number of active candidates) is robust to all conceivable voters' behaviors and it is not specific to the particularities of the current formulation. Independently of how voters vote, when infinite candidates enter the race, only a subset of them will enjoy an election probability larger than any fixed $c > 0$. That is, considering an arbitrarily small entry cost and that candidates enter if and only if their election probability is larger than this arbitrarily small entry cost, would be enough to eliminate the possibility of equilibria with arbitrarily many active candidates for every possible set of assumptions regarding voters' behavior.

Given that our infinite-player game is symmetric, it directly follows that if an equilibrium with $a \in \mathbb{N}^+$ active candidates exists, then infinitely many similar equilibria with $a \in \mathbb{N}^+$ active candidates should exist too, which would only differ in the identities of the active candidates and not in the strategies that the active candidates employ. If we assume that in an equilibrium with $a \in \mathbb{N}^+$ active candidates we have that $0 \leq y_1 \leq y_2 ... \leq y_a \leq 1$ and that $y_i = Out$ for every $i \geq a + 1$ then we can significantly reduce the complexity of exposition of the results that follow without any loss of generality.

Proposition 1 *If $k = 1$ then there is no BRE for almost every F (Osborne 1993).*

Proof. The proof of this result is a simple combination of our lemma 1 and Osborne (1993). ■

We now turn attention to k-vote rules with $k \geq 1$ and we first investigate possibility of a convergent equilibrium.

Lemma 2 *If $k \geq 1$ there is no BRE such that all active candidates offer the same platform.*

Proof. By lemma 1 we know that in every BRE a is finite. Consider that $(\hat{y}_1, \hat{y}_2, ...\hat{y}_a, \hat{y}_{a+1}, ...)$ is a BRE such that $\hat{y}_1 = \hat{y}_2 = ... = \hat{y}_a = \hat{y} \in [0, 1]$ and $\hat{y}_{a+1} = \hat{y}_{a+2} = ... = Out$. Then if player

$a+1$ deviates from *Out* to $\hat{y} \in [0,1]$ she will get an election probability of $\frac{1}{a+1} > 0$. That is, she is strictly better off by entering at $\hat{y}$ compared to staying *Out* and, hence, the strategy profile $(\hat{y}_1, \hat{y}_2, ...\hat{y}_a, \hat{y}_{a+1}, ...)$ cannot be a BRE. ■

This is a result that we know from Cox (1987); k-vote rules with $k \geq 1$ can give convergent equilibria if and only if the cardinality of the set of potential candidates is small enough compared to k (the exact condition is $k \geq \frac{\#N}{2}$). As it is evident, when the set of potential candidates is infinite no convergent equilibrium exists for any k-vote rule.

3.2 Symmetric two-location equilibria

In what follows we investigate the possibility of divergent equilibria. In most of our analysis we focus on symmetric two-location equilibria (half active candidates locate at one point and the other half at some other point; $n(y^1) = n(y^2) = \frac{a}{2}$) and we prove that every k-vote rule with $k \geq 2$ may support an equilibrium of this sort for a non-degenerate class of voters' preferences. The next lemma provides an upper and a lower bound for the cardinality of the set of active candidates in such an equilibrium.

Lemma 3 *Every symmetric two-location BRE must be such that: a)* $\frac{y^1+y^2}{2} = m$ *and b)* $a \in [2k+2, 4k]$.

Proof. The proof of the first part of the lemma is straightforward: if all voters never vote for a candidate whom they like strictly less than some other (top-voting), then the only way in which a profile with $n(y^1) = n(y^2) = \frac{a}{2}$ may be a Nash equilibrium is when $\frac{y^1+y^2}{2} = m$. Since a BRE is defined as such only if it is robust to every profile of minimally sincere voting behaviors (including the one considered here), it follows that in every symmetric two-location BRE it must be the case that $\frac{y^1+y^2}{2} = m$. As far as the second part of the proposition is concerned: if in a BRE we have $a \geq 2k$ and half candidates are located at the same position y^1 to the left of the median voter and the other half are located equidistantly to the right of the median voter, then for every profile of minimally sincere voting behaviors it is true that the vote-mass of each $i \in A$ is identical to $\frac{k}{a}$. Hence, it must be the case that $F(y^1) \leq \frac{k}{a}$ because if this were not true, a player could deviate from *Out* to a location to the left and arbitrarily close to y^1, get a vote-mass strictly larger than $\frac{k}{a}$ and win with certainty. Moreover, it should hold that $\frac{1}{2} - F(y^1) \leq \frac{k}{a}$ because otherwise a player could deviate from *Out* to a location to the right and arbitrarily close to y^1 and get a vote-mass strictly larger than $\frac{k}{a}$, and hence win with certainty. Combination of these two inequalities gives $a \leq 4k$. Assume now that $a = 2k$. Then, if all voters vote for all the k candidates they like best (full-voting), one of the k candidates located at y^1 can deviate to $y^1 + \varepsilon$ and increase her vote-mass while the vote-mass of each of the $k-1$ candidates who remain at y^1 is unchanged and the vote-mass of each of k candidates who are located at $2m - y^1$ decreases by $[F(\frac{\varepsilon+2m}{2}) - \frac{1}{2}]/k > 0$. That is, this is a profitable deviation since it leads to a certain election. Similar arguments rule out all cases with $a \leq 2k$. Hence, in a BRE it should be the case that $a > 2k$ or else that $a \geq 2k + 2$. ∎

The fact that such divergent equilibria allow for at most $4k$ active candidates is very important as it establishes not only that such equilibria rule out free-entry but, perhaps more importantly, that the maximum number of active candidates in such equilibria *does not relate to the cardinality of the set of potential candidates.* In contrast, the maximum number of active candidates of the divergent and no-free-entry equilibria of the standard runoff rule (Brusco et al. 2012) is essentially equal to the cardinality of the set of potential candidates.

Next, we fully characterize all symmetric two-location equilibria.

Proposition 2 *When the ideal policies of the society are distributed according to F and voting takes place according to a k-vote rule, a symmetric two-location BRE with $\frac{a}{2}$ active candidates at y^1 and $\frac{a}{2}$ active candidates at y^2 exists if and only if: A) $\frac{y^1+y^2}{2} = m$, B) $\max\{F(y^1), 1 - F(2m - y^1)\} \leq \frac{k}{a}$ and C) for every $y \in (y^1, y^2)$ we have $2k + 2 \leq a < \frac{2k \max\{F(\frac{y^1+y}{2}), 1-F(\frac{2m-y^1+y}{2})\}}{F(\frac{2m-y^1+y}{2})-F(\frac{y^1+y}{2})}$.*

Proof. We first deal with the *if* direction; that is, we first show that the above conditions are sufficient for the existence of a BRE. To this end we assume that $\frac{a}{2}$ active candidates locate at y^1 and $\frac{a}{2}$ active candidates locate at y^2, and that all three conditions of the above propositions hold. If a player deviates from *Out* to $y < y^1$ (or to $y > y^2$) she gets a vote-mass strictly less than $F(y^1)$ $(1 - F(2m - y^1))$ due to minimal sincerity and condition C ($2k + 2 \leq a$) and hence she loses with certainty because each active candidate at y^2 (y^1) gets a vote-mass at least equal to $\frac{k}{a}$ due to minimal sincerity and condition C ($2k + 2 \leq a$), which is larger or equal to $F(y^1)$ $(1 - F(2m - y^1))$ due to condition B. If a player deviates from *Out* to $y = y^1$ (or to $y = y^2$) she gets a vote-mass equal to $\frac{k}{a+2}$ due to minimal sincerity and condition C ($2k + 2 \leq a$) and hence she loses with certainty because each active candidate at y^2 (y^1) gets a vote-mass equal to $\frac{k}{a}$ due to minimal sincerity and condition C ($2k + 2 \leq a$). If a player deviates from *Out* to $y \in (y^1, y^2)$, she gets a vote-mass equal to $F(\frac{2m-y^1+y}{2}) - F(\frac{y^1+y}{2})$ due to minimal sincerity and condition C ($2k+2 \leq a$) and hence she loses with certainty because: a) if $1 - F(\frac{2m-y^1+y}{2}) \geq F(\frac{y^1+y}{2})$, then each active candidate at y^2 gets a vote-mass at least equal to $\frac{[1-F(\frac{2m-y^1+y}{2})]2k}{a}$ due to minimal sincerity and condition C ($2k + 2 \leq a$), which is strictly larger than $F(\frac{2m-y^1+y}{2}) - F(\frac{y^1+y}{2})$ due to condition C and b) if $1 - F(\frac{2m-y^1+y}{2}) < F(\frac{y^1+y}{2})$, then each active candidate at y^1 gets a vote-mass at least equal to $\frac{F(\frac{y^1+y}{2})2k}{a}$ due to minimal sincerity and condition C ($2k + 2 \leq a$), which is strictly larger than $F(\frac{2m-y^1+y}{2}) - F(\frac{y^1+y}{2})$ due to condition C. Hence, if all the three conditions hold, then nobody has incentives to deviate from *Out* to any $y \in [0, 1]$. Similar arguments rule out deviation of any of the a active candidates to other locations: if an active candidate deviates from y^1 (y^2) to some other y, her vote-mass will coincide with the vote-mass of a player who deviates from *Out* to the same y - which we have already computed - and it can be shown to be strictly smaller than the vote-mass of some other active candidate with arguments similar to the ones above. Finally, a deviation of an active candidate to *Out* is straightforwardly unprofitable since, in every profile that has the described characteristics, every active candidate has a positive election probability.

We now turn attention to the *only if* direction; that is, we now aim to establish that the described three conditions are necessary for a symmetric two-location BRE to exist. From lemma 3 we know that condition A and the first part of condition C ($2k+2 \leq a$) must hold in every symmetric two-location BRE and they are hence necessary conditions. Assume that there is a symmetric two-location BRE such that condition B does not hold - consider, without loss of generality, that $F(y^1) > \frac{k}{a}$. If all voters vote for all the k candidates they like best (full-voting), then a player may deviate from *Out* to $y < y^1$ but arbitrarily close to it and get a vote-mass strictly larger than $\frac{k}{a}$ and win with certainty. This is so because condition A and the first part of condition C ($2k+2 \leq a$) - which are already proved to be conditions that must hold in a BRE - along with full-voting, suggest that every other active candidate will get a vote-mass at most as large as $\frac{k}{a}$. Hence, if condition B does not hold then we are not in a BRE and it is therefore a necessary condition too. Finally, consider that there is a BRE such that the second part of condition C does not hold. That is, there exists $y \in (y^1, y^2)$ such that either $a \geq \frac{2kF(\frac{y^1+y}{2})}{F(\frac{2m-y^1+y}{2})-F(\frac{y^1+y}{2})}$ and $1 - F(\frac{2m-y^1+y}{2}) < F(\frac{y^1+y}{2})$ or $a \geq \frac{2k[1-F(\frac{2m-y^1+y}{2})]}{F(\frac{2m-y^1+y}{2})-F(\frac{y^1+y}{2})}$ and $1 - F(\frac{2m-y^1+y}{2}) \geq F(\frac{y^1+y}{2})$. In such a case a player may deviate from *Out* to this y and win with positive probability. This is so because if, for example, this y is such that $a \geq \frac{2kF(\frac{y^1+y}{2})}{F(\frac{2m-y^1+y}{2})-F(\frac{y^1+y}{2})}$ and $1 - F(\frac{2m-y^1+y}{2}) < F(\frac{y^1+y}{2})$, then a player who deviates from *Out* to this y gets a vote-mass equal to $F(\frac{2m-y^1+y}{2}) - F(\frac{y^1+y}{2})$ (due to minimal sincerity and the first part of condition C), while an active candidate located at y^1 gets a vote-mass equal to $\frac{F(\frac{y^1+y}{2})2k}{a}$ and an active candidate located at y^2 gets a vote-mass equal to $\frac{[1-F(\frac{2m-y^1+y}{2})]2k}{a}$ when no voter ever votes for a candidate whom she likes strictly less than some other (top-voting); each of these vote-masses are at most as large as $F(\frac{2m-y^1+y}{2}) - F(\frac{y^1+y}{2})$ when $a \geq \frac{2kF(\frac{y^1+y}{2})}{F(\frac{2m-y^1+y}{2})-F(\frac{y^1+y}{2})}$ and $1 - F(\frac{2m-y^1+y}{2}) < F(\frac{y^1+y}{2})$. Therefore, if the second part of condition C does not hold, there exist profiles of minimally sincere voting behaviors for which a player has incentives to deviate from *Out* to some $y \in (y^1, y^2)$ and, thus, we cannot be in a BRE. This proves that the second part of condition C is also necessary for the existence of a symmetric two-location BRE. ■

Next we state a direct corollary of the above proposition which will help us with our subsequent analysis.

Corollary 1 *If a symmetric two-location BRE with $\frac{a}{2}$ active candidates at y^1 and $\frac{a}{2}$ active candidates at y^2 exists then a symmetric two-location BRE with $k+1$ active candidates at y^1 and $k+1$ active candidates at y^2 exists too.*

The proof of this result is trivial and it is hence skipped.[17] The implication of this result, though, is of paramount importance in our quest to characterize the classes of preferences profiles for which each k-vote rule admits a symmetric two-location equilibrium. It suggests that we can concentrate our efforts on understanding which Fs admit such an equilibrium with $2k+2$ active candidates.

Moreover, we notice that, for a given a, if conditions A and C hold for some pair $(y^1, y^2) = (y^1, 2m - y^1)$ such that $\max\{F(y^1), 1 - F(2m - y^1)\} < \frac{k}{a}$ then they should also hold for the location pair $(\dot{y}^1, \dot{y}^2) = (\dot{y}^1, 2m - \dot{y}^1)$ which is such that $\max\{F(\dot{y}^1), 1 - F(2m - \dot{y}^1)\} = \frac{k}{a}$. These observations allow us to state the following simplified necessary and sufficient condition for existence of a symmetric two-location equilibrium.

Proposition 3 *When the ideal policies of the society are distributed according to F and voting takes place according to a k-vote rule, a symmetric two-location BRE exists if and only if there exists $\hat{y} < m$ such that: A) $\max\{F(\hat{y}), 1 - F(2m - \hat{y})\} = \frac{k}{2k+2}$ and B) for every $y \in (\hat{y}, 2m - \hat{y})$ we have $1 + \frac{1}{k} < \frac{\max\{F(\frac{\hat{y}+y}{2}), 1 - F(\frac{2m - \hat{y}+y}{2})\}}{F(\frac{2m-\hat{y}+y}{2}) - F(\frac{\hat{y}+y}{2})}$.*

Again, a formal proof is not necessary since the above proposition naturally follows if one combines proposition 2, corollary 1 and the fact that the fraction in the last part of condition C of proposition 2 is strictly increasing in y^1 for any fixed $y \in (y^1, 2m - y^1)$.

We notice that $\frac{k}{2k+2}$ is increasing in k and that $1 + \frac{1}{k}$ is decreasing in k. Moreover, we observe that $\max\{F(\hat{y}), 1 - F(2m - \hat{y})\}$ is increasing in $\hat{y} \in (0, m)$ and that $\frac{\max\{F(\frac{\hat{y}+y}{2}), 1 - F(\frac{2m-\hat{y}+y}{2})\}}{F(\frac{2m-\hat{y}+y}{2}) - F(\frac{\hat{y}+y}{2})} \leq \frac{\max\{F(\frac{\tilde{y}+y}{2}), 1 - F(\frac{2m-\tilde{y}+y}{2})\}}{F(\frac{2m-\tilde{y}+y}{2}) - F(\frac{\tilde{y}+y}{2})}$ when $0 < \hat{y} < \tilde{y} < y < m$. So if, given F, there exists $\hat{y} < m$ such that the two conditions of proposition 3 hold when voters have k votes, then there should also exist $\tilde{y} \in (\hat{y}, m)$ such that the two conditions hold when voters have $k+1$ votes. That is, if in society F a symmetric two-location equilibrium exists when voting takes place under a k-vote rule, then in the same society F a symmetric two-location equilibrium should also exist if voting took place under a $(k+1)$-vote rule. This brings us to our next result. Before we state it though we need to define Φ as the set of all admissible distributions on $[0, 1]$ and $\Phi_k \subseteq \Phi$ as the set which contains all admissible distributions for which a symmetric two-location BRE exists under a k-vote rule.

Proposition 4 *$\Phi_k \subseteq \Phi_{k+1}$ for every $k \in \mathbb{N}^+$ and $\lim_{k \to +\infty} \Phi_k = \Phi$.*

Proof. The first part of this proposition has been established by the arguments presented before its statement. If for some F a symmetric two-location BRE exists under a $\hat{k}$-vote rule, then a BRE exists under a k-vote rule too for every $k > \hat{k}$. Therefore, all Fs that belong to Φ_k must also

[17] It just hinges on the observation that, for a fixed pair (y^1, y^2), if conditions B and C hold for some a, then they also hold for $a = 2k+2$.

belong to Φ_{k+1}; $\Phi_k \subseteq \Phi_{k+1}$. The proof of the second part of this proposition is as follows. We notice that $\lim_{k\to+\infty} \frac{k}{2k+2} = \frac{1}{2}$. Hence: a) for every admissible F there exists $\hat{y} < m$ such that $\max\{F(\hat{y}), 1 - F(2m - \hat{y})\} = \frac{k}{2k+2}$, when k is sufficiently large, and b) $\lim_{k\to+\infty} \hat{y} = m$. Moreover, $\lim_{k\to+\infty}(1 + \frac{1}{k}) = 1$ and, considering that $y \in (\hat{y}, 2m - \hat{y})$, $\lim_{\hat{y}\to m^-} \frac{\max\{F(\frac{\hat{y}+y}{2}), 1-F(\frac{2m-\hat{y}+y}{2})\}}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})} = +\infty$. Therefore, for every $F \in \Phi$ there exists a large enough k, such that $F \in \Phi_k$. That is, $\lim_{k\to+\infty} \Phi_k = \Phi$. ■

This proposition formally establishes that the set of distributions of voters' ideal policies, for which symmetric two-location equilibria exist under a k-vote rule, is expanding in k and in the limit ($k \to +\infty$) it essentially includes all admissible distributions of ideal policies. Given though that for $k = 1$ (plurality rule) Osborne (1993) proved that an equilibrium exists for almost no distribution of voters' ideal policies, what is essential to be answered next, is how large need k be in order for such equilibria to exist for non-degenerate classes of voters' preferences. What we find is that k may be as small as two.

Proposition 5 *Φ_k is non-degenerate if and only if $k \geq 2$.*

Proof. We notice that when $k = 1$ the existence conditions of proposition 3 suggest that $F(\hat{y}) = 1 - F(2m - \hat{y}) = \frac{1}{4}$ (if, for example, $F(\hat{y}) < \frac{1}{4}$ and $1 - F(2m - \hat{y}) = \frac{1}{4}$ then condition B of proposition 3 is not satisfied for $y \to \hat{y}^+$). Consider that Φ_1 is non-degenerate and hence that it contains at least two admissible distribution functions F_1 and F_2 such that $F_1(x) < F_2(x)$ for every $x \in (0, 1)$. Then, we define $\hat{y}^{F_1}$ and m^{F_1} such that $F_1(\hat{y}^{F_1}) = \frac{1}{4} \iff F_1^{-1}(\frac{1}{4}) = \hat{y}^{F_1}$ and $F_1(m^{F_1}) = \frac{1}{2} \iff F_1^{-1}(\frac{1}{2}) = m^{F_1}$. Notice that every admissible $F(x)$, such that $F(x) \in [F_1(x), F_2(x)]$ for every $x \in [0, 1]$, should also belong to Φ_1. Hence, Φ_1 should contain an admissible distribution $\dot{F}(x)$ such that: a) $\dot{F}(x) = F_1(x)$ for $x \in [0, m^{F_1}]$, b) $\dot{F}(x) \in [F_1(x), F_2(x)]$ for every $x \in (\hat{y}^{F_1}, 2m^{F_1} - \hat{y}^{F_1})$ and c) $\dot{F}(x) = F_2(x)$ for $x \in [2m^{F_1} - \hat{y}^{F_1}, 1]$. If $\dot{F}$ admits a BRE under $k = 1$, then it should be such that two candidates locate at $\hat{y}^{F_1}$ and two candidates locate at $2m^{F_1} - \hat{y}^{F_1}$ (because $\dot{F}(\hat{y}^{F_1}) = F_1(\hat{y}^{F_1}) = \frac{1}{4}$ and $\dot{F}(m^{F_1}) = F_1(m^{F_1}) = \frac{1}{2}$). But if we are in a BRE then it should also be true that $2m^{F_1} - \hat{y}^{F_1} \in (m^{F_1}, 1)$ and that $1 - \dot{F}(2m^{F_1} - \hat{y}^{F_1}) = 1 - F_2(2m^{F_1} - \hat{y}^{F_1}) = \frac{1}{4}$ which may be true only if $F_1(2m^{F_1} - \hat{y}^{F_1}) = F_2(2m^{F_1} - \hat{y}^{F_1})$, and that is wrong by assumption. Therefore, our initial assumption is incorrect and hence Φ_1 cannot be non-degenerate.

To establish that Φ_k is non-degenerate for every $k \geq 2$, it is sufficient to prove that Φ_2 is non-degenerate (proposition 4). We consider that $k = 2$ and that $F(x) \in (x - \varepsilon, x + \varepsilon)$ for every $x \in [0, 1]$ and a sufficiently small but strictly positive ε. We assume that $y^1 = m - \frac{1}{5}$, $y^2 = m + \frac{1}{5}$ and $n(y^1) = n(y^2) = \frac{a}{2} = 3$ (condition A and the first part of condition C of proposition 2 hold). We observe that $\lim_{\varepsilon\to 0} \max\{F(y^1), 1 - F(2m - y^1)\} = \frac{3}{10} < \frac{1}{3} = \frac{k}{a}$ and hence, for sufficiently small ε, condition B of proposition 2 holds too. We moreover notice that for every $y \in (y^1, y^2)$ it is true that

$\lim_{\varepsilon\to 0}[F(\frac{2m-y^1+y}{2})-F(\frac{y^1+y}{2})]=\frac{1}{5}$ and $\lim_{\varepsilon\to 0}\max\{F(\frac{y^1+y}{2})\frac{2k}{a},[1-F(\frac{2m-y^1+y}{2})]\frac{2k}{a}\}\geq\frac{8}{20}\times\frac{4}{6}=\frac{4}{15}>\frac{1}{5}$. That is, for sufficiently small ε, the second part of condition C of proposition 2 also holds.

Since a BRE exists for every admissible $F(x)\in(x-\varepsilon,x+\varepsilon)$ when $\varepsilon>0$ is positive and sufficiently small, it follows that there exists a pair of admissible distribution functions F_1 and F_2 which are such that $x-\varepsilon<F_1(x)<F_2(x)<x+\varepsilon$ for every $x\in(0,1)$. Obviously, every $\hat{F}$ such that $\hat{F}(x)\in[F_1(x),F_2(x)]$ for every $x\in[0,1]$ also satisfies $\hat{F}(x)\in(x-\varepsilon,x+\varepsilon)$ and hence Φ_2 is non-degenerate. Since, $\Phi_2\subseteq\Phi_k$ for every $k\geq 3$ it follows that Φ_k is non-degenerate if $k\geq 2$. ■

But why is the plurality rule so different compared to every other k-vote rule with $k\geq 2$?

The reason why symmetric two-location equilibria almost never exist under the plurality rule, lies in the following facts: a) a candidate located at y^1 must be sharing this location with at least one other candidate because otherwise she would have incentives to move towards the right and gain votes and b) a candidate located at y^1 must be sharing this location with at most one other candidate. If $n(y^1)>2$ candidates share location y^1 then each of them is voted by strictly less than half of the constituency of y^1 (each is voted by a fraction $\frac{1}{n(y^1)}$ of the constituency of y^1). If the left (right) semi-constituency of y^1 is larger or equal to the right (left) semi-constituency of y^1, then a candidate by deviating from y^1 marginally to the left (right) increases her vote-mass and thus her election probability. But if exactly two candidates must share location y^1, then it should also be the case that the left semi-constituency of y^1 is exactly identical to the right semi-constituency of y^1 (otherwise one of the two candidates would be better off by marginally deviating towards the side of the larger semi-constituency of y^1). This leaves only one possibility for a y^1 in a symmetric two-location BRE - it must be such that $F(y^1)=\frac{1}{4}$. Since this first location is uniquely defined for every F and since the second location, y^2, has to satisfy at the same time $y^2=2m-y^1=2F^{-1}(\frac{1}{2})-F^{-1}(\frac{1}{4})$ and $1-F(y^2)=\frac{1}{4}\iff y^2=F^{-1}(\frac{3}{4})$ - which are both satisfied by almost no F - it trivially follows that such pairs of locations exist for almost no F and subsequently that Φ_1 cannot be non-degenerate (a similar reasoning rules out existence of any kind of equilibria for almost all Fs under the plurality rule).

When voters are allowed to vote for more than one candidate, though, things change dramatically. Consider, for example, that $k=2$ and that three candidates are located at y^1 and three at $y^2=2m-y^1$. Then each of the candidates located at y^1 is voted by a fraction $\frac{2}{3}$ of the constituency of y^1 (which is much larger than half of the constituency of y^1). This means that in equilibrium *the left semi-constituency of y^1 need not be precisely as large as the right semi-constituency of y^1*; as long as $F(y^1)\in(\frac{1}{6},\frac{1}{3})$ none of the three candidates located at y^1 has any incentives to deviate marginally to the left or marginally to the right of y^1. Since the admissible values for a y^1 are infinitely more compared to the plurality rule, it directly follows that equilibrium possibilities are infinitely more too. Of course more conditions on top of $F(y^1)\in(\frac{1}{6},\frac{1}{3})$ need to hold in order for the posited profile to constitute an equilibrium (conditions that guarantee that entry of other candidates and deviations far away from y^1 are also unprofitable). But the fact that this first condition specifies a non-degenerate range of locations that may be part of an equilibrium sums up the intuition why stability in electoral competition under a k-vote rule with $k \geq 2$ may actually be reached.

After having shown that stability in electoral competition is feasible[18] for each k-vote rule with $k \geq 2$, one would naturally want to have a better understanding of the sets of distribution functions that guarantee equilibrium existence. First, we show that every k-vote rule with $k \geq 2$ admits a symmetric two-location BRE when the distribution of voters' ideal policies is log-concave.[19] That is, existence of such equilibria is quite general since log-concavity is satisfied by many distributions (including the popular families of unimodal beta, truncated normal, distributions with linear densities and many other).

Proposition 6 *Φ_k contains all admissible distributions that satisfy log-concavity if and only if $k \geq 2$.*

Proof. Notice that $F(x) = x$ (uniform distribution) is an admissible distribution that satisfies log-concavity. When $k = 1$ condition A of proposition 3 suggests that $\hat{y} = \frac{1}{4}$. Moreover, $m = \frac{1}{2}$ and $2m - \hat{y} = \frac{3}{4}$. Hence, condition B of proposition 3 does not hold as $\frac{\max\{F(\frac{\hat{y}+y}{2}), 1-F(\frac{2m-\hat{y}+y}{2})\}}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})} < \frac{\frac{1}{2}}{\frac{1}{4}} = 1 + \frac{1}{k}$ for every $y \in (\hat{y}, 2m - \hat{y})$ and $k = 1$. In other words, Φ_1 does not contain all admissible distributions that satisfy log-concavity.

Now our aim is to show that, for every admissible distribution F with median m (fix, without loss of generality, $m \leq \frac{1}{2}$) that satisfies log-concavity and every $k \geq 2$, there exists $\hat{y} \in (0, m)$ that satisfies the two conditions of proposition 3 and hence a symmetric two-location BRE exists. First, we argue that when an admissible F is log-concave, then indeed there exists $\hat{y} \in (0, m)$ such that $\max\{F(\hat{y}), 1 - F(2m - \hat{y})\} = \frac{k}{2k+2}$ (condition A of proposition 3) for every $k \geq 2$. Notice that $\max\{F(x), 1 - F(2m - x)\}$ is strictly increasing in $x \in [0, m]$ and $\max\{F(m), 1 - F(2m - m)\} = \frac{1}{2}$. That is, the only possibility that there is no $\hat{y} \in (0, m)$, such that $\max\{F(\hat{y}), 1 - F(2m - \hat{y})\} = \frac{k}{2k+2} > \frac{1}{4}$, is when $\max\{F(0), 1 - F(2m)\} > \frac{k}{2k+2}$. We will show that when an admissible F is log-concave, then $\max\{F(0), 1 - F(2m)\} \leq \frac{1}{4}$ and, hence, $\max\{F(0), 1 - F(2m)\} < \frac{k}{2k+2}$. Assume, on the contrary, that $\max\{F(0), 1 - F(2m)\} > \frac{1}{4}$. Given that $F(0) = 0$ for every admissible distribution and that $m \leq \frac{1}{2}$ it follows that: a) $m < \frac{1}{2}$ (otherwise - that is, if $m = \frac{1}{2}$ - we would have $1 - F(2m) = 0 \Rightarrow \max\{F(0), 1 - F(2m)\} = 0 < \frac{1}{4}$, since every admissible distribution is such that $F(1) = 1$), and b) $1 - F(2m) > \frac{1}{4} \Leftrightarrow F(2m) < \frac{3}{4}$. These two observations suggest that F fails the

[18] The rules that we consider permit partial abstention (that is, a voter is not compelled to use all the k votes). Notice, though, that our genericity results directly apply to k-vote rules that do not allow partial abstention. This is so because not allowing for partial abstention is equivalent to allowing for partial abstention and voters using all their k votes. Since, in our setup voters may be characterized by any minimally sincere voting behavior, *including voting behaviors which involve the use of all k votes*, a BRE of our game remains an equilibrium when partial abstention is not allowed (that is, when the minimally sincere behaviors are restricted to those which involve the use of all k votes).

[19] We consider that F is log-concave if $\frac{\partial^2 \ln F(x)}{\partial x^2} < 0$ and $\frac{\partial^2 \ln[1-F(x)]}{\partial x^2} < 0$ for every $x \in (0, 1)$. Hence, our definition of log-concavity implies that F is twice-differentiable as well.

gradual escalating median (GEM) property that all log-concave distribution functions are guaranteed to have (see, for example, Haimanko et al. 2005).[20] Therefore, our assumption is wrong and hence $\max\{F(0), 1-F(2m)\} \leq \frac{1}{4} < \frac{k}{2k+2}$; condition A of proposition 3 holds when $k \geq 2$ for every admissible F that satisfies log-concavity.

As far as condition B of proposition 3 is concerned, we assume without loss of generality that $F(\hat{y}) = \frac{k}{2k+2}$ and $1 - F(2m - \hat{y}) \leq \frac{k}{2k+2}$ (now we drop the assumption that $m \leq \frac{1}{2}$). We notice that $\frac{\max\{F(\frac{\hat{y}+y}{2}), 1-F(\frac{2m-\hat{y}+y}{2})\}}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})} \geq \frac{F(\frac{\hat{y}+y}{2})}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})}$ for every $y \in (\hat{y}, 2m-\hat{y})$. So if we show that $1 + \frac{1}{k} < \frac{F(\frac{\hat{y}+y}{2})}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})}$ for every $y \in (\hat{y}, 2m-\hat{y})$ we are done. We observe that $\lim_{y \to \hat{y}^+} \frac{F(\frac{\hat{y}+y}{2})}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})} = \frac{F(\hat{y})}{\frac{1}{2}-F(\hat{y})} = \frac{\frac{k}{2k+2}}{\frac{1}{2}-\frac{k}{2k+2}} = k > 1+\frac{1}{k}$ for every $k > \frac{1}{2}(1+\sqrt{5}) \approx 1.62$. Moreover, we have that $\frac{F(\frac{\hat{y}+y}{2})}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})}$ is increasing in $y \in (\hat{y}, 2m-\hat{y})$ due to log-concavity of F. Hence, $\frac{F(\frac{\hat{y}+y}{2})}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})} > 1 + \frac{1}{k}$ for every $y \in (\hat{y}, 2m-\hat{y})$ and $k \geq 2$; and, thus, when an admissible F is log-concave then both conditions of proposition 3 hold and a symmetric two-location BRE is guaranteed to exist. ■

In the next part of our analysis we show that if elections take place under a k-vote rule, an equilibrium is guaranteed to exist for all distributions that are symmetric about their median (and for all distributions in the neighborhood of such symmetric distributions) *if and only if* $k \geq 3$. We consider that this result is of particular interest since: a) symmetric distributions are very popular in political economics literature (many seminal papers prove their results only by considering perfectly symmetric distributions - see for example Palfrey 1984) and b) our existence result is not limited to distributions being absolutely symmetric; it extends to the neighborhood of each symmetric distribution.

Proposition 7 *Φ_k contains all admissible distributions which are symmetric about their median and all admissible distributions in a neighborhood of such distributions if and only if $k \geq 3$.*

Proof. We assume that F is symmetric about its median ($F(x) = 1 - F(1-x)$ for every $x \in [0,1]$) and we show that: a) when $k = 1$ and when $k = 2$, it is possible that a BRE does not exist, b) when $k = 3$, a BRE always exists and c) when $k = 3$, a symmetric two-location BRE also exists for every F in a neighborhood of F.

If $k = 1$, then, according to the proof of proposition 6, there is no symmetric two-location BRE when F is uniform. But since a uniform distribution is also a symmetric one, we have that there exist symmetric admissible Fs for which a symmetric two-location BRE does not exist when $k = 1$.

If $k = 2$, then, according to proposition 3, a symmetric two-location BRE exists only if there exists $\hat{y} < m$ such that $\hat{y} = F^{-1}(\frac{1}{3})$ and for every $y \in (\hat{y}, 1-\hat{y})$ we have either $\frac{3}{2} < \frac{F(\frac{\hat{y}+y}{2})}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})}$ and

[20] The GEM property requires that as the subset $[t,1]$ of our society shrinks (that is, as t increases), its median, m_t (defined by $F(m_t) = \frac{1+F(t)}{2}$), increases slower than t. That is, GEM requires that $\frac{\partial m_t}{\partial t} < 1$ (GEM imposes symmetric restrictions on the median of the subset $[0,t]$ as well).

In our case, if $t = 0$ then $m_t = m < \frac{1}{2}$ and if $t = m$ then $m_t > 2m$, because $F(2m) < \frac{3}{4}$. In other words, t increased by m while m_t increased by strictly more than m, which suggests that there exists t for which GEM fails ($\frac{\partial m_t}{\partial t} > 1$).

$1-F(\frac{2m-\hat{y}+y}{2})<F(\frac{\hat{y}+y}{2})$ or $\frac{3}{2}<\frac{1-F(\frac{2m-\hat{y}+y}{2})}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})}$ and $1-F(\frac{2m-\hat{y}+y}{2})\geq F(\frac{\hat{y}+y}{2})$. If F is such that for some very small $\varepsilon>0$ we have $F(\frac{1}{2}+\varepsilon)-F(\frac{1}{2}-\varepsilon)=0.3$ and $\frac{1}{2}-\hat{y}>2\varepsilon$, then for $y=\frac{1}{2}\in(\hat{y},1-\hat{y})$ none of these conditions hold. This is so because, $F(\frac{2m-\hat{y}+\frac{1}{2}}{2})-F(\frac{\hat{y}+\frac{1}{2}}{2})\geq 0.3$ and $F(\frac{\hat{y}+\frac{1}{2}}{2})\leq 0.35$. Therefore, when $k=2$, it is possible that a symmetric two-location BRE does not exist, even when F is symmetric about its median.

To show that, if $k\geq 3$, a symmetric two-location BRE always exists when F is symmetric about its median, we first notice that condition B of proposition 3 always holds if $F(2m-\hat{y})-F(\hat{y})<\max\{F(\hat{y})\frac{k}{k+1},[1-F(2m-\hat{y})]\frac{k}{k+1}\}\iff F(\hat{y})>\frac{1+k}{2+3k}$.[21] Condition A of proposition 3 suggests that $F(\hat{y})=\frac{k}{2k+2}$. Hence, $\hat{y}<m$ which satisfies both conditions is guaranteed to exist if $\frac{k}{2k+2}>\frac{1+k}{2+3k}$. For k positive, this last inequality is equivalent to $k>1+\sqrt{3}$, which is true for every $k\geq 3$.

By the latter, it becomes straightforward that when F is symmetric about its median and $k\geq 3$, one may find $\dot{y}<m$ such that $\frac{1+k}{2+3k}<F(\dot{y})<\frac{k}{2k+2}$. That is, if F is symmetric about its median, then there exists a BRE with $k+1$ active candidates at $\dot{y}$ and $k+1$ active candidates at $1-\dot{y}$, where $\dot{y}<m$ is such that condition B of proposition 2 holds with a strict inequality. Now consider an admissible distribution $\tilde{F}$ with median $\tilde{m}$ such that $\tilde{F}(x)\in(F(x)-\varepsilon,F(x)+\varepsilon)$ for every $x\in(0,1)$ and a sufficiently small but strictly positive ε (that is, $\tilde{F}$ is a distribution in a neighborhood of this symmetric F) and a strategy profile with $k+1$ active candidates at $\tilde{m}-(\frac{1}{2}-\dot{y})$ and $k+1$ active candidates at $\tilde{m}+(\frac{1}{2}-\dot{y})$. When $\varepsilon\to 0$ we have that $\tilde{F}$ converges to F and that the two locations of the posited strategy profile converge to $(\dot{y},1-\dot{y})$. That is, for ε sufficiently small all conditions of proposition 2 hold and a symmetric two-location BRE is guaranteed to exist for every admissible distribution in a neighborhood of a symmetric one. ■

This result distinguishes the one-vote rule (plurality) and the two-vote rule from all k-vote rules with $k\geq 3$ in the sense that the first two are not guaranteed to generate stability in electoral competition, even when F is symmetric. We have to stress though that the failure of these two rules is far from being identical in magnitude. Given a symmetric F: a) $k=1$ need not admit a BRE when F is log-concave and, perhaps more importantly, even if it admits an equilibrium for a symmetric F, this existence does not extend to distributions in the neighborhood of F (see the discussion that follows the proof of proposition 5), while b) $k=2$ admits an equilibrium when F is log-concave (actually, a BRE fails to exist only in very special cases - namely, when there is at least one region to each side of the median that: i) has very little mass and ii) is surrounded by regions with very large masses) and, when it admits an equilibrium for a symmetric F, it generically admits one for every distribution in its neighborhood too.[22]

[21] This is so because condition B of proposition 3 can be written as $1<\frac{\max\{F(\frac{\hat{y}+y}{2})\frac{k}{k+1},[1-F(\frac{2m-\hat{y}+y}{2})]\frac{k}{k+1}\}}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})}$ and because $\frac{\max\{F(\hat{y})\frac{k}{k+1},[1-F(2m-\hat{y})]\frac{k}{k+1}\}}{F(2m-\hat{y})-F(\hat{y})}<\frac{\max\{F(\frac{\hat{y}+y}{2})\frac{k}{k+1},[1-F(\frac{2m-\hat{y}+y}{2})]\frac{k}{k+1}\}}{F(\frac{2m-\hat{y}+y}{2})-F(\frac{\hat{y}+y}{2})}$ for every $y\in(\hat{y},2m-\hat{y})$. Hence if $1<\frac{\max\{F(\hat{y})\frac{k}{k+1},[1-F(2m-\hat{y})]\frac{k}{k+1}\}}{F(2m-\hat{y})-F(\hat{y})}$, then condition B of proposition 3 must hold.

[22] This can be established by an argument similar to the one in the end of the proof of proposition 7.

To better demonstrate the difference between the two-vote rule and the plurality rule, we consider that F is a symmetric beta distribution with shape parameter $\beta > 0$[23] and by applying computational methods (see figures 1a and 1b) we get that: a) when $k = 1$ a symmetric two-location BRE exists if and only if $\beta \in (0, \hat{\beta})$ where $\hat{\beta} \approx 0.3$ (that is, when the society is very polarized) and b) when $k = 2$ a symmetric two-location BRE exists for every $\beta > 0$ (from proposition 7 it is straightforward that when $k \geq 3$ a symmetric two-location BRE exists for every $\beta > 0$).

[Insert figure 1 about here]

But how does the maximum number of votes that voters may cast and the distribution of their ideal policies affect their welfare? Given that we have fully characterized the class of symmetric two-location equilibria it should be possible to classify them according to some social welfare criterion. If we assume that social welfare increases when the distance between the implemented policy (the platform of the winner candidate) and the ideal policy of the median voter decreases,[24] then, given our formal results, a social welfare analysis is quite direct. For easier construction of the following arguments we define social welfare from an implemented policy x as $W(x)$ where W is strictly decreasing in $|m - x|$ and we consider that F is symmetric about its median and log-concave. By lemma 3 we know that in every symmetric two-location equilibrium it is the case that all candidates locate equidistantly away from the ideal policy of the median voter (half of the active candidates locate at $y^1 < m$ and the rest of them locate at $y^2 = 2m - y^1$). Hence, there is no uncertainty about social welfare in each of these equilibria: the smaller the $m - y^1$, the larger the social welfare $W(y^1)$. By corollary 1 it follows that, if for some k-vote rule an equilibrium with a active candidates exists such that $\frac{a}{2}$ locate at $y^1 < m$ and the rest of them locate at $y^2 = 2m - y^1$, then an equilibrium with $2k+2$ active candidates exists such that $k+1$ locate at $y^1 < m$ and the rest of them locate at $y^2 = 2m - y^1$. Hence, an equilibrium which delivers social welfare $W(y^1)$ exists if and only if: a) $y^1 \leq F^{-1}(\frac{k}{2k+2})$ (condition B of proposition 2 simplifies to this expression when F is symmetric and $a = 2k + 2$) and b) $y^1 > 2F^{-1}(\frac{1+k}{2+3k}) - \frac{1}{2}$.[25] That is, social welfare under a k-vote rule is strictly larger than $W(2F^{-1}(\frac{1+k}{2+3k}) - \frac{1}{2})$ and at most

[23] A shape parameter smaller than one corresponds to a bimodal symmetric beta distribution with modes at zero and one, a shape parameter equal to one corresponds to a uniform distribution and a shape parameter larger than one corresponds to a unimodal symmetric beta distribution with a mode at one half. We further note that a symmetric beta distribution is log-concave if and only if the shape parameter is larger or equal to one (Bagnoli and Bergstrom 2005).

[24] This is the case, for example, when voters' payoffs are identically and linearly decreasing in the distance between their ideal policies and the implemented one, and social welfare is defined as the unweighted sum of voters' payoffs.

[25] When F is symmetric and log-concave, then, for every $y^1 < m$, $\frac{F(\frac{y^1+y}{2})}{F(\frac{2m-y^1+y}{2})-F(\frac{y^1+y}{2})}$ is increasing at every $y \in (y^1, 2m - y^1)$ and $\frac{1-F(\frac{2m-y^1+y}{2})}{F(\frac{2m-y^1+y}{2})-F(\frac{y^1+y}{2})}$ is decreasing at every $y \in (y^1, 2m - y^1)$. That is, $\frac{\max\{F(\frac{y^1+y}{2}),1-F(\frac{2m-y^1+y}{2})\}}{F(\frac{2m-y^1+y}{2})-F(\frac{y^1+y}{2})} \geq \frac{F(\frac{y^1+\frac{1}{2}}{2})}{1-2F(\frac{y^1+\frac{1}{2}}{2})}$ for every $y \in (y^1, 2m - y^1)$. This suggests that for these distributions and $a = 2k + 2$, condition C of proposition 2 simplifies to $1 + \frac{1}{k} < \frac{F(\frac{y^1+\frac{1}{2}}{2})}{1-2F(\frac{y^1+\frac{1}{2}}{2})}$, and may be rewritten as $y^1 > 2F^{-1}(\frac{1+k}{2+3k}) - \frac{1}{2}$.

as large as $W(F^{-1}(\frac{k}{2k+2}))$. What we actually see is that by increasing k both the lower bound of equilibrium social welfare decreases ($W(2F^{-1}(\frac{1+(k+1)}{2+3(k+1)}) - \frac{1}{2}) < W(2F^{-1}(\frac{1+k}{2+3k}) - \frac{1}{2})$) and the upper bound of equilibrium social welfare increases ($W(F^{-1}(\frac{(k+1)}{2(k+1)+2})) > W(F^{-1}(\frac{k}{2k+2}))$). Hence, what we find, is that giving more votes to voters, increases the variability of equilibrium extremism[26] and subsequently the variability of equilibrium social welfare.

Finally, we try to understand how social welfare relates to the distribution of voters' ideal policies in these equilibria. The shaded areas of figures 2a, 2b and 2c show admissible values of y^1 as a function of the shape parameter, β, of a unimodal symmetric beta distribution for $k = 3$, $k = 5$ and $k = 10$ respectively (the larger the value of β, the larger the density of the distribution about the centre of the policy space). As we see, societies composed of like-minded voters (β large) enjoy policies that are on average nearer to the ideal policy of the median voter compared to more polarized societies (β small) and, hence, social welfare seems to react to changes in the distribution of voters' ideal policies in an intuitive manner.

[Insert figure 2 about here]

3.3 Other equilibria

We have analyzed in depth symmetric two-location equilibria under every k-vote rule and we have developed arguments that show that unlike the plurality rule, k-vote rules with $k \geq 2$ may lead to stability in electoral competition for a non-degenerate class of cases. We need now to explore possibility of other kinds of equilibria, in order to be sure that this identified advantage of k-vote rules with $k \geq 2$ compared to the plurality rule, extends to all possible configurations. One can follow steps similar to the ones we followed above and fully characterize all symmetric[27] r-location equilibria of the game with $r \geq 3$ under any k-vote rule. Actually one can even show that the set of admissible Fs for which a k-vote rule admits a symmetric r-location equilibrium with $r \geq 3$ is non-degenerate if and only if $k \geq 2$.

Regarding asymmetric equilibria (equilibria such that there are at least two occupied locations with unequal number of active candidates), one can trivially adapt arguments presented above and show that the set of admissible Fs for which the plurality rule admits an asymmetric r-location equilibrium, with $r \geq 2$ and a active candidates, cannot be non-degenerate for any $r \geq 2$ and $a \geq 1$ (Osborne 1993). One can further show that for k-vote rules, with k large, asymmetric two-location equilibria are guaranteed to exist for non-degenerate sets of voters' distributions.

In the Appendix we provide a complete characterization of all these equilibrium classes (symmetric equilibria with at least three occupied locations and asymmetric equilibria with at least two occupied

[26] See Dellis (2009) for a treatment of this question in the framework of the citizen-candidate model.

[27] Again, symmetric in the sense that $n(y^1) = n(y^2) = ... = n(y^r)$.

locations) for every k-vote rule, and we provide formal arguments in support of the claims made above. One needs to stress here though, that despite the possibility of equilibria with many occupied locations and/or asymmetric number of active candidates in each occupied location, symmetric two-location equilibria are the ones that face the fewest possible coordination issues[28] and the ones which exist under the simplest possible set of conditions, and under this perspective they represent the most likely players' behavior.

4 Extensions/Concluding remarks

4.1 Uncertainty about voters' preferences

Let us now extend the model by allowing potential candidates to be uncertain about voters' preferences. In specific we will assume that potential candidates believe that the real distribution of voters, $\tilde{F}(x)$, belongs to

$$S(F_1, F_2) = \{\bar{F} : [0,1] \to [0,1] | \bar{F}(x) = (1-\gamma)F_1(x) + \gamma F_2(x) \text{ for } \gamma \in [0,1]\} \subset \Phi$$

where F_1 and F_2 are two admissible distributions such that $F_1(x) < F_2(x)$ for every $x \in (0,1)$. Each candidate believes that the real distribution, $\tilde{F}(x)$, belongs to $S(F_1, F_2)$ and it is such that $\tilde{F}(x) \leq (1-\gamma)F_1(x) + \gamma F_2(x)$ for every $x \in [0,1]$ and some fixed $\gamma \in [0,1]$ with probability equal to $G(\gamma)$, where G is a continuous and strictly increasing distribution on $[0,1]$ with $G(0) = 0$.[29] Now define γ^m such that $G(\gamma^m) = \frac{1}{2}$ and $\hat{m}$ such that $(1-\gamma^m)F_1(\hat{m}) + \gamma^m F_2(\hat{m}) = \frac{1}{2}$; the probability with which the median of the real distribution is to the left (right) of $\hat{m}$ is equal to $\frac{1}{2}$. Without introducing unnecessary formalities we note that now candidates are uncertain about their exact vote-masses given a strategy profile, and hence computation of win probabilities becomes significantly more complex. Everything else about our model remains unchanged (voters' behavior, candidates' objectives, etc.).

Proposition 8 *Consider a k-vote rule with $k \geq 2$ and an admissible distribution F with median m such that, under perfect information, a symmetric two-location BRE exists with $2k+2$ active candidates and* $\max\{F(y^1), 1 - F(2m - y^1)\} < \frac{k}{2k+2}$*. Then there exists $\hat{\varepsilon} > 0$ such that, for every $\varepsilon \in (0, \hat{\varepsilon})$, the incomplete information game admits a symmetric two-location BRE for every admissible $S(F_1, F_2) \subset \{\bar{F} : [0,1] \to [0,1] | \bar{F}(x) \in (F(x) - \varepsilon, F(x) + \varepsilon)$ for every $x \in (0,1)\}$ and $G(\gamma)$.*

[28] The less coordination an equilibrium requires, the more likely it should be that players will play according to it. In our framework it is straightforward that, from all possible equilibrium configurations, symmetric two-location equilibria are the least demanding as far as players' coordination is concerned.

[29] In order to model "continuous" uncertainty we have employed a parametric specification (γ) in line with Roemer (1994).

Proof. The assumption with which we open the statement of the proposition just requires that the existence of a symmetric two-location BRE is not cut-edge for the considered F and k when the information that potential candidates have about voters' preferences is perfect (obviously, it is never satisfied when $k = 1$ for any F, since in a symmetric two-location BRE of the plurality rule we must have $F(y^1) = \frac{1}{4}$). Assume that there is incomplete information about voters' preferences given by $S(F_1, F_2) \subset \{\bar{F} : [0,1] \to [0,1] | \bar{F}(x) \in (F(x) - \varepsilon, F(x) + \varepsilon)$ for every $x \in (0,1)\}$ and $G(\gamma)$, that $k+1$ candidates are located at $\hat{m} - (m - y^1)$ and that $k+1$ candidates are located at $\hat{m} + (m - y^1)$.[30] Notice that if $\varepsilon > 0$ is small enough, then the probability with which each active candidate wins is $\frac{1}{2k+2}$ because: a) with probability $\frac{1}{2}$ the true median is to the left of $\hat{m}$ and hence one of the $k+1$ active candidates located at $\hat{m} - (m - y^1)$ will win with probability $\frac{1}{k+1}$ and each of the $k+1$ candidates located at $\hat{m} + (m - y^1)$ will win with probability zero and b) with probability $\frac{1}{2}$ the true median is to the right of $\hat{m}$ and hence one of the $k+1$ active candidates located at $\hat{m} + (m - y^1)$ will win with probability $\frac{1}{k+1}$ and each of the $k+1$ candidates located at $\hat{m} - (m - y^1)$ will win with probability zero. Moreover, observe that if $\varepsilon > 0$ is small enough no player has incentives to deviate from *Out*: a) to any $y \in [0, \hat{m} - (m - y^1))$ ($y \in (\hat{m} + (m - y^1), 1]$) since her vote-mass will be at most as large as $F_2(\hat{m} - (m - y^1))$ $(1 - F_1(\hat{m} - (m - y^1)))$ and hence strictly smaller than the smallest possible vote-mass of each active candidate located at $\hat{m} + (m - y^1)$ $(\hat{m} - (m - y^1))$, because $\lim_{\varepsilon \to 0} F_2(\hat{m} - (m - y^1)) = F(y^1) < \frac{k}{2k+2} = \lim_{\varepsilon \to 0}[1 - F_2(\hat{m})]\frac{k}{k+1}$, b) to any $y \in (\hat{m} - (m - y^1), \hat{m} + (m - y^1))$ as her vote-mass will be with probability one strictly smaller than the vote-mass either of each active candidate located at $\hat{m} + (m - y^1)$ or of each active candidate located at $\hat{m} - (m - y^1)$ (or both), because $\frac{2k \max\{\lim_{\varepsilon \to 0} F_1(\frac{\hat{m} - (m - y^1) + y}{2}), 1 - \lim_{\varepsilon \to 0} F_2(\frac{\hat{m} + (m - y^1) + y}{2})\}}{\lim_{\varepsilon \to 0}[F_2(\frac{\hat{m} + (m - y^1) + y}{2}) - F_1(\frac{\hat{m} - (m - y^1) + y}{2})]} = \frac{2k \max\{F(\frac{y^1 + y}{2}), 1 - F(\frac{2m - y^1 + y}{2})\}}{F(\frac{2m - y^1 + y}{2}) - F(\frac{y^1 + y}{2})} > 2k+2$ for every $y \in (\hat{m} - (m - y^1), \hat{m} + (m - y^1))$, and c) to $y = \hat{m} - (m - y^1)$ $(y = \hat{m} + (m - y^1))$ since her vote-mass will be with probability one strictly smaller than the vote-mass of each active candidate located at $\hat{m} + (m - y^1)$ $(\hat{m} - (m - y^1))$. Similar arguments rule out deviation of any of the $2k+2$ active candidates to other locations when $\varepsilon > 0$ is sufficiently small. That is, there exists $\hat{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \hat{\varepsilon})$ this incomplete information game admits a symmetric two-location BRE for every admissible $S(F_1, F_2) \subset \{\bar{F} : [0,1] \to [0,1] | \bar{F}(x) \in (F(x) - \varepsilon, F(x) + \varepsilon)$ for every $x \in (0,1)\}$ and $G(\gamma)$. ■

This result has non-negligible implications as far as robustness of the identified equilibria is concerned: unlike runoff rules whose equilibria collapse once we consider such continuous uncertainty about the exact distribution of voters' preferences (Matsushima 2007; Brusco et al.2012), the equilibria of k-vote rules survive in such a more realistic environment. Hence, if existence of equilibria is desirable and moreover we value equilibria that are robust to noise, then, at least in the framework of the Hotelling-Downs model, k-vote rules with $k \geq 2$ are found to outperform both the plurality and the runoff rule.

[30] If $\varepsilon > 0$ is sufficiently small, then both $\hat{m} - (m - y^1)$ and $\hat{m} + (m - y^1)$ are guaranteed to belong to $(0, 1)$.

4.2 M-winner elections

In reality, most of the times when voters are allowed to vote for more than one candidate, it is the case that there is more than one office at stake. In the so-called open-list systems with panachage or a unique candidates' list, voters are allowed to vote for up to k candidates and the M most voted candidates get elected in a council/parliament. In most of these systems, voters have as many votes as the offices at stake ($k = M$).[31] This observation, naturally, leads to the following question: Is our analysis relevant to such M-winner elections?

In the symmetric two-location equilibria that we fully characterized in this paper, at least $2k + 2$ candidates enter a race for a single office. As we have shown, all individual deviations were unprofitable for any of our players as: a) players which play *Out* in an equilibrium strategy profile, when they deviate to entry at any location, always get a vote-mass strictly smaller than the vote-mass of the $k + 1$ most voted candidates; and b) players which play y^1 (y^2) in an equilibrium strategy profile, when they deviate to any other strategy, always get a vote-mass strictly smaller than the vote-mass of the k most voted candidates. That is, any player who deviates finds herself having a vote-mass *strictly smaller than the vote-mass of the k-th most voted candidate.* This means that the described strategy profiles are also equilibria of M-winner elections with $M \leq k$.[32] So if we considered exactly the same model with the only difference that the $M \geq 1$ most voted candidates get elected and let all the other assumptions intact (voters' behavior, candidates' preferences etc.), we would get that the symmetric two-location equilibria of the one-office game that we characterized are equilibria of this extended game too, as long as $M \leq k$.

This observation, arguably, generalizes the empirical relevance of our analysis and offers alternative readings to our stability results. For instance, one may say that if the electoral rule constrains us to vote for as many candidates as the number of offices at stake (that is, if the constitution establishes an open-list system), then an increase in the number of offices at stake (for example, an increase in the size of the council/parliament) should also increase the prospects of stable outcomes in electoral competition (proposition 4). Namely, in the framework of such systems, stable outcomes should be more frequently encountered when candidates compete for many seats rather than for only a few. Finally, one could add that systems that permit voters to vote for up to $k \geq M$ candidates, endogenously lead to the emergence of candidates' clusters and hence to endogenous formation of political parties.

[31] For example, such rules are used in several Swiss cantons, including Zurich, in Luxemburg and in German local elections.

[32] We consider that an active candidate has positive probability of election if and only if the number of candidates with a vote-mass strictly larger than hers is smaller than M.

References

[1] Bagnoli, M., & Bergstrom, T. (2005). Log-concave probability and its applications. *Economic Theory*, 26(2), 445-469.

[2] Bouton, L., & Castanheira, M. (2012). One person, many votes: Divided majority and information aggregation. *Econometrica*, 80(1), 43-87.

[3] Brams, S. J., & Fishburn, P. C. (1978). Approval voting. *American Political Science Review*, 72(03), 831-847.

[4] Brusco, S., Dziubiński, M., & Roy, J. (2012). The Hotelling–Downs model with runoff voting. *Games and Economic Behavior*, 74(2), 447-469.

[5] Cahan, D., Slinko, A., & McCabe-Dansted, J. (2011). Nonconvergent Electoral Equilibria Under Scoring Rules: Beyond Plurality. *SSRN*, 1957790.

[6] Caplin, A., & Nalebuff, B. (1991). Aggregation and social choice: a mean voter theorem. *Econometrica*, 1-23.

[7] Cox, G. W. (1987). Electoral equilibrium under alternative voting institutions. *American Journal of Political Science*, 82-108.

[8] Debreu, G. (1952). A Social Equilibrium Existence Theorem. *Proceedings of the National Academy of Sciences of the U. S. A.*, 38.

[9] Dellis, A. (2009). Would letting people vote for multiple candidates yield policy moderation? *Journal of Economic Theory*, 144(2), 772-801.

[10] Dellis, A., & Oak, M. (2006). Approval voting with endogenous candidates. *Games and Economic Behavior*, 54(1), 47-76.

[11] Dellis, A., & Oak, M. (2015). Multiple Votes, Multiple Candidacies and Polarization. *Social Choice and Welfare*, doi:10.1007/s00355-015-0900-9.

[12] Dutta, B., Jackson, M. O., & Le Breton, M. (2001). Strategic candidacy and voting procedures. *Econometrica*, 69(4), 1013-1037.

[13] Haan, M., & Volkerink, B. (2001). A runoff system restores the principle of minimum differentiation. *European Journal of Political Economy*, 17(1), 157-162.

[14] Haimanko, O., Le Breton, M., & Weber, S. (2005). Transfers in a polarized country: bridging the gap between efficiency and stability. *Journal of Public Economics*, 89(7), 1277-1303.

[15] Matsushima, N. (2007). Uncertainty of voters' preferences and differentiation in a runoff system. *European Journal of Political Economy*, 23(4), 1185-1189.

[16] Myerson, R. B. (1999). Theoretical comparisons of electoral systems. *European Economic Review*, 43(4), 671-697.

[17] Osborne, M. J. (1993). Candidate positioning and entry in a political competition. *Games and Economic Behavior*, 5(1), 133-151.

[18] Palfrey, T. R. (1984). Spatial equilibrium with entry. *Review of Economic Studies*, 51(1), 139-156.

[19] Roemer, J. E. (1994). A theory of policy differentiation in single issue electoral politics. *Social Choice and Welfare*, 11(4), 355-380.

[20] Xefteris, D. (2014). Mixed equilibria in runoff elections. *Games and Economic Behavior*, 87, 619–623.

5 Appendix

5.1 Symmetric r-location equilibria

Here we explore the possibility of equilibria such that more than two locations are occupied. We focus on symmetric r-location equilibria ($n(y^i) = \frac{a}{r}$ for every $i \in \{1, 2, ..., r\}$) with $r \geq 3$. Transition from two-location strategy profiles to profiles with many occupied locations naturally brings along complications in many parts of the analysis but, surprisingly, it also simplifies some specific issues. For example, when we are in a two-location profile and we consider a deviation of a player from *Out* to some location in between the two occupied ones, it is obvious that the vote-masses of all active candidates will be affected. In case we are in an r-location profile, though, with $r \geq 3$, a deviation of a player from *Out* to any location need not affect the vote-masses of all active candidates. This is so because when each occupied position is shared by at least $k + 1$ candidates, each occupied location acts as a bulkhead which isolates adjacent constituencies: if there are at least $k + 1$ candidates at each occupied location and a player enters at $y \in [y^i, y^{i+1}]$, then it is impossible that this will affect voting behavior of voters with ideal policies to the right of $\frac{y^{i+1}+y^{i+2}}{2}$ (to the left of $\frac{y^i+y^{i-1}}{2}$) and, hence, vote-masses of players located at y^{i+2} (y^{i-1}) cannot be affected.

First, we will show that like in the two-location equilibria case, the number of active candidates in an r-location equilibrium is also bounded from above and from below. Then we will fully characterize the set of symmetric r-location equilibria for every admissible F and every k-vote rule and, finally we will show that the set admissible distributions for which a k-vote rule with $k \geq 2$ admits a symmetric r-location equilibrium is non-degenerate for every $r \geq 3$. Given that to establish these results one essentially has to repeat many arguments already presented in the proofs of the two-location case, we only highlight here the necessary steps that do not directly follow from our previous analysis.

Lemma 4 *Every symmetric r-location BRE must be such that: a)* $\frac{y^i+y^{i+1}}{2} = F^{-1}(\frac{i}{r})$ *for every* $i \in \{1, 2, ..., r-1\}$ *and b)* $a \in [rk + r, 2rk]$.

Proof. The argument that supports the second part of this lemma is identical to the argument used to support its $r = 2$ version in lemma 3: if $n(y^1) \leq k$ and voters vote for all their $n(y^1)$ top-ranked candidates then a candidate located at y^1 strictly increases her vote-mass by moving towards y^2 while at the same time the vote-masses of all other active candidates either decrease or remain unaffected. Thus, in a symmetric r-location BRE it has to be the case that $n(y^1) \geq k+1 \Longrightarrow a \geq rk+r$. Moreover, if $a \geq rk + r$ then in every symmetric r-location BRE it has to be the case that each constituency is equal to $\frac{1}{r}$. That is, we have for example $F(\frac{y^1+y^2}{2}) = \frac{1}{r}$ and $F(\frac{y^2+y^3}{2}) - F(\frac{y^1+y^2}{2}) = \frac{1}{r} \Longrightarrow F(\frac{y^2+y^3}{2}) = \frac{2}{r}$, and in general $F(\frac{y^i+y^{i+1}}{2}) = \frac{i}{r} \Longrightarrow \frac{y^i+y^{i+1}}{2} = F^{-1}(\frac{i}{r})$. Finally, if $a \geq rk + r$, then it also has to be the case that $\frac{k}{a} \geq \frac{1}{2} \times \frac{1}{r} \Longrightarrow a \leq 2rk$. This is so because in a symmetric r-location BRE a non-entrant can always deviate from *Out* to a location that gives her a vote-mass arbitrarily close to $\frac{1}{2} \times \frac{1}{r}$, and in such a case at least some of the active candidates will continue to receive a vote-mass equal to $\frac{k}{a}$. Hence for such a deviation not to be profitable the presented inequality has to hold. ■

Proposition 9 *When the ideal policies of the society are distributed according to F and voting takes place according to a k-vote rule, a symmetric r-location BRE with $\frac{a}{r}$ players located at y^i for every $i \in \{1, 2, ..., r\}$ and $r \geq 3$, exists if and only if: A)* $\frac{y^i+y^{i+1}}{2} = F^{-1}(\frac{i}{r})$ *for every* $i \in \{1, 2, ..., r-1\}$, *B)* $\max\{F(y^1), 1 - F(y^r)\} \leq \frac{k}{a}$ *and C) for every* $y \in (y^i, y^{i+1})$, *where* $i \in \{1, 2, ..., r-1\}$, *we have* $rk + r \leq a < \frac{k}{F(\frac{y^{i+1}+y}{2}) - F(\frac{y^i+y}{2})}$.

Proof. To prove this proposition one can follow very similar steps to the ones in the proof of proposition 2: first, by assuming that players use such a strategy profile, we can show that any deviation of any player is unprofitable (this establishes that the three conditions of the present proposition are sufficient for equilibrium existence) and then, given that condition A and the first part of condition C have already been proved to be necessary conditions by lemma 4, we can argue that condition B and the second part of condition C are equivalently necessary for existence of an equilibrium. The only difference compared to the two-location case is that when players play according to the posited profile and a player deviates from *Out* to any y, there is at least one active candidate who receives a vote-mass equal to $\frac{k}{a}$ and no active candidate who receives more than that. Hence, the vote-mass of the player who deviates from *Out* to any y - which is: a) equal to $F(\frac{y^{i+1}+y}{2}) - F(\frac{y^i+y}{2})$ for every $y \in (y^i, y^{i+1})$, where $i \in \{1, 2, ..., r-1\}$, b) equal to $F(\frac{y^1+y}{2})$ for every $y \in [0, y^1)$, c) equal to $1 - F(\frac{y^r+y}{2})$ for every $y \in (y^r, 1]$ and d) equal to $\frac{1}{r} \times \frac{k}{\frac{a}{r}+1} = \frac{k}{a+r}$ for every $y \in \{y^1, y^2, ..., y^r\}$ - must be strictly less than $\frac{k}{a}$

for the posited profile to be a BRE and this leads to condition C which is significantly simpler than the corresponding condition of the two-location case. ■

For the statement of the next result we need to define Φ_k^r as the set of admissible distributions for which a symmetric r-location BRE exists when voting takes place under a k-vote rule.

Proposition 10 *Φ_k^r is non-degenerate if and only if $r \geq 2$ and $k \geq 2$.*

Proof. From proposition 5 we know that Φ_k^2 is non-degenerate if and only if $k \geq 2$. Hence, here we focus on Φ_k^r for $r \geq 3$. To understand why Φ_1^r cannot be non-degenerate for any $r \geq 3$ we first note that in a BRE it has to be the case that $a = 2r$ (by lemma 4) and $F(y^1) = 1 - F(y^r) = \frac{k}{a} = \frac{1}{2r}$ - otherwise one of the two active candidates at y^1 (y^r) would have incentives to move marginally to the left or to the right and win with certainty. Therefore, if Φ_1^r is non-degenerate for some $r \geq 3$, it should contain at least two admissible distribution functions F_1 and F_2 such that $F_1(x) < F_2(x)$ for every $x \in (0,1)$. We denote $y^{1,F_1} = F_1^{-1}(\frac{1}{2r})$ and $y^{i,F_1} = 2F_1^{-1}(\frac{i-1}{r}) - y^{i-1,F_1}$ for every $i \in \{2, ..., r\}$, and $y^{1,F_2} = F_2^{-1}(\frac{1}{2r})$ and $y^{i,F_2} = 2F_2^{-1}(\frac{i-1}{r}) - y^{i-1,F_2}$ for every $i \in \{2, ..., r\}$. Moreover, Φ_1^r should contain at least one admissible distribution, $\dot{F}$, such that $\dot{F}(x) = F_1(x)$ for every $x \in [0, \frac{y^{r,F_1}+y^{r-1,F_1}}{2}]$ and $\dot{F}(x) = F_2(x)$ for every $x \in [y^{r,F_2}, 1]$.[33] But this suggests that $y^{1,\dot{F}} = \dot{F}^{-1}(\frac{1}{2r}) = y^{1,F_1}$ and $y^{i,\dot{F}} = 2\dot{F}^{-1}(\frac{i-1}{r}) - y^{i-1,\dot{F}} = y^{i,F_1}$ for every $i \in \{2, ..., r\}$, and, thus, that $1 - \dot{F}(y^{r,\dot{F}}) = 1 - F_2(y^{r,F_1}) < \frac{1}{2r}$. Therefore, there is no symmetric r-location BRE with $r \geq 3$ for such an $\dot{F}$ and therefore Φ_1^r cannot be non-degenerate.

To show that Φ_k^r is non-degenerate for any $r \geq 3$ and $k \geq 2$ we consider that $k+1$ candidates share each of the r occupied locations, that $y^1 = \frac{1}{2r}$, that $y^{i+1} = 2F^{-1}(\frac{i}{r}) - y^i$ for every $i \in \{1, 2, ..., r-1\}$ and that $F(x) \in (x - \varepsilon, x + \varepsilon)$ for some positive ε. We notice that $\lim_{\varepsilon \to 0} \max\{F(y^1), 1 - F(y^r)\} = \frac{1}{2r} < \frac{2}{3r} \leq \frac{k}{rk+r}$ when $k \geq 2$ and that $\lim_{\varepsilon \to 0}[F(\frac{y^{i+1}+y}{2}) - F(\frac{y^i+y}{2})] = \frac{1}{2r} < \frac{2}{3r} \leq \frac{k}{rk+r}$ when $k \geq 2$. That is, for every $k \geq 2$ a symmetric r-location BRE with $r \geq 3$ is guaranteed to exist when the distribution of ideal policies is sufficiently uniform. The argument that takes us from this observation to the conclusion that Φ_k^r is non-degenerate when $r \geq 3$ and $k \geq 2$ is identical to the one in the proof of proposition 5. ■

5.2 Asymmetric equilibria

Finally, we generalize our characterization results in order to take in account asymmetric equilibrium configurations as well - that is, equilibria such that the number of active candidates in each occupied location is not necessarily identical. Given that the arguments that support these characterization results are essentially simple adaptations of arguments presented in propositions 2 and 9, we skip the

[33] Notice that if Φ_1^r is non-degenerate for some $r \geq 3$ then it should contain F_1 and F_2 such that $F_1(x) < F_2(x)$ for every $x \in (0,1)$ and $\frac{y^{r,F_1}+y^{r-1,F_1}}{2} < y^{r,F_2}$.

proofs and only discuss some of their features that deserve our attention. We start by describing all two-location equilibria of the game.

Proposition 11 *When the ideal policies of the society are distributed according to F and voting takes place according to a k-vote rule, a two-location BRE with $n(y^1)$ active candidates at y^1 and $n(y^2)$ active candidates at y^2 $(n(y^1)+n(y^2)=a)$ exists if and only if: A)* $\frac{y^1+y^2}{2}=F^{-1}(\frac{n(y^1)}{a})$ *and* $\min\{n(y^1),n(y^2)\}\geq k+1$*, B)* $\max\{F(y^1),1-F(y^2)\}\leq\frac{k}{a}$ *and C) for every* $y\in(y^1,y^2)$ *we have* $F(\frac{y^2+y}{2})-F(\frac{y^1+y}{2})<\max\{F(\frac{y^1+y}{2})\frac{k}{n(y^1)},[1-F(\frac{y^2+y}{2})]\frac{k}{n(y^2)}\}$*.*

For k small, existence of asymmetric two-location equilibria is very hard when distribution of ideal policies is sufficiently symmetric. Consider, for example, that F is uniform and that $a=n(y^1)+n(y^2)$ players enter the race, with $n(y^1)<n(y^2)$. If this is a BRE then it should be the case that: a) $\frac{y^1+y^2}{2}\frac{k}{n(y^1)}=(1-\frac{y^1+y^2}{2})\frac{k}{n(y^2)}=\frac{k}{a}$ because otherwise not all entrants would have a positive election probability, b) $\frac{k}{a}\geq\max\{y^1,1-y^2,\frac{y^2-y^1}{2}\}$ because otherwise a player could deviate from *Out* to some $y\in[0,1]$ and win with probability one (this is a necessary, not a sufficient, condition for the posited profile to be an equilibrium) and c) $n(y^1)\geq k+1$ because otherwise $(n(y^1)\leq k)$ if voters use all the votes in their disposal (full-voting) a player located at y^1 could deviate to the right and strictly increase her vote-mass and win with certainty. These conditions do not hold at the same time, for example, for $k=2,3,4$ and, hence, when F is uniform (or nearly uniform) and the number of votes at voters' disposal is small, no asymmetric two-location equilibrium exists.

When k is large, though, an asymmetric equilibrium is guaranteed to exist for a non-degenerate set of voters' distributions. Consider, for example, that F is nearly uniform and that $2k+3$ candidates enter the race ($k+1$ locate at $y^1=0.4$ and $k+2$ locate at y^2 with $F(\frac{y^1+y^2}{2})=\frac{1+k}{3+2k}$). Then if k is sufficiently large (in specific, if $k\geq 12$): a) each active candidate receives a vote-mass equal to $\frac{k}{2k+3}$ and, hence, each is elected with equal probability, b) no candidate has incentives to deviate from *Out* to any $y\in[0,1]$ and c) no active candidate has any incentive to deviate to any other location.

Finally, we characterize all r-location equilibria for every k-vote rule with $k\geq 2$. For economy of space, one is referred to Osborne (1993) for the $k=1$ case.

Proposition 12 *When the ideal policies of the society are distributed according to F and voting takes place according to a k-vote rule with $k\geq 2$, an r-location BRE with $n(y^i)$ players located at y^i, for $i\in\{1,2,...,r\}$ and $r\geq 3$ $(\sum_{i=1}^{r}n(y^i)=a)$, exists if and only if: A)* $\frac{y^i+y^{i+1}}{2}=F^{-1}(\frac{\sum_{k=1}^{i}n(y^k)}{a})$ *for every* $i\in\{1,2,...,r-1\}$ *and* $\min\{n(y^1),n(y^2),...,n(y^r)\}\geq k+1$*, B)* $\max\{F(y^1),1-F(y^r)\}\leq\frac{k}{a}$ *and C) for every* $y\in(y^i,y^{i+1})$*, where* $i\in\{1,2,...,r-1\}$*, we have* $F(\frac{y^{i+1}+y}{2})-F(\frac{y^i+y}{2})<\frac{k}{a}$*.*

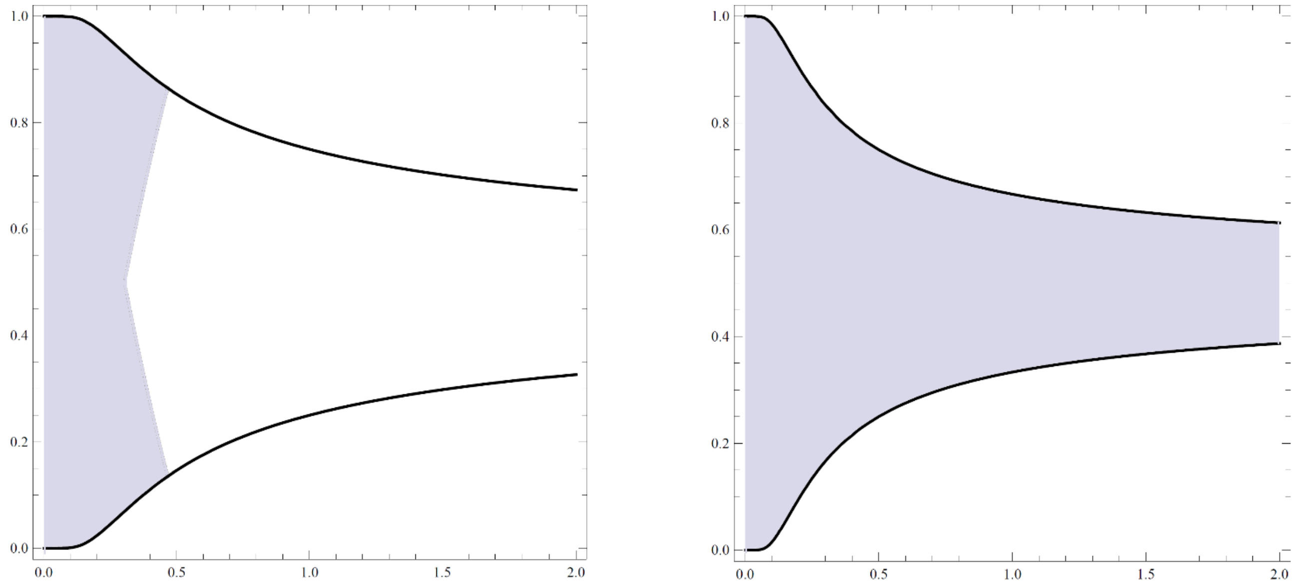
Here the only thing that deserves some attention is why $n(y^i)\geq k+1$ for every $i\in\{2,...,r-1\}$. In previous proofs we only needed to show that in a BRE $\min\{n(y^1),n(y^r)\}\geq k+1$ and the argument was straightforward. If, for example, $n(y^1) \leq k$ and voters use all their k votes (full-voting) then an active candidate located at y^1 has incentives to move marginally to the right as such a motion unambiguously increases her vote-mass. But why should it be the case that $n(y^i) \geq k+1$ for every $i \in \{2, ..., r-1\}$ when $k \geq 2$?

As we know in every BRE we must have $n(y^1) \geq k+1$. Consider that, for some $k \geq 2$, there exists a BRE such that $n(y^2) \in \{2, ..., k\}$ and that voters behave in the following way: they always vote for their top-ranked candidate(s) and, in case they have some spare votes, they vote for the candidate who offers the third most-leftist platform as long as she is distinctly-positioned (that is, as long as she does not share her position with anybody else).[34] If we are in equilibrium it must be the case that all candidates get the same vote-mass. If a candidate located at y^2 deviates to $y^3 - \varepsilon$ then: a) her vote-mass strictly increases (she is voted at least by those voters who were voting for her when she was at y^2 and by the voters with ideal policies between $\frac{y^2+y^3}{2}$ and $y^3 - \frac{\varepsilon}{2}$) and b) the vote-mass of each of the other active candidates either remains unchanged or decreases. This is so, because all other candidates are voted only by voters who rank them first. That is, there can be no BRE with $n(y^2) \in \{2, ..., k\}$ when $k \geq 2$.

Now consider that, for some $k \geq 2$, there exists a BRE such that $n(y^2) = 1$ and that voters behave in the following way: they always vote for their top-ranked candidate(s) and, in case the have some spare votes, they vote for the candidate who offers the second most-leftist platform as long as she is distinctly-positioned. Since in a BRE every candidate receives exactly the same vote-mass, these assumptions suggest that $n(y^i) \geq k$ for every $i \neq 2$. If this were not true, a candidate located at $y^i \neq y^2$ would be voted only by the constituency of y^i while the candidate located at y^2 would be voted both by the constituency of y^2 and by the constituency of y^i. This suggests that in a BRE with $n(y^2) = 1$ the candidate located at y^2 would be voted only her constituency and, hence, the vote-mass of each active candidate should be equal to the constituency of y^2. But since $k \geq 2$, a player could deviate from *Out* to y^2 and get voted by all the constituency of y^2 without affecting the vote-masses of all other active candidates. That is, a player deviating from *Out* to y^2 would get a positive election probability and, subsequently, no BRE with $n(y^2) = 1$ may exist when $k \geq 2$.

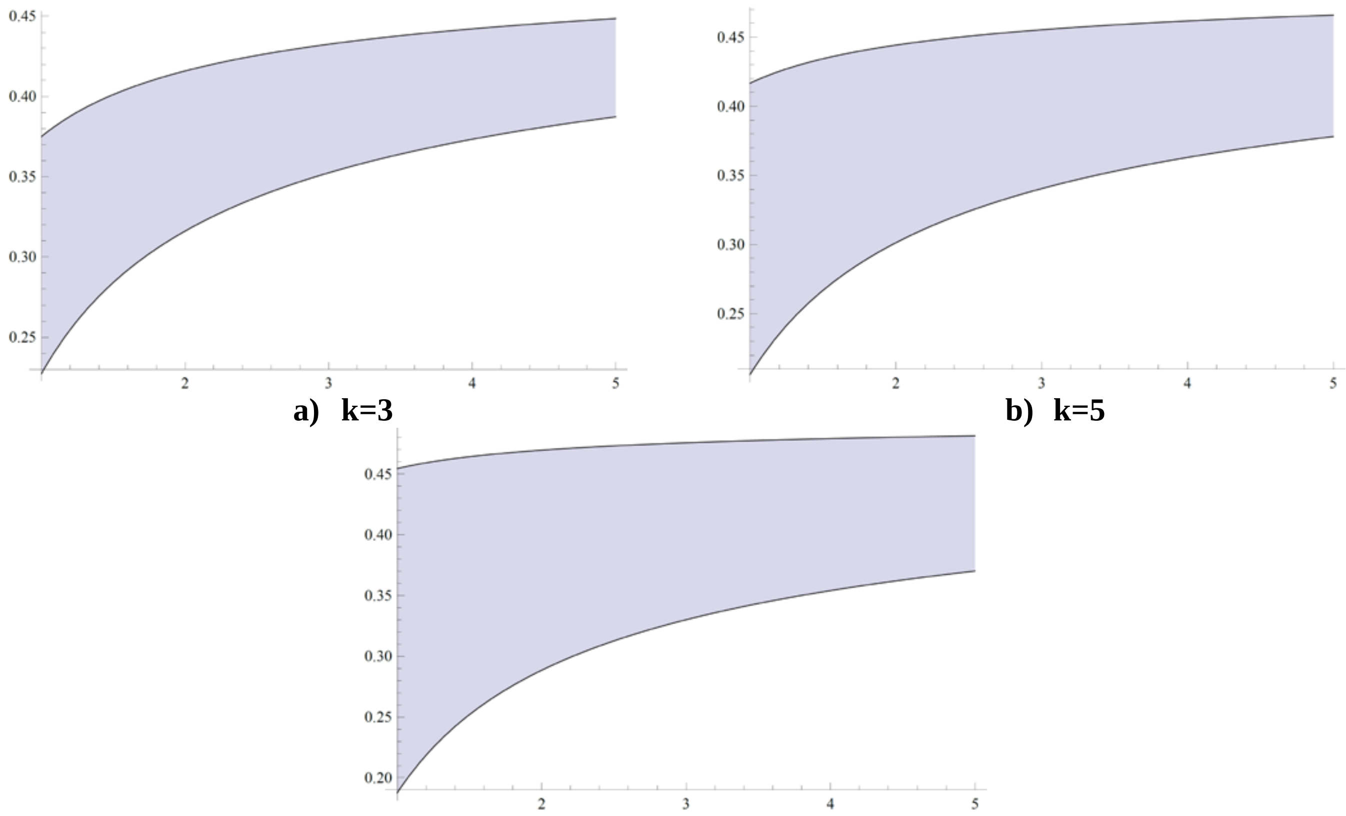
Summing up all the above, we conclude that in a BRE it must be the case that $n(y^2) \geq k+1$. Applying similar arguments we can establish that in a BRE it must be the case that every occupied location is shared by at least $k+1$ candidates.

[34] This voting behavior might not be the most intuitive one but it is consistent with minimal sincerity.



a) k=1 **b) k=2**

Figure 1. Locations for which condition B of proposition 3 holds (shaded area) as a function of the shape parameter, β, of a symmetric beta distribution when $k=1$ and $k=2$ (the black curves represent $\hat{y}$ and $2m - \hat{y}$).



a) k=3 **b) k=5**

c) k=10

Figure 2. Admissible values of y^1 (shaded area) as a function of the shape parameter, β, of a unimodal symmetric beta distribution when $k=3$, $k=5$ and $k=10$ respectively.