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Strategic vote trading in power-sharing systems

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Strategic vote trading in power-sharing systems

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Abstract

This paper studies decentralized vote trading in a power-sharing system that follows the rules of strategic market games. In particular, we study a two-party election, in which prior to the voting stage voters are free to trade votes for money. Voters hold private information about both their ordinal and cardinal preferences, whereas their utilities are proportionally increasing in the vote share of their favorite party. In this framework we prove generic existence of a unique full trade equilibrium (an equilibrium in which nobody refrains from vote trading). We moreover argue that vote trading in such systems unambiguously improves voters' welfare.

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Keywords: Vote trading; Strategic market games; Power sharing.

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1 Introduction

Despite many critiques on ethical and philosophical grounds, vote markets have been a research issue for economists for quite a long time, because they reflect voters' intensity of preferences over policies in the same way that regular markets reflect consumers' preferences over commodities. However, vote-trading models are not immune to criticism on economics' grounds as well, because of votes' distinct characteristics when compared to other commodities.[1] For example, vote trading embodies externalities on third parties and in many occasions leads to undesirable welfare results. Furthermore, in many settings the existence of equilibrium faces serious problems that have prevented the wide adoption of a general model of decentralized vote trading. This paper contributes to the literature on vote markets by departing from standard approaches in a dual manner; first, by analyzing the market for votes in a power-sharing system (instead of the majoritarian framework) and second, by introducing to a decentralized vote-trading model the rules of strategic market games.

The first distinctive feature of our analysis is that decision-making power is not attributed as a whole to the plurality winner, but is distributed among competing parties in a proportional manner. In other words, a voter's utility is proportionally increasing in the vote share of her favorite party and hence she should prefer that this party receives as many votes as possible. Similar frameworks have been employed to analyze a variety of issues in political economics literature[2] and provide us with all the necessary tools to avoid the deficiencies associated with simple plurality rule. In particular, our model does not involve severe discontinuities in voters' utilities and hence, at the right price, there is always demand for any single vote offered in exchange for money. Moreover, our analysis is relevant to many modern democratic societies in which policy outcomes are not determined by the plurality winner but instead are shaped by all competing parties. Indeed, several parliamentary systems are characterized by grand coalition governments and proportional electoral rules (Switzerland, Belgium and Denmark are only a few examples of countries that are governed in this manner). In such power-sharing[3]

[1] See, for instance, Ferejohn (1974), Philipson and Snyder (1996), Casella, Llorente-Saguer, and Palfrey (2012) for a review of the issue.

[2] See, for instance, Lijphart (1984), Austen-Smith and Banks (1988), Ortuño-Ortín (1997), Grossman and Helpman (1999), Baron and Diermeier (2001), Morelli (2004), Llavador (2006), Sahuguet and Persico (2006), Merrill and Adams (2007), De Sinopoli and Iannantuoni (2007), Herrera, Morelli, and Palfrey, (2014), Saporiti (2014), Iaryczower and Mattozzi (2013) and Matakos, Troumpounis, and Xefteris (2015) among others.

[3] The term is due to Lijphart (1984) and describes systems such that no group is left out of decision making

settings, all parties have significant role in decision making, with the higher the number of votes that a party receives the more influence it has on the implemented policy.[4] Hence, the whole distribution of votes is crucial for the determination of policies. Given the significant extent to which power-sharing procedures are used, we believe that it is imperative to study how vote trading operates under such conditions. In addition, given the debate about the ideal governance system,[5] where both types of democracy (power-sharing and majoritarian) have been contested in terms of their applicability and performance, our paper adds a new aspect (the efficiency of vote trading) into the discussion.

Our approach also employs alternative methodological tools, as vote trading is modeled as a strategic market game. Strategic market games are a class of non-cooperative games that are characterized by an explicit market mechanism, which maps individuals' actions to trades and prices (see, for example, Shapley and Shubik, 1977; Peck, Shell, and Spear, 1992). Hence, our approach, instead of assuming price-taking behavior with all its resultant inadequacies, allows individuals to strategically trade votes for money, with their market actions having a clear impact on prices. Given the variety of issues in political economics in which vote trading has been used for, we believe that the development of a strategic version of the trading process is very important as it helps us to evaluate which of the conclusions of the literature, if any, carry over to non-competitive markets. Moreover, the rules of these games allow us to study vote trading using a conventional solution concept (Bayesian Nash equilibrium) that is more familiar to voting literature.[6] Earlier papers on decentralized vote trading (Casella, Llorente-Saguer, and Palfrey, 2012; Casella, Palfrey, and Turban, 2014; Casella and Turban, 2014) employed a price-taking approach and, hence had to adopt solution concepts that associate with such a framework (for example, ex ante competitive equilibrium in Casella, Llorente-Saguer, and Palfrey, 2012).

Combining the above features we study a game, in which prior to the voting stage voters

(grand coalitions) and the power of each group is equal to its size (proportional electoral rule).

[4]In fact, following Iaryczower and Mattozzi (2013), the implemented policy may be simply interpreted as the result of a probabilistic compromise between the competing alternatives, with the probability of implementation of each policy alternative being equal to its vote share.

[5]A debate that started with the seminal work of Lijphart (1968), who is the primary advocate of the superiority of power-sharing regimes over majoritarian ones.

[6]The use of game-theoretic tools also associates our analysis with the literature of vote buying through centralized bargaining (e.g., Baron and Ferejohn, 1989) and decentralized bargaining (e.g., Iaryczower and Oliveros, 2016).

are free to trade votes for money. We assume that voters are heterogeneous not just in terms of ordinal preferences but also as far as cardinal preferences are concerned and that a voter's preferences (both ordinal and cardinal) are her private information. We moreover consider that if a voter decides to sell her vote she has to sell it as a whole, and that if a voter decides to buy votes she has to bid a predetermined amount of money - which we fix to one dollar without loss of generality. These assumptions, combined with the market mechanism of strategic market games, suggest that the price of a vote is equal to the ratio of the number of vote buyers over the number of vote sellers. Of course, since voters' preferences are private information at the vote-trading stage and all players have to decide simultaneously whether to buy, sell or do nothing it is straightforward that when they make their choices they are uncertain regarding how many of them choose what and hence they are uncertain about the price of a vote. In this setup, we wish to address the following questions: is a Bayesian Nash equilibrium with positive trade guaranteed to exist? Does the persistent small number of vote buyers in equilibrium as reported in the price-taking literature (for example, in Casella, Llorente-Saguer, and Palfrey, 2012 there are at most two vote buyers and everybody else is a vote seller) carry over to this strategic trading context? What are the welfare implications of these equilibrium outcomes, and in particular, is the vote trading option more efficient than the no-trade alternative, in which individuals simply cast their votes?

Our results give explicit answers to these questions. Initially, we establish in Proposition 1 that vote trading occurs in every almost strict Bayesian Nash equilibrium.[7] In each such equilibrium, voters who are relatively indifferent between the two parties offer their votes in exchange for money, whereas voters who have relatively more asymmetric valuations of the parties offer their money in exchange for votes. Next we demonstrate that there is a *unique* full trade equilibrium, that is an equilibrium in which all individuals engage in vote trading (i.e., no single voter prefers to abstain from it). Indeed, Proposition 2 defines for this equilibrium the unique preference intensity parameter that groups individuals into vote sellers and vote buyers. Finally, given that vote trading is preferred to the *just cast your vote* alternative by all individuals, we show in Proposition 3 that allowing for vote trading *unambiguously* improves social welfare. Therefore, it should receive law-makers' attention.

[7]A Bayesian Nash equilibrium in which almost all players' types strictly prefer their equilibrium strategy to any other.

It should be noted that our results offer a novel positive perspective on decentralized vote trading as they stand in profound contrast to existing ones in the literature. For instance, let us consider the pioneering study of Casella, Llorente-Saguer, and Palfrey (2012). That paper considers an election under a simple plurality rule, where individuals place stochastic demands for votes. In that setup, the market for votes clears ex ante in expectation, whereas ex post an anonymous rationing rule determines which trades are executed in case of imbalance between the realized demands and supplies. The questions of Casella, Llorente-Saguer, and Palfrey (2012) are similar to ours, their results though are quite different as they demonstrate that an equilibrium with active vote trading is always characterized by small number of buyers (i.e., only the two voters with the most intense preferences are willing to buy votes and all other voters sell their votes). This fact leads to dictatorship, as a single voter acquires a majority position. Moreover, due to dictatorship, if the number of voters is large (or if the distribution of valuations is not very skewed) vote trading is welfare decreasing when compared to plurality rule without vote trading. Similar results in decentralized competitive vote markets also appear in Casella and Turban (2014) and Casella, Palfrey, and Turban (2014), which consider two groups of unequal size supporting different policies and show that only the most intense voter of each group demands votes, with the minority's voter being more aggressive. As a result, the minority's favorite policy is implemented with higher probability than the efficient level, which suggests that vote trading can be welfare inferior to simple plurality rule without vote trading.

The strong differences though between the present and earlier approaches to decentralized vote trading should come at no surprise because: a) the transition from price-taking to strategic behavior has non-trivial implications on equilibrium outcomes in most trade environments and b) power-sharing systems involve a completely distinct set of strategic incentives as far as voters' behavior is concerned compared to simple plurality rule. In order to fully apprehend the source of these differences, one should also have an idea how plurality performs in our vote-trading environment. For that purpose, in the last part of the paper we explore features of the equilibrium outcomes under plurality rule. As a complement to our results, we demonstrate that a full trade equilibrium is not guaranteed to exist in majoritarian settings, whereas the welfare results are no longer unambiguous as in our original approach (in fact, vote trading could be welfare reducing).

We proceed to develop the model in Section 2. Next we present the results in Section 3. The differences with plurality rule follow in Section 4. In Section 5 we summarize our conclusions and we discuss some possible extensions.

2 The model

We consider a society $Q = \{1, 2, ..., n\}$ of $n > 2$ voters and two fixed alternatives, A and B. Each voter $i \in Q$ is characterized by her ordinal preferences, $t_i \in \{(A \succ B), (B \succ A)\}$, and an intensity parameter, $w_i \in [0, +\infty)$.[8] Each voter is assumed to have one vote (which she can trade for money) and one unit of money (numeraire). If we denote by $v_A \in [0, 1]$ the vote share of alternative A and by $v_B = 1 - v_A$ the vote share of alternative B, then the utility of voter i after the election is given by

$$u_i = \begin{cases} v_A \times w_i + m_i & \text{if } t_i = (A \succ B) \\ v_B \times w_i + m_i & \text{if } t_i = (B \succ A) \end{cases},$$

where $m_i \geq 0$ is the amount of money voter i ends up having. Notice that this formulation of voters' preferences is perfectly compatible with other papers studying power-sharing systems (see, for example, Herrera, Morelli, and Palfrey, 2014; Iaryczower and Mattozzi, 2013). What we do is simply to normalize the utility one receives from her least preferred party to zero. As it will be evident in the following section, this is very helpful in conducting a tractable analysis and is obviously without any loss of generality.

Each voter $i \in Q$ knows t_i and w_i but is uncertain about the ordinal preferences and the intensity parameters of her fellow citizens. The beliefs of voter $i \in Q$ regarding the ordinal preferences of $j \in Q - \{i\}$ are modelled by a Bernoulli distribution with parameter $\frac{1}{2}$ and support $\{(A \succ B), (B \succ A)\}$ and beliefs of voter $i \in Q$ regarding the intensity parameter of $j \in Q - \{i\}$ are given by an absolutely continuous distribution F with support $[0, +\infty)$ and which is twice differentiable in its support. The symmetric environment (each voter supports each of the two parties with equal probability and all intensity parameters are drawn from the

[8] When $t_i = (A \succ B)$ and $w_i = 0$ or when $t_i = (A \prec B)$ and $w_i = 0$, voter i is essentially indifferent between the two alternatives.

same distribution) makes feasible the comparison with the welfare results in Casella, Llorente-Saguer, and Palfrey (2012), which is the only other paper in the literature that considers private information regarding both ordinal and cardinal preferences.[9]

As far as the timing of the game is concerned we make the following assumptions: initially vote trading takes place; next players who haven't sold their votes vote in order to maximize their expected utilities, while players who sold their votes remain inactive; finally the payoffs of all players are computed. In particular, as far as the vote-trading stage is concerned, we assume that each player chooses an action x_i from the set $\{s, b, y\}$ where s stands for "sell vote", b stands for "buy votes" and y stands for "neither sell nor buy votes". Players who choose action s sacrifice their whole vote and players who choose b sacrifice the whole unit of money that they have. That is, individuals are not free to choose the quantities they submit for exchange and hence the submitted quantities of votes and money have to be integers.[10] If we denote by B the set of players who choose action b and by S the set of players who choose action s then, following the trading technology of strategic market games and conditional on both B and S being non-empty, the amount of money that is assigned to each player who chose s is

$$p = \frac{\#B}{\#S}$$

and the amount of extra votes that are assigned to each player who chose b is

$$h = \frac{\#S}{\#B}.$$

In our framework, votes are perfectly divisible and, hence, a player might end up having a non-integer number of votes. Notice that we study a model of power sharing - all that matters is the share of votes that each alternative receives - and hence voting is merely an attribution

[9] Casella, Llorente-Saguer, and Palfrey (2012) find that vote trading has negative impact on voters' welfare, and this result in derived in a symmetric environment like the present paper.

[10] Requiring that all purchasing bids are of the same monetary amount is made to ensure tractability of our equilibrium analysis: allowing different players bid different amounts would make the analytical computation of expected utilities implausible. Notice though that this assumption does not interfere in any way with the main distinctive feature of vote markets compared to markets for standard goods (the utility of non-traders in vote markets is affected by traders' decisions, whereas no such externalities arise in standard markets). In our framework it is clear that the expected vote share of a non-trader's preferred party - and subsequently her utility - depends, on the relationship between type A and type B sellers/buyers and hence the vote market distinctiveness is preserved intact.

of additional weight to one's preferred alternative. That is, there is no reason at all why these weights cannot be non-integers. What we should stress though is that the above is strictly conditional on both B and S being non-empty. When at least one of these sets is empty then *no trade takes place* and all players keep their money and their vote.

Since the behavior of players who can vote (the ones who haven't sold their votes) is completely unambiguous - attributing all votes that one has to her preferred alternative is a dominant strategy - we essentially have an one-shot game and, hence we define an equilibrium only in terms of players' strategies and beliefs in the vote-trading stage of the game. The obvious solution concept for such one-shot games of private information with a continuum of types is Bayesian Nash equilibrium (BNE) in pure strategies. We will focus on almost strict BNE in pure strategies, that is on equilibria such that a measure one of players' types strictly prefer their equilibrium strategy to any other. Moreover, given the symmetry assumption, we focus on equilibria in which players of the same preference intensity choose the same vote-trading strategy (irrespective of whether they prefer alternative A or B). Hence, when we use the term equilibrium we refer to this particular subset of BNE.

3 Results

We directly proceed to the statement of the formal results of the paper.

Proposition 1 *In every equilibrium trade occurs with positive probability.*

Proof. Assume that there exists an almost strict BNE such that trade does not occur with positive probability. If in such an equilibrium a measure zero of types chooses b and a non-degenerate measure, $z_s \in (0,1]$, of types chooses s then there exists a $\hat{w}$ such that all players with $w_i > \hat{w}$ are better off choosing b than any other action. This is so because in such a case the expected utility of a player $i \in Q$ with intensity parameter w_i when choosing s or y is given by

$$Eu_i(s) = Eu_i(y) = (\tfrac{1}{2} + \tfrac{1}{2n})w_i + 1$$

while when choosing b it is given by

$Eu_i(b) = (1 - z_s)^{n-1}[(\frac{1}{2} + \frac{1}{2n})w_i + 1] + \sum_{k=1}^{n-1}[\binom{n-1}{k} z_s^k (1 - z_s)^{n-1-k} E(v_{J_i} \mid x_i = b,\ \#B = 1$ and $\#S = k)]w_i$

where $J_i = A$ if $t_i = (A \succ B)$ and $J_i = B$ if $t_i = (B \succ A)$.

We notice that

$$E(v_{J_i} \mid x_i = b,\ \#B = 1 \text{ and } \#S = k) = \frac{k + 1 + \frac{1}{2}(n - k - 1)}{n}$$

and hence we get that

$$Eu_i(b) = (1 - z_s)^{n-1} + \sum_{k=0}^{n-1}[\binom{n-1}{k} z_s^k (1 - z_s)^{n-1-k} \frac{k + 1 + \frac{1}{2}(n - k - 1)}{n}]w_i$$

which, by applying the Binomial theorem, can be shown[11] to be equivalent to

$$Eu_i(b) = (1 - z_s)^{n-1} + \frac{w_i(1 + n + (n - 1)z_s)}{2n}.$$

Standard algebraic manipulations yield that for every $w_i > \hat{w} = \frac{2n((1-z_s)^n + z_s - 1)}{(n-1)(z_s-1)z_s}$ (which is strictly positive for any $n > 1$ and $z_s > 0$)[12] it is the case that $Eu_i(b) > Eu_i(s) = Eu_i(y)$. Obviously, the measure of types for which $w_i > \hat{w}$ is equal to $1 - F(\hat{w})$ and it is hence non-degenerate for any $z_s \in (0, 1]$. Thus, it is not possible that in an almost strict BNE a measure zero of types chooses b and a positive measure of types chooses s. One can similarly show that in an almost strict BNE it is not possible that a measure zero of types chooses s and a positive measure of types chooses b. Finally, it is trivial to see that no trade (a measure one of players' types choose y) is a BNE. If a measure zero of players' types is expected to choose s (b) then nobody strictly prefers b (s) over y. In fact every player is absolutely indifferent among all actions. Precisely because all players' types are indifferent among all actions, no trade is not an almost strict BNE. These prove that in every almost strict BNE it must be the case that trade occurs with positive probability. ■

The above proposition establishes that vote trading takes place in every equilibrium; it does not guarantee that an equilibrium actually exists. The next proposition does that by proving existence of a unique full trade equilibrium for every possible parameter values. In a full trade

[11] Notice that $\binom{n-1}{k}k = \binom{n-2}{k-1}(n-1)$ and hence $\sum_{k=0}^{n-1}[\binom{n-1}{k} z_s^k (1 - z_s)^{n-1-k} \frac{k + 1 + \frac{1}{2}(n - k - 1)}{n}] = \frac{1}{2n}\sum_{k=0}^{n-1}[\binom{n-1}{k} z_s^k (1-z_s)^{n-1-k} k] + \frac{n+1}{2n}\sum_{k=0}^{n-1}[\binom{n-1}{k} z_s^k (1-z_s)^{n-1-k}] = \frac{1}{2n}\sum_{k=1}^{n-1}[\binom{n-1}{k} z_s^k (1-z_s)^{n-1-k} k] + \frac{n+1}{2n} = \frac{n-1}{2n}\sum_{k=1}^{n-1}[\binom{n-2}{k-1} z_s^k (1 - z_s)^{n-1-k}] + \frac{n+1}{2n} = \frac{n-1}{2n} z_s \sum_{h=0}^{n-2}[\binom{n-2}{h} z_s^h (1 - z_s)^{n-2-h}] + \frac{n+1}{2n} = \frac{n-1}{2n} z_s + \frac{n+1}{2n}$.

[12] For $z_s \to 1$ we have that $\hat{w} \to \frac{2n}{n-1} > 0$.

equilibrium almost no one refrains from vote trading (a measure one of players' types choose either to sell or to buy votes) and hence only players who decided to buy votes actually get to vote. This means that each of these voters carries as many votes as any other and thus the vote-trading game that we analyze essentially leads to a pay-to-vote mechanism: players who pay one dollar get the right to vote and players who do not pay to vote get an equal share of the amount gathered by those who paid to vote.

Proposition 2 *For any admissible F there exists a unique full trade equilibrium and it is such that all players with intensity parameters smaller than w''' sell their votes and all players with intensity parameters larger than w''' buy votes, where w''' is uniquely defined by $2n([1-F(w''')]^n - 1) = w'''F(w''')^n - (2n+w''')F(w''')$.*

Proof. Step 1 We first argue that in every almost strict BNE there exists: a) w' such that all players with $w_i < w'$ are better off choosing s than y and all players with $w_i > w'$ are better off choosing y than s and b) w'' such that all players with $w_i > w''$ are better off choosing b than y and all players with $w_i < w''$ are better off choosing y than b.

Since in every almost strict BNE trade occurs with positive probability it follows that a measure $z_s > 0$ of types choose s and a measure $z_b > 0$ of types choose b. If this is the case, then the expected utility of a player $i \in Q$ with intensity parameter w_i when choosing s is given by

$$Eu_i(s) = (1-z_b)^{n-1}[(\tfrac{1}{2}+\tfrac{1}{2n})w_i + 1] + [1-(1-z_b)^{n-1}][\tfrac{1}{2}w_i + E(p \mid x_i = s \text{ and } \#B > 0) + 1]$$

when choosing b it is given by

$$Eu_i(b) = (1-z_s)^{n-1}[(\tfrac{1}{2}+\tfrac{1}{2n})w_i + 1] + [1-(1-z_s)^{n-1}]E(v_{J_i} \mid x_i = b \text{ and } \#S > 0)w_i$$

while when choosing y it is given by

$$Eu_i(y) = (\tfrac{1}{2}+\tfrac{1}{2n})w_i + 1$$

where $J_i = A$ if $t_i = (A \succ B)$ and $J_i = B$ if $t_i = (B \succ A)$.

We observe that all the above expected utilities are linear functions of intensity parameter w_i. Hence, since $E(p \mid x_i = s \text{ and } \#B > 0) > 0$ we have that all players with $w_i < w' =$

$2nE(p \mid x_i = s$ and $\#B > 0)$ are better off choosing s than y and all players with $w_i > w'$ are better off choosing y than s. Similarly, since $E(v_{J_i} \mid x_i = b$ and $\#S > 0) > (\frac{1}{2} + \frac{1}{2n})$ we have that all players with $w_i < w'' = 1/[E(v_{J_i} \mid x_i = b$ and $\#S > 0) - (\frac{1}{2} + \frac{1}{2n})]$ are better off choosing y than b and all players with $w_i > w''$ are better off choosing b than y.

Step 2 In a full trade almost strict BNE it must be the case that $0 < w'' \leq w'$ and hence that $z_s = F(w''') = 1 - z_b$ where $w''' > 0$ is the solution of $Eu_i(s) = Eu_i(b)$. This is so because if we had $0 < w' < w''$ then a set of types of positive measure $F(w'') - F(w') > 0$ would prefer action y over b and over s.

Assume that such an equilibrium exists. Then all players with $w_i > w'''$ choose b and that all players with $w_i < w'''$ choose s. If the posited behavior is an equilibrium then no player should have incentives to deviate. In this case we have that

$$E(p \mid x_i = s \text{ and } \#B > 0) = \frac{\sum_{k=0}^{n-2}[\frac{n-1-k}{k+1}\binom{n-1}{k}F(w''')^k[1-F(w''')]^{n-1-k}]}{1-F(w''')^{n-1}}$$

and

$$E(v_{J_i} \mid x_i = b \text{ and } \#S > 0) = \frac{\sum_{k=0}^{n-2}[(\frac{1}{2}+\frac{1}{2+2k})\binom{n-1}{k}[1-F(w''')]^k F(w''')^{n-1-k}]}{1-[1-F(w''')]^{n-1}}.$$

The above help us write the expected utilities in a much more convenient form.[13] That is,

$$Eu_i(s) = \frac{2-2[1-F(w''')]^n + w_iF(w''')}{2F(w''')} + \frac{w_iF(w''')^{n-1}}{2n}$$

and

$$Eu_i(b) = \frac{w_i[nF(w''') + F(w''')^n - n - 1]}{2n[F(w''') - 1]} + [1-F(w''')]^{n-1}.$$

We observe that: a) $Eu_i(s) = Eu_i(b)$ if and only if $w_i = \frac{2n([1-F(w''')]^n + F(w''')-1)}{F(w''')^n - F(w''')}$, b) $Eu_i(s) = Eu_i(y)$ if and only if $w_i = \frac{2n([1-F(w''')]^n + F(w''')-1)}{F(w''')^n - F(w''')}$ and c) $Eu_i(b) = Eu_i(y)$ if and only if $w_i = \frac{2n([1-F(w''')]^n + F(w''')-1)}{F(w''')^n - F(w''')}$. These show that in a full trade almost strict BNE it must be the case

[13]Notice that $\frac{n-1-k}{k+1}\binom{n-1}{k} = \binom{n-1}{k+1}$ and, by applying the Binomial theorem, we get $\sum_{k=0}^{n-2}[\frac{n-1-k}{k+1}\binom{n-1}{k}F(w''')^k[1-F(w''')]^{n-1-k}] = \sum_{k=0}^{n-2}[\binom{n-1}{k+1}F(w''')^k[1-F(w''')]^{n-1-k}] = \frac{1-F(w''')}{F(w''')}\sum_{k=0}^{n-2}[\binom{n-1}{k+1}F(w''')^{k+1}[1-F(w''')]^{n-1-k-1}] = \frac{1-F(w''')}{F(w''')}\sum_{h=1}^{n-1}[\binom{n-1}{h}F(w''')^h[1-F(w''')]^{n-1-h}] = \frac{1-F(w''')}{F(w''')}[\sum_{h=0}^{n-1}[\binom{n-1}{h}F(w''')^h[1-F(w''')]^{n-1-h}] - [1-F(w''')]^{n-1}] = \frac{1-F(w''')}{F(w''')}[1-[1-F(w''')]^{n-1}] = \frac{1-[1-F(w''')]^n - F(w''')}{F(w''')}$. Similarly, it can be shown that $\sum_{k=0}^{n-2}[(\frac{1}{2}+\frac{1}{2+2k})\binom{n-1}{k}[1-F(w''')]^kF(w''')^{n-1-k}] = \frac{1}{2}\sum_{k=0}^{n-1}[\binom{n-1}{k}[1-F(w''')]^kF(w''')^{n-1-k}] + \frac{1}{2}\sum_{k=0}^{n-1}[\frac{1}{k+1}\binom{n-1}{k}[1-F(w''')]^kF(w''')^{n-1-k}] - (\frac{1}{2}+\frac{1}{2n})[1-F(w''')]^{n-1} = \frac{1}{2} + \frac{1-F(w''')^n}{2n[1-F(w''')]} - (\frac{1}{2}+\frac{1}{2n})[1-F(w''')]^{n-1}$.

that $0 < w' = w'' = w''' = \frac{2n([1-F(w''')]^n + F(w''') - 1)}{F(w''')^n - F(w''')}$. So if we show the existence of a unique $w''' > 0$ such that $w''' = \frac{2n([1-F(w''')]^n + F(w''') - 1)}{F(w''')^n - F(w''')}$, we essentially establish both existence and uniqueness of a full trade almost strict BNE.

We define $R(x) = \frac{2n((1-x)^n + x - 1)}{x^n - x}$ and observe that: a) $\lim_{x \to 0} R(x) = 2n(n-1) > 0$, b) $\lim_{x \to 1} R(x) = 2n/(n-1) \in (0, \lim_{x \to 0} R(x))$ and $\frac{\partial R(x)}{\partial x} < 0$ for every $x \in (0,1)$. That is, $w''' = \frac{2n([1-F(w''')]^n + F(w''') - 1)}{F(w''')^n - F(w''')}$, which may be rewritten as $2n([1-F(w''')]^n - 1) = w''' F(w''')^n - (2n + w''') F(w''')$, is guaranteed to have a unique solution for every admissible F and hence the game admits a unique full trade almost strict BNE. ■

As far as comparative statics of this equilibrium are concerned we notice that the threshold value w''' is "roughly" increasing in n. By "roughly" increasing we mean that w''' becomes arbitrarily large when n takes arbitrarily large values, whereas for small values of n their relationship need not be monotonic. This is illustrated in Figure 1, where we plot w''' for various values of n considering that: a) F is a log-normal distribution with parameters zero and one (that is, with mean $e^{1/2}$ and variance $(e-1)e$); and b) F is Pareto distribution with scale parameter one, shape parameter one and an adjusted support $[0, +\infty]$ to coincide with our model's assumptions.

[Insert Figure 1 about here]

Observe that as the population becomes arbitrarily large it must be the case that the intensity threshold must become arbitrarily large too. If the equilibrium threshold is below some $w^{\max} > 0$ for every $n \in \mathbb{R}$, then, in expected terms, the vote buyers should represent a share of the total population at least as large as $1 - F(w^{\max})$. If vote buyers represent such a non-degenerate measure, it should be the case that their expected absolute number should converge to infinity when $n \to +\infty$ and, hence, the effect of individual vote buying on parties' expected vote shares should become infinitesimal. That is, for n sufficiently large, if at least a share $1 - F(w^{\max})$ of players buy votes (and at most a share $F(w^{\max})$ of players sell votes), a player with parameter equal to $w_i > w^{\max}$ should strictly prefer to sell her vote rather than buy. This contradicts the assumption that the threshold is below some $w^{\max} > 0$ for every $n \in \mathbb{R}$ and suggests that the equilibrium threshold should be increasing in n, at least for relatively large values of n.

To see how the threshold changes with n, when n is small, let us turn attention to the exact effects of adding a player in the game. Assume that, initially, there are n players and that we are in equilibrium: all players with intensity parameter smaller than w^n sell and all players with intensity parameter larger than w^n buy. If we keep the threshold constant and we add a new player (that is, if we add a player but consider that each of these $n+1$ players sells if her intensity is smaller than w^n and buys otherwise), the following force arises: the expected effect of individual vote buying on parties' expected vote shares decreases as there exist, in expected terms, more vote buyers. This makes vote buying less appealing and pushes the threshold to increase. If the threshold increases then a new force appears: the expected price of each vote decreases as more people are expected to sell. This makes vote selling less appealing and pushes the threshold to decrease. In other words, there are opposite forces from adding a new player on the equilibrium threshold and depending on the model's exact parametrization (that is, depending on F and the initial value of n) we might have an increase or a decrease of the threshold value.

We finally argue in the following proposition that in any equilibrium of this game players' welfare is unambiguously larger compared to the no-trade scenario. This holds both in social and in individual terms and most importantly it is true both under the veil of ignorance and when players are fully aware of their preferences. The intuition why allowing for vote trading in the framework of a power-sharing system unambiguously improves welfare lies in the fact that the no-trade action delivers to an individual the same expected utility *for all possible equilibrium beliefs*. Hence, a player that neither sells nor buys votes expects the same utility both when vote trading is allowed (that is, when some players might be expected to engage in it) and when vote trading is not allowed (that is, when nobody is expected to engage in vote trading).

Proposition 3 *All (a positive measure of) players' types expect weakly (strictly) larger utility in every equilibrium of this game compared to when vote trading is not allowed.*

Proof. The proof is straightforward. We know that the expected utility of a player $i \in Q$ with intensity parameter w_i when choosing y is given by $Eu_i(y) = (\frac{1}{2} + \frac{1}{2n})w_i + 1$. We notice that this is independent of the exact share of vote buyers over vote sellers. Hence, this should be the expected utility of this player even when nobody is expected to trade and thus in the variation of the game in which trade is not allowed. By Proposition 1 we know that in every almost strict BNE it has to be the case that a positive measure of types trades and by Proposition 2 we know that an almost strict BNE actually exists. Therefore, in every almost strict BNE it is the case that a positive measure of players' types expect strictly larger utility than $Eu_i(y)$ and no player expects lower utility than $Eu_i(y)$. ■

This establishes that once a voter knows her preferences and given the equilibrium expectations regarding what other players will do, she is better off (all players' types weakly and a positive measure of them strictly) when vote trading is allowed compared to when it is not. Since this holds for every possible player type it should trivially extend: a) to the society as a whole and b) to a possible constitutional design pre-stage in which voters are not yet aware of their preferences. That is, if voters were somehow asked to choose under a veil of ignorance whether they would like to allow vote trading or not, they would unanimously approve vote trading.

Finally, let us comment on the robustness of our welfare results with respect to more general bidding frameworks (that is, when different players may bid different monetary amounts). It is perfectly possible to formally argue that our positive welfare results directly extend to such a general case, despite the fact that a full characterization of an equilibrium is not plausible. Indeed, given the symmetric structure of the voters' blocks[14] and the employed equilibrium notion, players' strategies in equilibrium should depend only on their cardinal preferences and hence the expected utility of a non-trader is $Eu_i(y) = (\frac{1}{2} + \frac{1}{2n})w_i + 1$ independently of the exact bid of each buyer. That is, in every equilibrium, the no-trade action makes a player at least as happy as in the case in which trade is not allowed and hence everybody is weakly better off when vote trading is allowed.

[14] If a party is expected to be more popular and/or intensities are drawn from different distributions depending on which party one ranks first, then the effect of vote trading on social welfare need not be unambiguously positive as in our case. There is no reason, however, to assert that welfare improvement collapses in mildly asymmetric environments. The power-sharing approach that we consider here is quite smooth in the sense that expecting a slightly lower or larger vote share for your party does not change much your incentives to trade. This allows us to conjecture that, at least when the environment is not too asymmetric, there will be only a relatively small proportion of the population that might endure some relatively small losses from the introduction of vote trading.

4 Comparison with plurality rule

In this part of the paper we discuss the effects of substituting the power-sharing institutions with majoritarian ones in our vote-trading environment. This is crucial as it will help us clarify and disentangle the effects of the two main elements of our analysis (power-sharing institutions and norms of strategic market games) on our formal results (generic existence of a welfare improving full trade equilibrium). Voters are characterized by the ordinal/cardinal preferences and beliefs that were described earlier, there are no changes in terms of the trading rules and the only difference concerns the adoption of plurality decision rule.

In this setting we denote by $h(m, p, x)$ the probability mass function of a binomial distribution with parameters m (number of trials) and p (probability of success) at x. That is, $h(m, p, x)$ is the probability of x successes out of n trials when the success probability of each trial is equal to p. When m is even (E) and $p = \frac{1}{2}$, the probability that, out of m trials, the number of successes is at least $\frac{m}{2}$ is denoted by $H^{E,m} = \sum_{x=\frac{m}{2}}^{m} h(m, \frac{1}{2}, x)$. Similarly, when m is odd (O) and $p = \frac{1}{2}$, the probability that, out of m trials, the number of successes is at least $\frac{m+1}{2}$ is denoted by $H^{O,m} = \sum_{x=\frac{m+1}{2}}^{m} h(m, \frac{1}{2}, x)$.

Considering that n is odd, we assume that a full trade equilibrium exists and it is such that all players with intensity parameters smaller than v sell their votes and all players with intensity parameters larger than v buy votes. Then, the expected utility of individual i from choosing s is given by:

$$Eu_i(s) = \sum_{k=0}^{n-2}[(\tfrac{w_i}{2} + \tfrac{n-1-k}{k+1} + 1) \times h(n-1, F(v), k)] + (H^{E,n-1} \times w_i + 1) \times h(n-1, F(v), n-1).$$

The first part of this expression corresponds to the cases in which vote trading takes place. For vote trading to occur in this case it is necessary that at least one player chooses to buy. Notice that $k = 0$ corresponds to the case that zero other players sell (that is, all other voters buy) and $k = n-2$ corresponds to the case that $n-2$ other players sell (that is, only one player buys). The second term of the expression corresponds to the eventuality that vote trading does not occur (that is, when all the $n-1$ other players decide to sell too).

The expected utility of individual i from choosing b is given by:

$Eu_i(b) = \sum_{k=0}^{\frac{n-3}{2}} [H^{E,2k} \times w_i \times h(n-1, 1-F(v), 2k)] + \sum_{k=0}^{\frac{n-3}{2}} [(H^{O,2k+1} + \frac{1}{2} h(2k+1, \frac{1}{2}, k)) \times w_i \times h(n-1, 1-F(v), 2k+1)] + (H^{E,n-1} \times w_i + 1) \times h(n-1, 1-F(v), n-1).$

The first sum of this expression corresponds to the cases in which the number of vote buyers, excluding player i, is an even number and vote trading takes place (that is, there is at least one player that sells her vote). Hence, the number of vote buyers, except player i, is even and ranges from zero to $n-3$. Similarly, the second sum corresponds to the cases in which the number of vote buyers, excluding player i, is an odd number and vote trading takes place. The last part of the expression corresponds to the eventuality that vote trading does not occur (that is, when all the $n-1$ other players decide to buy too).

Finally, the expected utility of individual i from choosing y is given by:

$Eu_i(y) = \sum_{k=1}^{\frac{n-1}{2}} [(H^{E,2k} \times w_i + 1) \times h(n-1, 1-F(v), 2k)] + \sum_{k=0}^{\frac{n-3}{2}} [(\frac{w_i}{2} + 1) \times h(n-1, 1-F(v), 2k+1)] + (H^{E,n-1} \times w_i + 1) \times h(n-1, 1-F(v), 0)$

The first sum of this expression corresponds to the cases in which the number of players who chose b is an even positive number. If $k \in \{1, ..., \frac{n-3}{2}\}$ vote trading takes place and each player who chose b holds more votes than player i, who just holds her one vote; and if $k = \frac{n-1}{2}$ vote trading does not take place and each player that chose b holds one vote, exactly like player i. Notice though, that in all such cases, voter i is pivotal if and only if the number of players who chose b are evenly split between the two parties. Hence, when the number of vote buyers is even, it does not matter that i (in most cases) holds less votes than the rest of the voters: she can influence the result as often as if she held exactly as many votes as each vote buyer. On the contrary, when the number of vote buyers is odd (this corresponds to the second sum) player i is never pivotal. This is so because vote trading always takes place - each vote buyer holds strictly more than one vote and, given that the number of vote buyers is odd, the difference in parties' vote shares, without taking in account the vote of player i, is always strictly larger than one. In the last part of the expression, we treat the case in which no player chooses b. In this case, vote trading does not occur (exactly like when all other players choose b) and voter i is pivotal if and only if the rest of the players are evenly split between the two parties.

This full trade equilibrium should be such that: a) $Eu_i(y) \leq \max\{Eu_i(s), Eu_i(b)\}$, for every $w_i \in [0, +\infty]$, and b) $Eu_i(s) \neq Eu_i(b)$, for almost every $w_i \in [0, +\infty]$. Given that both $Eu_i(s)$

and $Eu_i(b)$ are linear in w_i, it follows that $v = w_i$ if and only if $Eu_i(s) = Eu_i(b)$. We consider that $F = \ln N(0, 1)$ and two distinct population sizes; $n = 7$ and $n = 9$. When $n = 7$ (in this case $v \approx 3.685$) we find that: a) indeed, a full trade equilibrium exists, but b) some players' types (those that are relatively indifferent between b and s) expect *strictly lower utility* in this equilibrium compared to when vote trading is not allowed. Moreover, when $n = 9$ (in this case $v \approx 3.7$) a full trade equilibrium *does not exist* since those that are relatively indifferent between b and s, strictly prefer y to both of these actions. All these findings are presented on Figure 2.

[Insert Figure 2 about here]

Hence, existence of a full trade equilibrium is not generic under plurality rule and, even when such an equilibrium exists, it does not guarantee unambiguously larger expected payoffs to almost all players compared to when vote trading is not allowed. These observations show that the general findings of our main analysis (generic existence of a unique welfare improving full trade equilibrium), are not solely due to the strategic market approach that we employed, but they closely relate to the power-sharing institutions that we consider.

5 Concluding remarks

This paper studies a simple power-sharing environment with strategic players in an attempt to overcome some shortcomings of existing models of decentralized vote trading. For instance, our approach does not feature the discontinuities in payoffs associated with plurality rule and allows prices to be determined by the actions of all vote traders. Above all, we provide clear-cut results about the existence of equilibrium involving trade. Moreover, unlike most findings in the literature, our equilibrium outcomes allow for the dispersion of votes to many buyers and provide positive welfare results as we exhibit that vote trading makes voters better off. Therefore, the superiority of vote trading compared to the no-trade option is easily justified.

We believe that our approach offers interesting insights into decentralized vote trading; yet there are still many central questions in the literature, so we hope that our attempt will pave the way for new studies. An interesting way to go forward would possibly be to extend our model to multiple issues. Vote markets in such environments allow individuals to buy (sell) votes on issues they care the most (less) and the strategic market game mechanism will allow prices to reflect the differences in the intensities of preferences. Finally, given that experimental strategic market games have already been implemented (e.g., Huber, Shubik, and Sunder, 2010; Duffy, Matros, and Temzelides, 2011), we believe that it is challenging for future research to test our theoretical results by conducting laboratory experiments in the same way as Casella, Llorente-Saguer, and Palfrey (2012) and Casella, Palfrey, and Turban (2014).

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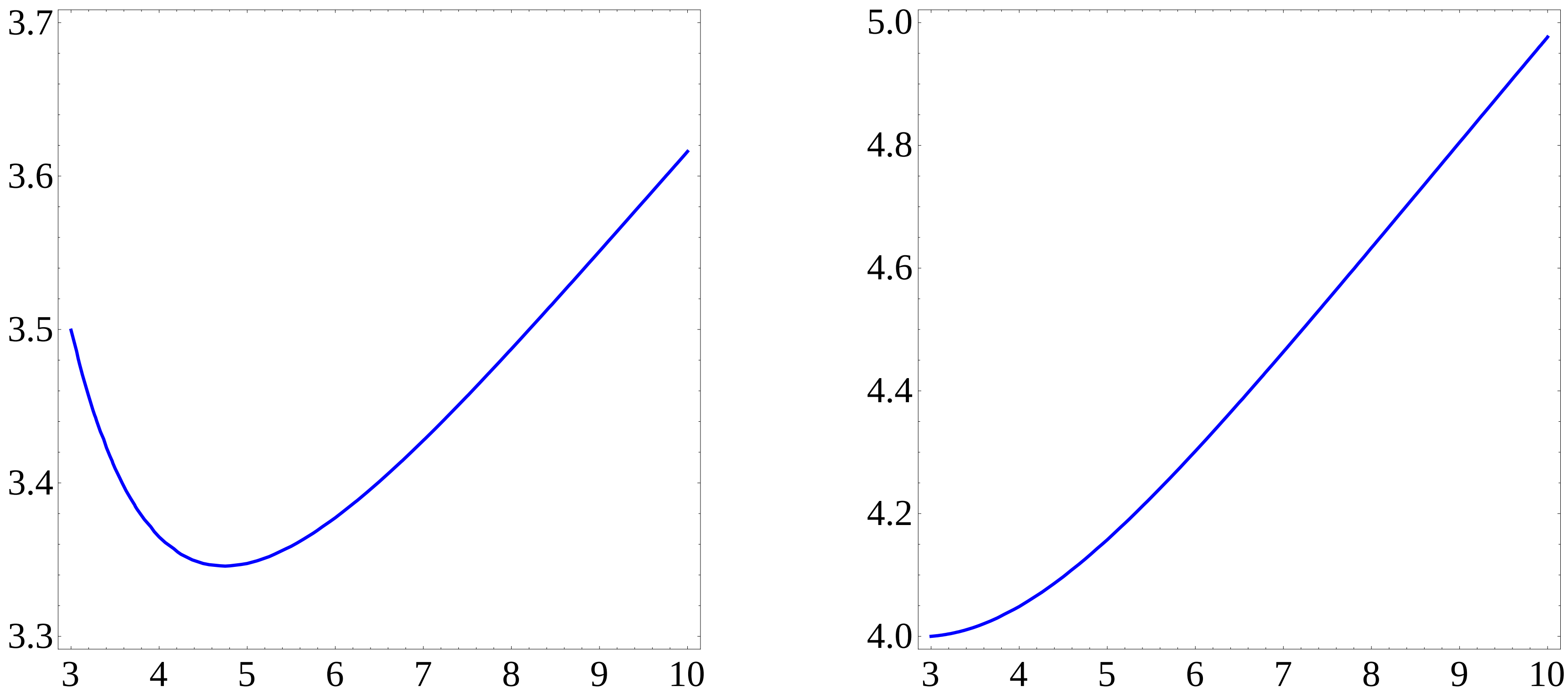

Figure 1. The threshold value, w''', as a function of n considering that $F = lnN(0,1)$ (left) and that $F(x) = 1 - \frac{1}{x+1}$ (right).

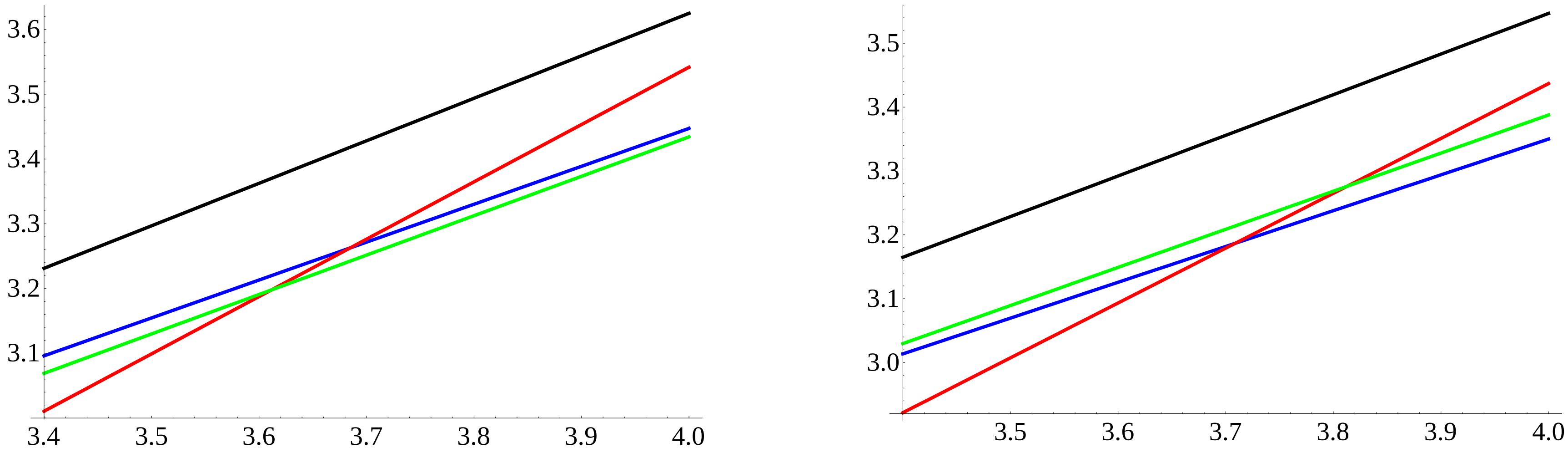

Figure 2. Plots of $Eu_i(s)$ (blue), $Eu_i(b)$ (red), $Eu_i(y)$ (green) and of the expected utility of an individual when no trade is allowed (black) as functions of $w_i \in [3.4,4]$, when $F = lnN(0,1)$, considering that $n = 7 \rightarrow v \approx 3.685$ (left); and $n = 9 \rightarrow v \approx 3.7$ (right).