University of Cyprus
Department of Economics

Working Paper 01-2015

Multidimensional electoral competition between differentiated candidates

Dimitrios Xefteris

Department of Economics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus
Tel.: +357-22893700, Fax: +357-22895028, Web site: www.econ.ucy.ac.cy

Multidimensional electoral competition between differentiated candidates

Dimitrios Xefteris

Department of Economics,

University of Cyprus,

PO Box 20537,

1678, Nicosia, CYPRUS.

tel: +357 2289 3689.

email: xefteris.dimitrios@ucy.ac.cy.

January, 2015

Abstract

It is known that multidimensional Downsian competition fails to admit an equilibrium in pure strategies unless very stringent conditions on the distribution of voters' bliss points are imposed (Plott 1967). This paper revisits this problem considering that the two vote share maximizing candidates are *differentiated*. That is, candidates strategically decide positions only in some of the n dimensions while in the rest their positions are assumed to be fixed. These fixed dimensions may be viewed as candidates' immutable characteristics (race, religion, culture, etc.). We find that if candidates are sufficiently differentiated - if in the fixed dimensions their positions are sufficiently different - then a unique Nash equilibrium in pure strategies is guaranteed to exist *for any distribution of voters' bliss points*. Perhaps more importantly, we show that this is true even if there exists a unique fixed dimension and candidates instrumentally decide their positions in all other $n-1$ dimensions.

Keywords: electoral competition; multidimensional model; equilibrium existence; differentiated candidates.

JEL classification: D72

1 Introduction

Spatial representation of sets of policy alternatives and preference profiles is very popular in contemporary economics and political science literature. The intuition behind the ideas that an alternative may be well represented by a point in some space and that the utility of an individual is decreasing in the distance between her ideal alternative and the applied one is very strong. It goes hand in hand with the common perception regarding what policy alternatives are and what is the mental process that we use to evaluate them. When there exist n distinct political issues then each political issue may be interpreted as a separate dimension in a Euclidean space. Therefore, a) the policy platform of a candidate should be a vector consisting of n policies - one for each of the n issues - and b) the utility of a voter from this candidates' platform should be decreasing in the distance between the candidate's platform and her bliss point - the vector of her ideal policies for each of the n issues.

In its basic unidimensional ($n = 1$) version the spatial model delivers very strong results as far as electoral competition between two office motivated candidates is concerned (or simply, as far as Downsian competition is concerned). For any distribution of voters' bliss points there exists a unique pure strategy equilibrium and it is such that both candidates promise to implement the bliss point of the median voter (Downs 1957; Black 1958). Once we depart from the unidimensional world though and we arrive to more complex spaces, the "curse of multidimensionality" appears (the term was proposed by Bernheim and Slavov 2009); for almost all distributions of voters' bliss points there exists no equilibrium in pure strategies (Plott 1967; Kramer 1972; Davis et al. 1972; McKelvey and Wendell 1976).

This extremely negative result gave rise to the need of identifying conditions under which it could be overturned. To that effect, alternative candidates' objectives were considered (see Calvert 1985, for example, for policy motivation) and, moreover, possibility of candidates using mixed strategies was also introduced in the model (see Banks et al. 2002, Duggan and Jackson 2004 and Banks et al. 2006). In this paper we choose an alternative strategy to overcome the issue of equilibrium inexistence. In line with Downsian tradition, we still assume that candidates are purely office motivated and we only allow candidates to use pure strategies[1] but, unlike the original approach, we consider that the two competing candidates are *not identical*. That is, we assume that the two competing candidates have fixed positions in some of the dimensions and that they strategically decide positions in the remaining dimensions to maximize vote shares. Such models of electoral competition between differentiated candidates have become quite popular recently as they incorporate two very realistic features of electoral competition. First, they allow candidates to be instrumental in only a subset of the dimensions that are relevant for voters' choices. Indeed, voters decide which candidate to support not only on the basis of what candidates decide during a campaign but also on the basis of candidates' immutable characteristics (race, religion, culture, etc.). Second, candidates are hardly ever identical as far as their immutable characteristics are concerned. Thus, bearing in mind that immutable characteristics are relevant in voters' choices, discarding such existing asymmetries between candidates might lead to results which are not empirically relevant.

Most papers which consider electoral competition between differentiated candidates (see for instance Lindbeck and Weibull 1987; Dziubiński and Roy 2011; Krasa and Polborn 2012; Krasa and Polborn 2014; Matakos and Xefteris 2014) provide equilibrium existence results for the case in which candidates are instrumental *in only one dimension*. That is, strategy-wise these games are still unidimensional; a candidate's strategy is a real number - not a vector of real numbers. To the author's knowledge the only paper in this literature which studies existence of a Nash equilibrium in pure strategies considering that candidates are instrumental in two or more dimensions is Krasa and Polborn (2010). In that paper though the nature of positioning is binary; it is assumed that for each issue there are only two policy alternatives. Hence, there is no paper that deals with equilibrium existence in multidimensional Downsian competition between differentiated candidates in its generic form; in a non-binary policy framework. This is precisely the gap in the literature that this paper aims to fill.

We consider that the policy space is an n-dimensional continuous Euclidean space and that voters' bliss points are distributed in any arbitrary manner on this space. Candidates propose

[1] For this reason, whenever we use the term equilibrium we specifically refer to a pure strategy Nash equilibrium.

policy platforms which are vectors of n coordinates - one number/policy for each dimension/policy-issue - but they can strategically choose coordinates only for $n-k$ specific policy issues; the remaining $k \geq 1$ coordinates of each candidate's platform are fixed. Voting is deterministic: each voter observes candidates' platforms and votes for the one which is nearer to her bliss point. That is, as it is standardly assumed, a voter's utility from a candidate is decreasing in the distance between this candidate's platform and the voter's bliss point. The two competing candidates are purely Downsian; their unique objective is to maximize expected vote share. Our main finding is the following: if candidates are sufficiently differentiated in the fixed dimensions, then a unique Nash equilibrium in pure strategies is guaranteed to exist in this model *for any* distribution of voters' bliss points. This result reads even stronger if one bears in mind that the number of fixed dimensions may be as small as one.

As far as characterization of this unique equilibrium is concerned, we have that a) in the general case this unique equilibrium is interior and convergent - in each political issue candidates make the same non-extreme promise - and b) in the special case in which voters' ideal policies are independently distributed for each issue, this unique equilibrium may be fully characterized - candidates locate precisely at the mean policy of each dimension. The latter is compatible with seminal contributions by Caplin and Nalebuff (1991) and Schofield (2007) who also argue that in multidimensional settings the best a candidate can do is to locate at the mean ideal policy of each issue.[2]

A natural question regarding the robustness of the present existence result is whether the sufficient degree of candidate differentiation is unrealistically large or not. To deal with this question, we study a particular example and we show that the sufficiently large degree of candidate differentiation need be actually very small.[3] Our analysis may, hence, be viewed as a positive argument in favor of representative democracy in the sense that under a fair in many cases assumption -

[2]Our findings also relate to the ones of the probabilistic voting literature and in specific to works by Hinich et al. (1972), Enelow and Hinich (1989), Lindbeck and Weibull (1993) and Banks and Duggan (2005) both in terms of how they are derived (existence of an equilibrium is established by showing that candidates' payoff functions are quasi-concave in own strategies) and in terms of implications (equilibria in these probabilistic models - whenever they exist - are convergent).

[3]In the strategic voting literature (see, for example, Bouton and Castanheira 2012) which also derives equilibrium existence results assuming that a certain parameter - the expected number of voters - is sufficiently large this is also the case; in many cases a sufficiently large number of voters need be actually very small.

sufficient candidate differentiation in immutable characteristics (such as race, religion, culture etc.) - stability in multi-issue electoral competition may be achieved. In what follows we present the model (Section 2), the formal results (Section 3) and finally (Section 3) we conclude with the explicit presentation of a representative example.

2 The Model

The policy space is $X = \mathbb{R}^n$. The bliss points of a unit mass of voters are distributed on $\mathbb{R}^n$ according to an absolutely continuous and twice differentiable distribution function $F : \mathbb{R}^n \to [0, 1]$ with support $S = \prod_{j=1}^{n}[\psi_j, \omega_j]$ where $n \geq 2$, $\psi_j \in \mathbb{R}$, $\omega_j \in \mathbb{R}$ and $\psi_j < \omega_j$. The density of this distribution, f, is assumed to be positive valued on S. Given the above $x = (x_1, x_2, x_3, ..., x_n) \in S$ is a vector of policies for all issues, $x_{-1} = (x_2, x_3, ..., x_n) \in S_{-1} = \prod_{j=2}^{n}[\psi_j, \omega_j]$ is a vector of policies for all issues except for the first one, $F_1(x_1|x_{-1} = \hat{x}_{-1})$ is the conditional distribution of ideal policies[4] regarding the first issue given ideal policies regarding all other issues, $\hat{x}_{-1}$, with corresponding density $f_1(x_1|x_{-1} = \hat{x}_{-1})$, $F_j(x_j)$ is the marginal distribution of ideal policies regarding issue j with corresponding density $f_j(x_j)$ and $F_{-1}(x_{-1})$ is the marginal distribution of ideal policies regarding all issues except for the first one with corresponding density $f_{-1}(x_{-1})$. Notice that since F is twice differentiable on S it is the case that all conditional and marginal densities are twice differentiable on their domains.

There are two candidates, A and B, who propose policy platforms $a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n$ and $b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$. We consider that a_1 and b_1 are fixed parameters of the model[5] and that $F_1(\frac{a_1+b_1}{2}|x_{-1} = \hat{x}_{-1}) \in (0, 1)$ for every $\hat{x}_{-1} \in S_{-1}$.[6] Moreover, without loss of generality, we further consider that $a_1 < 0 < b_1$ and $\frac{a_1+b_1}{2} = 0$. The remaining $n - 1$ coordinates are strategically chosen by the candidates in order to maximize vote shares. That is, the platform of candidate A should

[4]To avoid confusion we note here that a voter's bliss point is a vector of policies, $x \in \mathbb{R}^n$, while the ideal policy of a voter regarding issue j is a real number, $x_j \in \mathbb{R}$.

[5]For the derivation of our results we work under the assumption that there exists a unique fixed dimension. It will become straightforward by our formal arguments that the results trivially extend to the case of an arbitrary number of fixed dimensions.

[6]This is a very mild assumption which only ensures that in every group of voters who agree on the last $n - 1$ issues, at least some are leaning towards candidate A and at least some towards candidate B.

be such that $a = (a_1, a_2, ..., a_n) \in \{a_1\} \times \mathbb{R}^{n-1}$ - the strategy set of player A is $\{a_1\} \times \mathbb{R}^{n-1}$ - and the platform of candidate B should be such that $b = (b_1, b_2, ..., b_n) \in \{b_1\} \times \mathbb{R}^{n-1}$ - the strategy set of player B is $\{b_1\} \times \mathbb{R}^{n-1}$. A voter with bliss point $x = (x_1, x_2, ..., x_n)$ will derive utility $u(d(a, x))$ if candidate A is elected and utility $u(d(b, x))$ if candidate B is elected, where $u : [0, +\infty) \to \mathbb{R}$ is strictly decreasing and $d(y, x)$ denotes the Euclidian distance between $y \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Hence, the closer a candidate's platform to a voter's bliss point the larger the utility that this voter derives from this candidate's platform. The distribution of voters' bliss points and the objectives of the candidates are common information.

The timing of the game is as follows. In the first stage of the game candidates simultaneously decide platforms from their strategy sets. In the second stage these platforms become common information and each voter votes for the candidate whose platform is closer to her bliss point. When a voter is indifferent between two platforms she splits her vote between the two candidates. In the third stage players' payoffs are computed. Since the behavior of voters is unambiguous in this model, we define an equilibrium only in terms of candidates' strategies. The equilibrium concept that we apply is Nash equilibrium in pure strategies.[7]

Given that a voter's utility is decreasing in the distance between her bliss point and a candidate's platform, a voter who is indifferent between the platforms of the two candidates should have a bliss point $i = (i_1, i_2, ..., i_n)$ such that

$$d(a, i) = d(b, i).$$

Because $a_1 < b_1$ and $\frac{a_1 + b_1}{2} = 0$, the last equality may be written as

$$i_1 = \frac{\sum_{j=2}^{n} [(b_j - i_j)^2 - (a_j - i_j)^2]}{2d}$$

where $d = b_1 - a_1$; the degree of candidate differentiation. Hence, a voter with ideal policy, $x \in \mathbb{R}^n$, votes for A if

[7] In spite of focusing only on pure strategies, after the presentation of our formal results we argue that the equilibrium that we identify is unique even if one permits candidates to use mixed strategies.

$$u(d(a,x)) > u(d(b,x)) \Longleftrightarrow x_1 < \frac{\sum_{j=2}^{n}[(b_j-x_j)^2-(a_j-x_j)^2]}{2d}$$

votes for B if

$$u(d(a,x)) < u(d(b,x)) \Longleftrightarrow x_1 > \frac{\sum_{j=2}^{n}[(b_j-x_j)^2-(a_j-x_j)^2]}{2d}$$

and splits her vote between the two otherwise.

The above allow us to define the payoffs of candidate A - her vote share - as a function of candidates' strategies a and b. We have

$$V_A(a,b:F,d) = \int\limits_{S_{-1}} F_1\left(\frac{\sum_{j=2}^{n}[(b_j-i_j)^2-(a_j-i_j)^2]}{2d}|i_{-1} = \hat{\imath}_{-1}\right) \times f_{-1}(\hat{\imath}_{-1})d\hat{\imath}_{-1}$$

where $\int\limits_{S_{-1}}$ stands for $\int\limits_{\psi n}^{\omega_n} ... \int\limits_{\psi_3}^{\omega_3}\int\limits_{\psi_2}^{\omega_2}$ and $d\hat{\imath}_{-1}$ stands for $d\hat{\imath}_2 d\hat{\imath}_3 ... d\hat{\imath}_n$.

Obviously, the payoffs of candidate B - her vote share - are $V_B(a,b:F,d) = 1 - V_A(a,b:F)$.

3 Results

We directly proceed to the main result of the paper

Proposition 1 *For every admissible F there exists a real number $\hat{d}(F) > 0$ such that whenever $d > \hat{d}(F)$ a unique Nash equilibrium in pure strategies, (a^*, b^*), is guaranteed to exist. This equilibrium is convergent, that is, $a^*_{-1} = b^*_{-1}$ and interior, that is, $a^*_{-1} \in \prod_{j=2}^{n}(\psi_j, \omega_j)$.*

Proof We consider a constrained version of the game. That is we assume that the strategy set of player A is $\{a_1\} \times S_{-1}$ and that the strategy set of player B is $\{b_1\} \times S_{-1}$. We prove our results for this constrained version of the game in three steps. In the first step we deal with equilibrium existence. In the second step we establish uniqueness. In the third step we provide a partial characterization (convergence and interiority). Finally, we argue that everything would continue to hold in the unconstrained version of our game.

Step 1 (Equilibrium existence) We observe that the strategy sets are convex and compact and that the players' payoff functions are continuous in own strategies. By Glicksberg (1952) it follows that this game admits a Nash equilibrium in mixed strategies. If we moreover derive conditions that guarantee that the players' payoff functions, $V_A(a, b : F, d)$ and $V_B(a, b : F, d)$, are quasi-concave in own strategies then by Debreu (1952) it will follow that this constrained game admits a Nash equilibrium in pure strategies. To that effect we will demonstrate that when candidate differentiation, $d > 0$, is sufficiently large, $V_A(a, b : F, d)$ is strictly concave on $\{a_1\} \times S_{-1}$ for every $b \in \{b_1\} \times S_{-1}$ and, hence, strictly quasi-concave. A symmetric argument can guarantee that when candidate differentiation, $d > 0$, is sufficiently large, $V_B(a, b : F, d)$ is strictly concave on $\{b_1\} \times S_{-1}$ for every $a \in \{a_1\} \times S_{-1}$ and, hence, strictly quasi-concave.

To establish that $V_A(a, b : F, d)$ is strictly concave on $\{a_1\} \times S_{-1}$ for every $b \in \{b_1\} \times S_{-1}$ we first need to make sure that $V_A(a, b : F, d)$ is twice differentiable on $\{a_1\} \times S_{-1}$ for every $b \in \{b_1\} \times S_{-1}$. We observe that

$$\lim_{d \to +\infty} \frac{\sum_{j=2}^{n} [(b_j - i_j)^2 - (a_j - i_j)^2]}{2d} = 0$$

for every admissible strategy profile (a, b) and every $i_{-1} \in S_{-1}$. That is, when d is large enough it must be the case that $F_1(\frac{\sum_{j=2}^{n} [(b_j - i_j)^2 - (a_j - i_j)^2]}{2d} | i_{-1} = \hat{\imath}_{-1}) \in (0, 1)$ for every admissible strategy profile (a, b) and every $\hat{\imath}_{-1} \in S_{-1}$. This suggests that when d is large enough $F_1(\frac{\sum_{j=2}^{n} [(b_j - i_j)^2 - (a_j - i_j)^2]}{2d} | i_{-1} = \hat{\imath}_{-1})$ is twice differentiable on $\{a_1\} \times S_{-1}$ for every $b \in \{b_1\} \times S_{-1}$ and every $\hat{\imath}_{-1} \in S_{-1}$ and hence that $V_A(a, b : F, d)$ is twice differentiable on $\{a_1\} \times S_{-1}$ for every $b \in \{b_1\} \times S_{-1}$.

Considering that d is large enough for $F_1(\frac{\sum_{j=2}^{n} [(b_j - i_j)^2 - (a_j - i_j)^2]}{2d} | i_{-1} = \hat{\imath}_{-1})]$ to be twice differentiable on $\{a_1\} \times S_{-1}$ for every $b \in \{b_1\} \times S_{-1}$ and every $\hat{\imath}_{-1} \in S_{-1}$we compute

$$\frac{\partial^2 V_A(a,b:F,d)}{\partial a_h^2} = \frac{1}{d} \int\limits_{S_{-1}} [\frac{(a_h - \hat{\imath}_h)^2}{d} f_1'(\frac{\sum_{j=2}^{n} [(b_j - i_j)^2 - (a_j - i_j)^2]}{2d} | i_{-1} = \hat{\imath}_{-1}) -$$

$$-f_1(\frac{\sum_{j=2}^{n}[(b_j-i_j)^2-(a_j-i_j)^2]}{2d}|i_{-1}=\hat{\imath}_{-1})]\times f_{-1}(\hat{\imath}_{-1})d\hat{\imath}_{-1}$$

and

$$\frac{\partial^2 V_A(a,b:F,d)}{\partial a_h \partial a_z}=\frac{1}{d}\int\limits_{S_{-1}}\frac{(a_h-\hat{\imath}_h)(a_z-\hat{\imath}_z)}{d}f_1'(\frac{\sum_{j=2}^{n}[(b_j-i_j)^2-(a_j-i_j)^2]}{2d}|i_{-1}=\hat{\imath}_{-1})\times f_{-1}(\hat{\imath}_{-1})d\hat{\imath}_{-1}.$$

Hence, the determinant of the k-th order principal minor of the Hessian matrix of $V_A(a,b:F,d)$ is given by

$$D_{A,k}=(\frac{1}{d})^k\times\Xi_{A,k}$$

where

$$\Xi_{A,k}=\begin{vmatrix} g_{11}(a,b:F,d) & g_{12}(a,b:F,d) & \dots & g_{1k}(a,b:F,d) \\ g_{21}(a,b:F,d) & g_{22}(a,b:F,d) & \dots & g_{2k}(a,b:F,d) \\ \vdots & \vdots & \ddots & \vdots \\ g_{k1}(a,b:F,d) & g_{k2}(a,b:F,d) & \dots & g_{kk}(a,b:F,d) \end{vmatrix}$$

and

$$g_{hh}(a,b:F,d)=d\times\frac{\partial^2 V_A(a,b:F,d)}{\partial a_h^2},\ g_{hz}(a,b:F,d)=d\times\frac{\partial^2 V_A(a,b:F,d)}{\partial a_h\partial a_z}\text{ for }h\neq z.$$

Therefore, $D_{A,k}>0$ if and only if $\Xi_{A,k}>0$ and $D_{A,k}<0$ if and only if $\Xi_{A,k}<0$.

We notice that

$$\lim_{d\to+\infty} g_{hh}(a,b:F,d)=-\int\limits_{S_{-1}} f_1(0|i_{-1}=\hat{\imath}_{-1})\times f_{-1}(\hat{\imath}_{-1})d\hat{\imath}_{-1}=-F_1(0)<0$$

for every $h\in\{2,3,...,n\}$ and that

$$\lim_{d\to+\infty} g_{hz}(a,b:F,d) = -\int_{S_{-1}} 0 \times f_{-1}(\hat{\imath}_{-1})d\hat{\imath}_{-1} = 0$$

for every $h, z \in \{2, 3, ..., n\}$ such that $h \neq z$.

Therefore, when $d \to +\infty$ we have that

$$\Xi_{A,k} \to \begin{vmatrix} -F_1(0) & 0 & ... & 0 \\ 0 & -F_1(0) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & -F_1(0) \end{vmatrix} = [-F_1(0)]^k$$

and, hence, when d is sufficiently large $V_A(a, b : F, d)$ is strictly concave on $\{a_1\} \times S_{-1}$ for every $b \in \{b_1\} \times S_{-1}$. A similar argument guarantees that when d is sufficiently large $V_B(a, b : F, d)$ is strictly concave on $\{b_1\} \times S_{-1}$ for every $a \in \{a_1\} \times S_{-1}$. Therefore when d is large enough the payoff functions of the players are strictly quasi-concave in own strategies and by Debreu (1952) a Nash equilibrium in pure strategies is guaranteed to exist in this constrained game.

Step 2 (Uniqueness) Notice that this game is constant sum. This suggests that if the constrained game admits two distinct equilibria, (a^*, b^*) and (a^{**}, b^{**}), such that $a^* \neq a^{**}$ then (a^{**}, b^*) should be an equilibrium too.[8] But this contradicts the fact that when d is sufficiently large $V_A(a, b^* : F, d)$ is strictly concave on $\{a_1\} \times S_{-1}$ because it suggests that $V_A(a, b^* : F, d)$ admits two maxima. Hence, when d is sufficiently large a pure strategy equilibrium exists and it is unique.

Step 3 (Characterization) Consider that the two candidate promise the same policies in the dimensions in which they are instrumental, that is, $a_{-1} = b_{-1}$. In such a case $\frac{\sum_{j=2}^{n}[(b_j - i_j)^2 - (a_j - i_j)^2]}{2d} = 0$ and hence

$$\frac{\partial V_A(a,b:F,d)}{\partial a_j}\Big|_{a_{-1}=b_{-1}} = \int_{S_{-1}} \frac{\hat{\imath}_j - a_j}{d} f_1(0|i_{-1} = \hat{\imath}_{-1}) \times f_{-1}(\hat{\imath}_{-1})d\hat{\imath}_{-1}.$$

[8] This is without loss of generality since for (a^*, b^*) and (a^{**}, b^{**}) to be distinct it should be the case that either $a^* \neq a^{**}$ or $b^* \neq b^{**}$ - or both.

We observe that a) when $a_j = b_j = \psi_j$ it is true that $\frac{\partial V_A(a,b:F,d)}{\partial a_j}|_{a_{-1}=b_{-1}} > 0$, when $a_j = b_j = \omega_j$ it is true that $\frac{\partial V_A(a,b:F,d)}{\partial a_j}|_{a_{-1}=b_{-1}} < 0$ and c) $\frac{\partial V_A(a,b:F,d)}{\partial a_j}|_{a_{-1}=b_{-1}}$ is continuous in $a_j \in [\psi_j, \omega_j]$ and independent of a_k for every $k \neq j$. Therefore there exists $a^*_{-1} = b^*_{-1} \in \prod_{j=2}^{n}(\psi_j, \omega_j)$ - convergent and interior - such that $\frac{\partial V_A(a,b:F,d)}{\partial a_j}|_{a^*_{-1}=b^*_{-1}} = 0$ for every $j \in \{2, 3, ..., n\}$. Given that when d is sufficiently large $V_A(a, b : F, d)$ is strictly concave on $\{a_1\} \times S_{-1}$ for every $b \in \{b_1\} \times S_{-1}$ it follows that this convergent and interior point in which all first order conditions are equal to zero must be the unique pure strategy equilibrium of the game.

Now let us argue why this identified strategy profile is the unique Nash equilibrium of the unconstrained version of our game too - that is, of the version of the game in which the strategy set of player A is $\{a_1\} \times \mathbb{R}^{n-1}$ and the strategy set of player B is $\{b_1\} \times \mathbb{R}^{n-1}$. If B selects the identified equilibrium strategy and d is sufficiently large then A gets a payoff equal to $\frac{1}{2}$ if she selects the identified equilibrium strategy and a payoff strictly small than $\frac{1}{2}$ if she selects any other strategy from $\{a_1\} \times S_{-1}$. Now consider two strategy profiles $a' = (a_1, a_2, ..., a_{j-1}, a'_j, a_{j+1}, ..., a_n)$ and $a'' = (a_1, a_2, ..., a_{j-1}, a''_j, a_{j+1}, ..., a_n)$ which are identical in everything except from the policy regarding issue j. Moreover consider that $a'_j < \psi_j$ and that $a''_j = \psi_j$. If we take the derivative of $\frac{\sum_{j=2}^{n}[(b_j-i_j)^2-(a_j-i_j)^2]}{2d}$ with respect to a_j we will get $\frac{i_j-a_j}{d}$. So if $a_j \leq \psi_j$ and, hence, $a_j \leq i_j$ for every $i_j \in [\psi_j, \omega_j]$ it follows that $\frac{\sum_{j=2}^{n}[(b_j-i_j)^2-(a_j-i_j)^2]}{2d}$ is weakly increasing in a_j. Therefore, if $a_j \leq \psi_j$ it should also be the case that $F_1(\frac{\sum_{j=2}^{n}[(b_j-i_j)^2-(a_j-i_j)^2]}{2d}|i_{-1} = \hat{\imath}_{-1})$ is weakly increasing in a_j and, thus that $V_A(a, b : F, d)$ is weakly increasing in a_j too. This suggests that $V_A(a', b^* : F, d) \leq V_A(a'', b^* : F, d)$. One can similarly show that if a' and a'' differ only in the policy regarding issue j and $a'_j > \omega_j$ and that $a''_j = \omega_j$ then it should hold that $V_A(a', b^* : F, d) \leq V_A(a'', b^* : F, d)$. Hence, if in a there are more than one $a_j \notin [\psi_j, \omega_j]$, candidate A can substitute each of them with ψ_j if $a_j < \psi_j$ or with ω_j if $a_j > \omega_j$ and get a weakly larger vote share. But this new platform will belong in $\{a_1\} \times S_{-1}$ - in the strategy set of the constrained game - and as we know it will deliver to candidate A a strictly lower payoff than the identified equilibrium strategy of the constrained game. That is, the equilibrium of the constrained game will be an equilibrium of the unconstrained game too. Finally, observe that the unconstrained game is also a constant sum game. Since we argued that $\arg\max V_A(a, b^* : F, d) = \{a^*\}$ - a singleton - it trivially follows that the identified equilibrium is guaranteed to be unique. *QED*

Notice than when candidates are sufficiently differentiated in the fixed dimension and a unique Nash equilibrium in pure strategies exists then this equilibrium would also be unique even if players were allowed to use mixed strategies as well. This trivially follows from the constant sum nature of the game which dictates that if a player has both pure and mixed minimaximizer strategies then the support of every mixed minimaximizer should be a subset of the set of the pure minimaximizer strategies. Hence, when a unique pure minimaximizer strategy exists for each player - as in our case - the unique mixed minimaximizer that exists is the one which coincides with the pure minimaximizer. Moreover, we observe that the assumptions on the nature of F, despite the fact that they are already quite general, they may be relaxed even more. As it became evident by the proof above, connectedness of the support or global continuity and differentiability of F are not necessary for the result: what is necessary is that each conditional distribution of ideal policies regarding the first issue given ideal policies regarding all other issues behaves well (it has a positive valued, continuous and differentiable density) only about $\frac{a_1+b_1}{2}$.

We now turn attention to the special case in which voters' ideal policies for each issue are independently distributed. In this case we have that

$$F_1(x_1|x_{-1}=\hat{x}_{-1})=F_1(x_1) \text{ and } f_{-1}(x_{-1})=f_2(x_2)\times f_3(x_3)...\times f_n(x_n).$$

Hence, the equilibrium first order conditions, $\frac{\partial V_A(a,b:F,d)}{\partial a_j}|_{a=b=a^*}=0$ for every $j\in\{2,3,...,n\}$, should become

$$\int\limits_{S_{-1}}\frac{\hat{\imath}_j-a_j^*}{d}f_1(0)\times f_2(\hat{\imath}_2)\times f_3(\hat{\imath}_3)...\times f_n(\hat{\imath}_n)d\hat{\imath}_{-1}=0\Longrightarrow a_j^*=\mu_j$$

where $\mu_j=\int\limits_{\psi_j}^{\omega_j}\hat{\imath}_jf_j(\hat{\imath}_j)d\hat{\imath}_j$; the mean ideal policy regarding issue j.

Therefore, without the need of further formalities we can state the second result of the paper.

Proposition 2 *For every admissible F with density $f(x_1,x_2,...,x_n)=f_1(x_1)\times f_2(x_2)\times f_3(x_3)...\times f_n(x_n)$ there exists a real number $\hat{d}(F)>0$ such that whenever $d>\hat{d}(F)$ a unique Nash equilibrium* *in pure strategies,* (a^{*}, b^{*})*, is guaranteed to exist and it is such that* $a_j^{*} = b_j^{*} = \mu_j$ *for every* $j \in \{2, 3, ...\}$.

4 Concluding remarks/Example

A question that naturally accompanies these formal results is the following: how large is this sufficiently large degree of candidate differentiation which guarantees equilibrium existence? In this last part of the paper we study a simple example in order to deal with this question using the convenient result of Proposition 2 which allows us to fully characterize a Nash equilibrium in pure strategies. The study of this example proves rather reassuring in the sense that it demonstrates that a small degree of candidate differentiation should be enough for an equilibrium to exist.

Consider the case in which there are three policy issues/dimensions and that voters' ideal policies are independently distributed on each dimension. The marginal distribution of voters' ideal policies regarding the first issue is uniform on the unit interval, the marginal distribution of voters' ideal policies regarding the second issue is uniform on the unit interval and the marginal distribution of voters' ideal policies regarding the third issue is triangular on the unit interval with a peak at one. That is $f_1(x_1) = f_2(x_2) = 1$ and $f_3(x_3) = 2x_3$ when $x_j \in [0, 1]$ for every $j \in \{1, 2, 3\}$ and $f_j(x_j) = 0$ otherwise. This three-dimensional distribution fails the necessary and sufficient conditions of McKelvey and Wendell (1976)[9] and, hence, when two identical candidates are instrumental in all three dimensions a pure strategy equilibrium fails to exist. More importantly notice that if the two candidates have fixed positions in the first dimension, $a_1 \in [0, 1]$ and $b_1 \in [0, 1]$, such that $a_1 = b_1$ then still an equilibrium fails to exist. In such circumstances we essentially have two identical candidates competing in a two-dimensional policy space on which voters' ideologies are distributed independently for each issue; uniformly for one and triangularly for the other. Again, this two-dimensional distribution fails the sufficient and necessary conditions of McKelvey and Wendell (1976) and this is why an equilibrium fails to exist in this two-dimensional case too.[10]

[9] McKelvey and Wendell (1976) prove that when candidates are not differentiated an equilibrium exists if and only if there exists a point $m \in \mathbb{R}^n$ such that m is the median voter projection of voters' bliss points on every line which passes through m.

[10] In this case the McKelvey and Wendell (1976) conditions suggest that the unique candidate for equilibrium is

But what if $a_1 < b_1$ and candidates are instrumental in the remaining two dimensions? In Figure 1 we plot the vote share of A when B locates at the mean ideology of each dimension, $\mu_2 = \frac{1}{2}$ and $\mu_3 = \frac{2}{3}$, $V_A((a_1, a_2, a_3), (b_1, \frac{1}{2}, \frac{2}{3}) : F, d)$, as a function of her policy choices in the second and third dimension, a_2 and a_3, considering first that $a_1 = 0.4$ and $b_1 = 0.6$ and then that $a_1 = 0.4$ and $b_1 = 0.5$. One can trivially show in a formal manner that $V_A((a_1, a_2, a_3), (b_1, \frac{1}{2}, \frac{2}{3}) : F, d)$ admits a local maximum at $(a_2, a_3) = (\frac{1}{2}, \frac{2}{3})$ and that non-marginal deviations around this point deliver strictly lower vote share to candidate A. Hence the graphs are enough to prove that in such cases of non-extreme differentiation existence of a pure strategy equilibrium becomes possible.

[Insert Figures 1 and 2 about here]

If we consider that candidate B locates at the mean of each dimension then, in the version of the game in which candidates are not differentiated ($a_1 = b_1$), we notice that candidate A still prefers to locate at the mean of each location than very far away from it. The equilibrium inexistence problem in this case is because A has profitable deviations close by (see Figure 2) in what is usually referred at by the term uncovered set (see Laffond et al. 1993, Banks et al. 2002 and Duggan and Jackson 2004 among others). By studying Figures 1 and 2 in detail we observe that if we fix $a_1 = 0.4 \leq b_1$ and we start increasing b_1, the subset of $[0,1]^2$ on which $V_A((a_1, a_2, a_3), (b_1, \frac{1}{2}, \frac{2}{3}) : F, d)$ is strictly concave smoothly increases around $(a_2, a_3) = (\frac{1}{2}, \frac{2}{3})$. But since far away deviations are always unprofitable - both when candidates are similar and when they are differentiated - it follows that $V_A((a_1, a_2, a_3), (b_1, \frac{1}{2}, \frac{2}{3}) : F, d)$ starts accepting a unique maximum at $(a_2, a_3) = (\frac{1}{2}, \frac{2}{3})$ before candidate differentiation becomes very large. Hence the equilibrium existence result is arguably quite robust.

As a final note we add that our analysis carries through a) to the case in which candidates maximize probability to win instead of vote shares and b) to the case in which each voter might care to a different degree about each issue. When candidates maximize probability to win then existence and characterization results trivially continue to hold: if at $a^*_{-1} = b^*_{-1}$ there is no deviation

$(\frac{1}{2}, \frac{1}{\sqrt{2}})$; the medians of each dimension. But as one can easily verify $(\frac{1}{2}, \frac{1}{\sqrt{2}})$ is *not* the median voter projection of voters' bliss points on the line $x_3 = 0.8 + (\sqrt{2} - 1.6)x_2$, for example, which passes through $(\frac{1}{2}, \frac{1}{\sqrt{2}})$. Hence, no point satisfies the McKelvey and Wendell (1976) conditions and thus no equilibrium exists.

for A which gives her a larger vote share then there is no deviation which gives her a larger win probability. What need not hold though is the uniqueness of a pure strategy equilibrium. If we allow voters to care to a different degree about each policy then each voter will not be characterized only by a bliss point $(x_1, x_2, ..., x_n)$ but also by a vector of weights $(\gamma_1, \gamma_2, ..., \gamma_n) \in (0, \gamma^{\max})^n$. In this case the distribution of voters should be $F : \mathbb{R}^{2n} \rightarrow [0,1]$ with support $S = (\prod_{j=1}^{n} [\psi_j, \omega_j]) \times (0, \gamma^{\max})^n$. If the utility of a voter with bliss point $(x_1, x_2, ..., x_n)$ and weights $(\gamma_1, \gamma_2, ..., \gamma_n)$ from policy vector $(y_1, y_2, ..., y_n)$ is a strictly decreasing function of $\sqrt{\sum_{j=1}^{n} [\gamma_j (x_j - y_j)^2]}$ then a voter with weights $(\hat{\gamma}_1, \hat{\gamma}_2, ..., \hat{\gamma}_n)$ who is indifferent between the platforms of the two candidates should have a bliss point $i = (i_1, i_2, ..., i_n)$ such that

$$i_1 = \frac{\sum_{j=2}^{n} \hat{\gamma}_j [(b_j - i_j)^2 - (a_j - i_j)^2]}{2\hat{\gamma}_1 d}.$$

Hence, by appropriately defining a conditional distribution of ideal policies regarding the first issue given ideal policies regarding all other issues and given weights, one may straightforwardly replicate all steps of the proof on Proposition 1 and establish existence of an equilibrium in this generalized framework too.

References

[1] Banks, J. S. and J. Duggan 2005. Probabilistic voting in the spatial model of elections: the theory of office motivated candidates. In D. Austen-Smith and J. Duggan, editors, *Social choice and strategic decisions: essays in honor of Jeffrey S. Banks*. Springer Verlag.

[2] Banks, J. S., Duggan, J., and M. Le Breton 2002. Bounds for mixed strategy equilibria and the spatial model of elections. *Journal of Economic Theory*, 103(1), 88-105.

[3] Banks, J. S., Duggan, J. and M. Le Breton 2006. Social choice and electoral competition in the general spatial model. *Journal of Economic Theory*, 126(1), 194-234.

[4] Bernheim, B. D. and S. N. Slavov 2009. A solution concept for majority rule in dynamic settings. *Review of Economic Studies*, 76(1), 33-62.

[5] Black, D. 1958. *The theory of committees and elections*. Cambridge: University Press.

[6] Bouton, L. and M. Castanheira 2012. One person, many votes: Divided majority and information aggregation. *Econometrica*, 80(1), 43-87.

[7] Calvert, R. L. 1985. Robustness of the multidimensional voting model: Candidate motivations, uncertainty, and convergence. *American Journal of Political Science*, 39, 69-95.

[8] Caplin, A. and B. Nalebuff 1991. Aggregation and social choice: a mean voter theorem. *Econometrica*, 59, 1-23.

[9] Davis, O. A., DeGroot, M. H. and M.J. Hinich 1972. Social preference orderings and majority rule. *Econometrica*, 40, 147-157.

[10] Debreu, G. 1952. A Social Equilibrium Existence Theorem. *Proceedings of the National Academy of Sciences of the U. S. A.*, 38.

[11] Downs, A. 1957. *An Economic Theory of Democracy*. New York: Harper and Row.

[12] Duggan, J. and M. Jackson 2004. Mixed Strategy Equilibrium and Deep Covering in Multi-dimensional Electoral Competition. *mimeo.*

[13] Dziubiński, M. and J. Roy 2011. Electoral Competition in 2-Dimensional Ideology Space with Unidimensional Commitment. *Social Choice and Welfare,* 36(1), 1-24.

[14] Enelow, J. M. and M. J. Hinich 1989. A general probabilistic spatial theory of elections. *Public Choice*, 61(2), 101-113.

[15] Glicksberg, I.L. 1952. A Further Generalization of the Kakutani Fixed Point Theorem, with Application to Nash Equilibrium Points. *Proceedings of the American Mathematical Society*, 3(3), 170–174.

[16] Hinich, M. J., Ledyard, J. O. and P.C. Ordeshook 1972. Nonvoting and the existence of equilibrium under majority rule. *Journal of Economic Theory*, 4(2), 144-153.

[17] Kramer, G. H. 1972. Sophisticated voting over multidimensional choice spaces. *Journal of Mathematical Sociology*, 2(2), 165-180.

[18] Krasa, S. and M. Polborn 2010. The Binary Policy Model. *Journal of Economic Theory,* 145(2), 661-88.

[19] Krasa, S. and M. Polborn 2012. Political Competition between Differentiated Candidates. *Games and Economic Behavior*, 76(1), 249–71.

[20] Krasa, S. and M. Polborn 2014. Social Ideology and Taxes in a Differentiated Candidates Framework. *American Economic Review*, 104(1), 308-322.

[21] Laffond, G., Laslier, J. F. and M Le Breton 1993. The bipartisan set of a tournament game. *Games and Economic Behavior*, 5(1), 182-201.

[22] Lindbeck, A. and J. Weibull 1987. Balanced-budget Redistribution as the Outcome of Political Competition. *Public Choice*, 52 (2): pp. 273-297

[23] Lindbeck, A. and J. Weibull 1993. A model of political equilibrium in a representative democracy. *Journal of Public Economics*, 51(2), 195-209.

[24] Matakos, K. and D. Xefteris 2014. Divide and Rule: Redistribution in a Model with Differentiated Candidates. *LSE Department of Government - PSPE Working Papers.*

[25] McKelvey, R. D. and R.E. Wendell 1976. Voting equilibria in multidimensional choice spaces. *Mathematics of Operations Research*, 1(2), 144-158.

[26] Plott, C. R. 1967. A notion of equilibrium and its possibility under majority rule. *American Economic Review*, 787-806.

[27] Schofield, N. J. 2007. The mean voter theorem: necessary and sufficient conditions for convergent equilibrium. *Review of Economic Studies*, 74, 965–980.

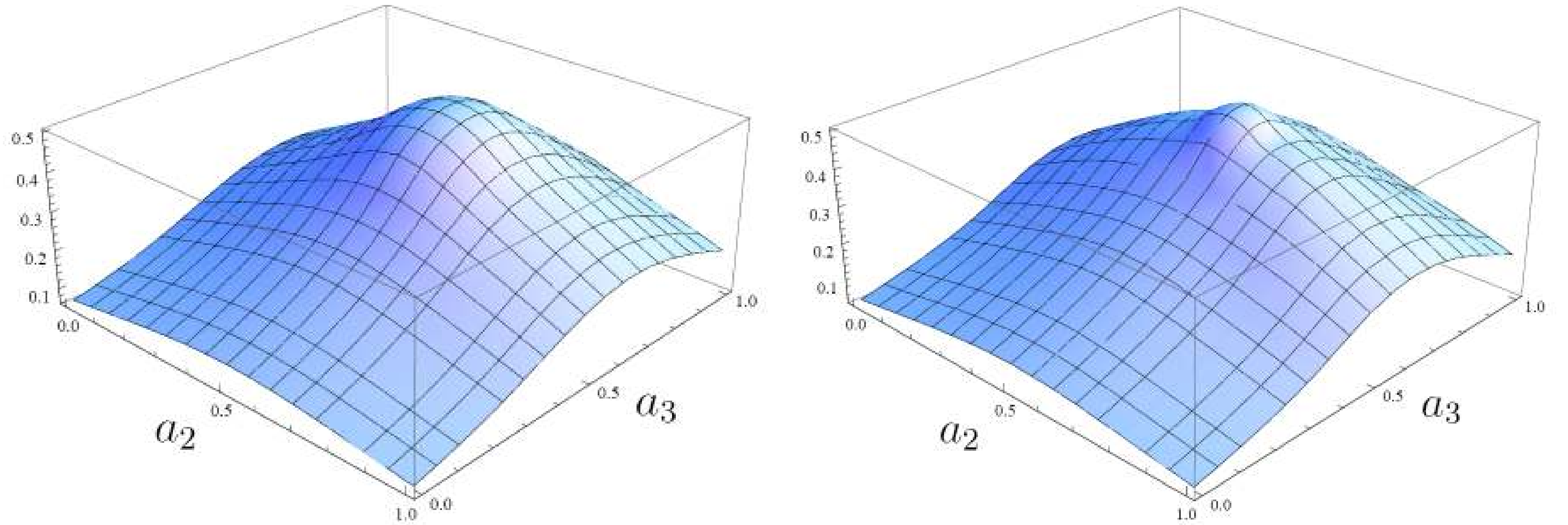

Figure 1. The vote share of candidate A when $(b_2, b_3) = (\frac{1}{2}, \frac{2}{3})$ as a function of her positions in the second and the third issue for $a_1 = 0.4$ and $b_1 = 0.6$ (left) and for $a_1 = 0.4$ and $b_1 = 0.5$ (right).

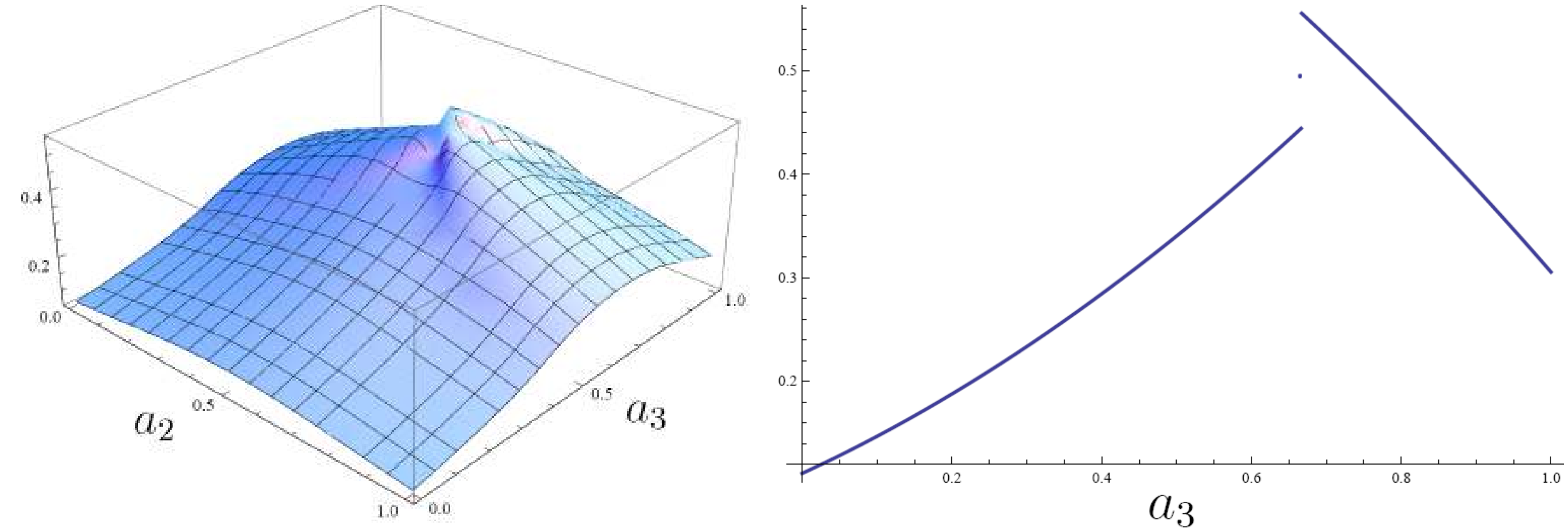

Figure 2. The vote share of candidate A when $(b_2, b_3) = (\frac{1}{2}, \frac{2}{3})$ as a function of her positions in the second and the third issue for $a_1 = b_1$ (left) and only as a function of her position in the third issue for $a_1 = b_1$ and $a_2 = \frac{1}{2}$ (right).